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从调和 Zygmund 型空间到调和 Bloch 型空间的复合算子差分

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摘 要: 本文研究了单位圆盘上调和 Zygmund 型空间 $\mathcal{Z}_{\mathcal{H}}^{\alpha}$ ($\alpha > 1$) 到调和 Bloch 型空间 $\mathcal{B}_{\mathcal{H}}^{\beta}(0 < \beta < \infty)$ 的复合算子差分的性质. 利用调和函数空间的性质、Stirling 公式和检验函数等工具，获得了复合算子 $C_\varphi : \mathcal{Z}_{\mathcal{H}}^{\alpha} \to \mathcal{B}_{\mathcal{H}}^{\beta}$ 的有界性与紧性的充分必要条件，进而建立了复合算子差分 $C_\varphi - C_\psi : \mathcal{Z}_{\mathcal{H}}^{\alpha} \to \mathcal{B}_{\mathcal{H}}^{\beta}$ 的有界性与紧性的等价刻画.

关 键 词: 差分; 复合算子; 调和 Zygmund 型空间; 调和 Bloch 型空间
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1 引言

设 $\mathbb{D}$ 为复平面 $\mathbb{C}$ 中的单位圆盘, $H(\mathbb{D})$ 为 $\mathbb{D}$ 上所有解析函数构成的集合, $S(\mathbb{D})$ 表示 $\mathbb{D}$ 上解析自映射构成的集合. 定义在 $\mathbb{D}$ 上的调和函数 f 是复值函数且满足

$$\Delta f := 4\frac{\partial^2 f}{\partial z \partial \overline{z}} \equiv 0,$$

记 $\mathcal{H}ar(\mathbb{D})$ 为 $\mathbb{D}$ 上所有调和函数组成的集合. 由调和函数和解析函数的关系知 $f \in \mathcal{H}ar(\mathbb{D})$. 当且仅当 f 有唯一分解 $f = g + \overline{h}$, 其中 $g, h \in H(\mathbb{D})$ 且 $h(0) = 0$. 记 $\varphi_a(z) := (a-z)/(1-\bar{a}z)$ 为 $\mathbb{D}$ 上交换 0 和 $a \in \mathbb{D}$ 的对合自同构. 对于给定的 $z, w \in \mathbb{D}$, z 与 w 之间的伪双曲度量为

$$\rho(z,w) := |\varphi_w(z)| = \left|\frac{z-w}{1-\bar{w}z}\right|.$$

为简便起见, 对于 $\varphi, \psi \in S(\mathbb{D})$, 记 $\rho(z) := \rho(\varphi(z), \psi(z))$. 对于每个 $\varphi \in S(\mathbb{D})$ 都可以定义一个复合算子 C_φ 为 $C_\varphi f = f \circ \varphi$, $f \in \mathcal{H}ar(\mathbb{D})$. 此时这样的算子保持了调和性. 对于任意两个赋范线性空间 X 和 Y, 线性算子 $T : X \longrightarrow Y$ 是有界的, 当且仅当存在 $C > 0$ 使得对于任意 $f \in X$ 成立 $\|Tf\|_Y \leq C\|f\|_X$. 如果 T 把 X 中的有界集映成 Y 中的列紧集, 则 T 是紧算子. 长期以来, 一些全纯函数空间上复合算子 C_φ 与复合算子差分 $C_\varphi - C_\psi$ 的算子性质得到了广泛的研究, 可参考 [1–15]. 但是对于调和函数构成的空间上相关的研究仍然有限. 因此, 本文主要研究从 (小) 调和 Zygmund 型空间到 (小) 调和 Bloch 型空间的复合算子 C_φ 与复合算子差分 $C_\varphi - C_\psi$ 的有界性与紧性.

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对于 $0<\alpha<\infty$, 调和 Zygmund 型空间 $\mathcal{Z}_{\mathcal{H}}^{\alpha}$ 包含所有 $f\in\mathcal{H}ar(\mathbb{D})$ 满足范数

$$\|f\|_{\mathcal{Z}_{\mathcal{H}}^{\alpha}}=|f(0)|+\left|\frac{\partial f}{\partial z}(0)\right|+\left|\frac{\partial f}{\partial \bar{z}}(0)\right|+\sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha}\left(\left|\frac{\partial^2 f}{\partial z^2}(z)\right|+\left|\frac{\partial^2 f}{\partial \bar{z}^2}(z)\right|\right)<\infty$$

成立. 进一步地, 小调和 Zygmund 型空间 $\mathcal{Z}_{\mathcal{H},0}^{\alpha}$ 定义为

$$\mathcal{Z}_{\mathcal{H},0}^{\alpha}=\left\{f\in\mathcal{Z}_{\mathcal{H}}^{\alpha}:\lim_{|z|\to1}(1-|z|^2)^{\alpha}\left(\left|\frac{\partial^2 f}{\partial z^2}(z)\right|+\left|\frac{\partial^2 f}{\partial \bar{z}^2}(z)\right|\right)=0\right\}.$$

对于 $0<\beta<\infty$, 调和 Bloch 型空间 $\mathcal{B}_{\mathcal{H}}^{\beta}$ 包含所有 $f\in\mathcal{H}ar(\mathbb{D})$ 使得范数

$$\|f\|_{\mathcal{B}_{\mathcal{H}}^{\beta}}=|f(0)|+\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\beta}\left(\left|\frac{\partial f}{\partial z}(z)\right|+\left|\frac{\partial f}{\partial \bar{z}}(z)\right|\right)<\infty$$

成立. 进而小调和 Bloch 型空间 $\mathcal{B}_{\mathcal{H},0}^{\beta}$ 定义为

$$\mathcal{B}_{\mathcal{H},0}^{\beta}=\left\{f\in\mathcal{B}_{\mathcal{H}}^{\beta}:\lim_{|z|\to1}\left(1-|z|^2\right)^{\beta}\left(\left|\frac{\partial f}{\partial z}(z)\right|+\left|\frac{\partial f}{\partial \bar{z}}(z)\right|\right)=0\right\}.$$

特别地, 当 $f\in H(\mathbb{D})$ 时, $\frac{\partial f}{\partial z}=f'$ 且 $\frac{\partial f}{\partial \bar{z}}=\frac{\partial^2 f}{\partial \bar{z}^2}=0$. 于是, 对于所有的 $0<\alpha<\infty$, $\mathcal{Z}_{\mathcal{H}}^{\alpha}$ 中所有解析函数组成的空间是经典的 Zygmund 型空间 $\mathcal{Z}^{\alpha}$, 其范数为

$$\|f\|_{\mathcal{Z}^{\alpha}}=|f(0)|+|f'(0)|+\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\alpha}|f''(z)|.$$

对于所有的 $0<\beta<\infty$, $\mathcal{B}_{\mathcal{H}}^{\beta}$ 中所有解析函数组成的空间是经典的 Bloch 型空间 $\mathcal{B}^{\beta}$, 其范数为 $\|f\|_{\mathcal{B}^{\beta}}=|f(0)|+\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\beta}|f'(z)|$. 由 [16, 定理 19] 知, 对于 $0<\alpha<\infty, f\in\mathcal{H}ar(\mathbb{D})$,

$$\|f\|_{\mathcal{B}_{\mathcal{H}}^{\alpha}}\approx\|f\|_{\mathcal{Z}_{\mathcal{H}}^{\alpha+1}}. \tag{1.1}$$

关于上述空间的更多资料, 请参阅 [13,14,17–22].

本文的结构如下：在第二节中, 我们引用了一些相关引理为后续证明做铺垫；在第三节中刻画了 $\alpha>1,\beta>0$ 时复合算子 $C_{\varphi}:\mathcal{Z}_{\mathcal{H}}^{\alpha}\to\mathcal{B}_{\mathcal{H}}^{\beta}$ 的有界性与紧性；在第四节中建立了复合算子差分 $C_{\varphi}-C_{\psi}:\mathcal{Z}_{\mathcal{H}}^{\alpha}\to\mathcal{B}_{\mathcal{H}}^{\beta}$ 相关性质的等价刻画.

在本文中, 符号 $A\lesssim B$ 与 $A\gtrsim B$ 意味着存在正整数 C（其确切值可能不同）, 使得 $A\leqslant CB$ 与 $A\geqslant CB$ 分别成立. $A\approx B$ 当且仅当 $A\lesssim B$ 且 $A\gtrsim B$.

2 相关引理

在本节中, 我们提供了几个引理用于证明第二节和第三节中的定理.

引理 2.1[8, 引理 2.1] 设 $1<\alpha<\infty$. 对于任意 $f\in\mathcal{Z}^{\alpha}$, 有

$$\left|\left(1-|z|^2\right)^{\alpha-1}f'(z)-\left(1-|w|^2\right)^{\alpha-1}f'(w)\right|\lesssim\|f\|_{\mathcal{Z}_{\alpha}}\rho(z,w),\quad z,w\in\mathbb{D}.$$

对 [21, 定理 1.14] 稍作修改可得到如下引理.

引理 2.2 设 $\alpha > 1, \beta > 0$, 且 $T : \mathcal{Z}_{\mathcal{H}}^{\alpha} \to \mathcal{B}_{\mathcal{H}}^{\beta}$ 是一个有界线性算子, 则 T 是紧算子当且仅当函数序列 $\{f_n\}$ 在 $\mathcal{Z}_{\mathcal{H}}^{\alpha}$ 上是有界的, 且当 f_n 在 $\mathbb{D}$ 的紧子集上一致收敛到 0 时, 有 $\|Tf_n\|_{\mathcal{B}_{\mathcal{H}}^{\beta}} \to 0, n \to \infty$.

引理 2.3[23, 引理 1.1] 设 $\alpha > 1$, 则对于每个 $f \in \mathcal{Z}^{\alpha}$, 有 $|f'(z)| \lesssim \frac{\|f\|_{\mathcal{Z}^{\alpha}}}{(1-|z|^2)^{\alpha-1}}$.

3 $C_{\varphi} : \mathcal{Z}_{\mathcal{H}}^{\alpha} \to \mathcal{B}_{\mathcal{H}}^{\beta}$ 的有界性与紧性

在本节中, 主要研究了复合算子 $C_{\varphi} : \mathcal{Z}_{\mathcal{H}}^{\alpha} \to \mathcal{B}_{\mathcal{H}}^{\beta}$ 的相关性质, 其结论将会应用于下一节复合算子差分 $C_{\varphi} - C_{\psi} : \mathcal{Z}_{\mathcal{H}}^{\alpha} \to \mathcal{B}_{\mathcal{H}}^{\beta}$ 的相关性质中. 对于 $\alpha > 1$, $\beta > 0$ 与 $\varphi \in S(\mathbb{D})$, 记

$$\varphi_{\#}^{\alpha,\beta}(z) := \frac{(1-|z|^2)^{\beta}\, \varphi'(z)}{(1-|\varphi(z)|^2)^{\alpha-1}}.$$

定理 3.1 设 $\alpha > 1$, $\beta > 0$ 且 $\varphi \in S(\mathbb{D})$, 则下列条件等价:

(1)$C_{\varphi} : \mathcal{Z}_{\mathcal{H}}^{\alpha} \to \mathcal{B}_{\mathcal{H}}^{\beta}$ 是有界的.

(2)$\sup\limits_{j\in\mathbb{N}} \|C_{\varphi}P_j\|_{\mathcal{B}_{\mathcal{H}}^{\beta}} < \infty, P_j = (z^j + \overline{z}^j)j^{\alpha}, j \in \mathbb{N}$.

(3)$\sup\limits_{z\in\mathbb{D}} |\varphi_{\#}^{\alpha,\beta}(z)| < \infty$.

证 首先证明 (1) $\Rightarrow$ (2). 假设 $C_{\varphi} : \mathcal{Z}_{\mathcal{H}}^{\alpha} \to \mathcal{B}_{\mathcal{H}}^{\beta}$ 是有界的, 从而对任意 $f \in \mathcal{Z}_{\mathcal{H}}^{\alpha}$, 有 $||C_{\varphi}f||_{B_{\mathcal{H}}^{\beta}} \lesssim ||f||_{\mathcal{Z}_{\mathcal{H}}^{\alpha}}$. 由计算得 $P_j \in \mathcal{Z}_{\mathcal{H}}^{\alpha}$, 于是对任意 $j \in \mathbb{N}$ 成立 $||C_{\varphi}P_j||_{B_{\mathcal{H}}^{\beta}} \lesssim ||P_j||_{\mathcal{Z}_{\mathcal{H}}^{\alpha}} < \infty$. 故 (2) 成立.

现假设 (3) 成立. 设 $f \in \mathcal{Z}_{\mathcal{H}}^{\alpha}$, 根据 (1.1), 有

$$\begin{aligned}
\|C_{\varphi}f\|_{\mathcal{B}_{\mathcal{H}}^{\beta}} &= |f(\varphi(0))| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\left(\left|\frac{\partial(C_{\varphi}f)}{\partial z}(z)\right| + \left|\frac{\partial(C_{\varphi}f)}{\partial \overline{z}}(z)\right|\right)\\
&= |f(\varphi(0))| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\,|\varphi'(z)|\left(\left|\frac{\partial f}{\partial z}(\varphi(z))\right| + \left|\frac{\partial f}{\partial \overline{z}}(\varphi(z))\right|\right)\\
&\le |f(\varphi(0))| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\,|\varphi'(z)|\,\frac{\|f\|_{\mathcal{B}_{\mathcal{H}}^{\alpha-1}}}{(1-|\varphi(z)|^2)^{\alpha-1}}\\
&\approx |f(\varphi(0))| + \|f\|_{\mathcal{Z}_{\mathcal{H}}^{\alpha}} \cdot \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\beta}\,|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}} \lesssim \|f\|_{\mathcal{Z}_{\mathcal{H}}^{\alpha}}.
\end{aligned} \tag{3.1}$$

于是 $C_{\varphi} : \mathcal{Z}_{\mathcal{H}}^{\alpha} \to \mathcal{B}_{\mathcal{H}}^{\beta}$ 有界. 因此 (3) $\Rightarrow$ (1) 成立.

下面证明 (2) $\Rightarrow$ (3). 假设 $L := \sup_{j\in\mathbb{N}} \|C_{\varphi}P_j\|_{\mathcal{B}_{\mathcal{H}}^{\beta}} < \infty$. 因为 $C_{\varphi}P_1 = \varphi + \overline{\varphi}$, 所以对任意 $z \in \mathbb{D}$, $\left|\frac{\partial[C_{\varphi}P_1(z)]}{\partial z}\right| = \left|\frac{\partial[C_{\varphi}P_1(z)]}{\partial \overline{z}}\right| = |\varphi'(z)|$. 于是

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|\varphi'(z)| \le \frac{1}{2}\|C_{\varphi}P_1\|_{\mathcal{B}_{\mathcal{H}}^{\beta}} \le \frac{L}{2}. \tag{3.2}$$

令 $0 < s < 1$, 当 $|\varphi(z)| \le s$ 时,

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\beta}|\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^{\alpha-1}} \le \frac{L}{2(1-s^2)^{\alpha-1}} < \infty.$$

当 $|\varphi(z)|>s$ 时, 对于固定的 $b\in\mathbb{D}$, 定义

$$F_b^\alpha(z):=\frac{(1-|b|^2)^3}{(1-\bar{b}z)^{\alpha+1}}+\frac{(1-|b|^2)^3}{(1-b\overline{z})^{\alpha+1}}. \tag{3.3}$$

当 $|b|\to 1$ 时, $F_b^\alpha(z)$ 在 $\mathbb{D}$ 的紧子集上一致收敛到 0. 由 Stirling 公式, $F_b^\alpha(z)$ 的级数展开式为

$$F_b^\alpha(z)=(1-|b|^2)^3\sum_{j=0}^{\infty}\frac{\Gamma(j+\alpha+1)}{j!\Gamma(\alpha+1)}\Big\{(\bar{b}z)^j+(b\overline{z})^j\Big\}\approx(1-|b|^2)^3\sum_{j=0}^{\infty}j^\alpha\Big\{(\bar{b}z)^j+(b\overline{z})^j\Big\}. \tag{3.4}$$

于是, 对于 $z\in\mathbb{D}$, $\|C_\varphi F_{\varphi(z)}^\alpha\|_{\mathcal{B}_{\mathcal{H}}^\beta}\lesssim(1-|\varphi(z)|^2)^3\sum\limits_{j=0}^{\infty}|\varphi(z)|^j\|C_\varphi P_j\|_{\mathcal{B}_{\mathcal{H}}^\beta}\le L\sum\limits_{j=0}^{\infty}|\varphi(z)|^j\lesssim L$. 通过直接计算, 对任意 $z\in\mathbb{D}$, 我们得到

$$\frac{\partial\left[C_\varphi F_{\varphi(z)}^\alpha\right]}{\partial z}(z)=\frac{(\alpha+1)\,\overline{\varphi(z)}\varphi'(z)}{\left(1-|\varphi(z)|^2\right)^{\alpha-1}},\quad \frac{\partial\left[C_\varphi F_{\varphi(z)}^\alpha\right]}{\partial\overline{z}}(z)=\frac{(\alpha+1)\,\varphi(z)\,\overline{\varphi'(z)}}{\left(1-|\varphi(z)|^2\right)^{\alpha-1}}.$$

于是

$$\begin{aligned}\frac{(1-|z|^2)^\beta|\varphi'(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}}&\le\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|\varphi'(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}}\\&\lesssim\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\left(\left|\frac{\partial\left[C_\varphi F_{\varphi(z)}^\alpha\right]}{\partial z}(z)\right|+\left|\frac{\partial\left[C_\varphi F_{\varphi(z)}^\alpha\right]}{\partial\overline{z}}(z)\right|\right)\\&\le\|C_\varphi F_{\varphi(z)}^\alpha\|_{B_{\mathcal{H}}^\beta}\lesssim L.\end{aligned} \tag{3.5}$$

由于 $|\varphi(z)|>s$, 对任意 $z\in\mathbb{D}$,

$$\frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}}\lesssim\frac{L}{|\varphi(z)|}<\frac{L}{s}<\infty,$$

综上所述, $\sup_{z\in\mathbb{D}}|\varphi_\#^{\alpha,\beta}(z)|<\infty$. 由此证明了 (2) $\Rightarrow$ (3).

下面讨论 $C_\varphi:\mathcal{Z}_{\mathcal{H}}^\alpha\to\mathcal{B}_{\mathcal{H}}^\beta$ 的紧性.

定理 3.2 设 $\alpha>1$, $\beta>0$ 且 $\varphi\in S(\mathbb{D})$, 则下列条件等价:

(1)$C_\varphi:\mathcal{Z}_{\mathcal{H}}^\alpha\to\mathcal{B}_{\mathcal{H}}^\beta$ 是紧的.

(2)$\lim\limits_{j\to\infty}\|C_\varphi P_j\|_{\mathcal{B}_{\mathcal{H}}^\beta}=0, P_j=(z^j+\overline{z}^j)j^\alpha, j\in\mathbb{N}$.

(3)$\lim\limits_{|\varphi(z)|\to 1}|\varphi_\#^{\alpha,\beta}(z)|=0$.

证 先证 (1) $\Rightarrow$ (2). 有界集 $\{P_j\}\subseteq\mathcal{Z}_{\mathcal{H}}^\alpha$ 在 $\mathbb{D}$ 的紧子集上一致收敛到 0. 因为 $C_\varphi:\mathcal{Z}_{\mathcal{H}}^\alpha\to\mathcal{B}_{\mathcal{H}}^\beta$ 是紧的, 根据引理 2.2, 有 $\lim\limits_{j\to\infty}\|C_\varphi P_j\|_{\mathcal{B}_{\mathcal{H}}^\beta}=0$.

现假设 (2) 成立, 则 $L := \sup\limits_{j\in\mathbb{N}} \|C_\varphi P_j\|_{\mathcal{B}^\beta_\mathcal{H}} < \infty$. 且对任意 $\varepsilon > 0$, 存在 $N \in \mathbb{N}$ 使得 $\|C_\varphi P_j\|_{\mathcal{B}^\beta_\mathcal{H}} < \varepsilon, j \geq N$. 利用 (3.3) 中的检验函数 $F^\alpha_b(z)$, 对任意 $z \in \mathbb{D}$, 有

$$\begin{aligned}\left\|C_\varphi F^\alpha_{\varphi(z)}\right\|_{\mathcal{B}^\beta_\mathcal{H}} &\lesssim (1-|\varphi(z)|^2)^3\left[\left(\sum_{j=0}^{N-1}+\sum_{j=N}^{\infty}\right)|\varphi(z)|^j\|C_\varphi P_j\|_{\mathcal{B}^\beta_\mathcal{H}}\right]\\ &\leq (1-|\varphi(z)|^2)^3\sum_{j=0}^{N-1}\|C_\varphi P_j\|_{\mathcal{B}^\beta_\mathcal{H}}+\varepsilon\sum_{j=N}^{\infty}|\varphi(z)|^j\\ &\lesssim (1-|\varphi(z)|^2)^3 NL+\varepsilon.\end{aligned}$$

取 $s = \left[1-\left(\frac{\varepsilon}{NL}\right)^{\frac{1}{3}}\right]^{\frac{1}{2}} \in (0,1)$, 对任意 $|\varphi(z)| > s$ 有 $\left\|C_\varphi F^\alpha_{\varphi(z)}\right\|_{\mathcal{B}^\beta_\mathcal{H}} < 2\varepsilon$. 由 (3.5) 知,

$$\frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}} \lesssim \frac{\left\|C_\varphi F^\alpha_{\varphi(z)}\right\|_{\mathcal{B}^\beta_\mathcal{H}}}{|\varphi(z)|} < \frac{2\varepsilon}{s} \lesssim \varepsilon,$$

从而

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}} = 0.$$

因此, (2) $\Rightarrow$ (3) 成立.

下证 (3) $\Rightarrow$ (1). 假设 $\lim\limits_{|\varphi(z)|\to 1}|\varphi^{\alpha,\beta}_\#(z)| = 0$, 则对任意 $\varepsilon > 0$, 存在 $s \in (0,1)$, 当 $s < |\varphi(z)| < 1$ 有 $|\varphi^{\alpha,\beta}_\#(z)| = \frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}} < \varepsilon$. 取函数列 $\{f_j\} \subseteq \mathcal{Z}^\alpha_\mathcal{H}$ 满足 $M := \sup_{j\in\mathbb{N}}\|f_j\|_{\mathcal{Z}^\alpha_\mathcal{H}} < \infty$, 且在 $\mathbb{D}$ 的紧子集上一致收敛到 0, 只需证 $\lim\limits_{j\to\infty}\|C_\varphi f_j\|_{\mathcal{B}^\beta_\mathcal{H}} = 0$. 由柯西估计知, 函数列 $\left\{\frac{\partial f_j}{\partial z}\right\}$ 与 $\left\{\frac{\partial f_j}{\partial \bar{z}}\right\}$ 也在 $\mathbb{D}$ 的紧子集上一致收敛到 0. 于是在 $\mathbb{D}$ 的紧子集上, 对上述 $\varepsilon > 0$, 存在 $N \in \mathbb{N}^+$, 对任意 $j > N$, 有

$$\left|\frac{\partial f_j}{\partial z}(\varphi(z))\right| < \varepsilon, \quad \left|\frac{\partial f_j}{\partial \bar{z}}(\varphi(z))\right| < \varepsilon.$$

根据 (1.1) 与 (3.2),

$$\begin{aligned}&\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\left(\left|\frac{\partial[C_\varphi f_j(z)]}{\partial z}\right|+\left|\frac{\partial[C_\varphi f_j(z)]}{\partial \bar{z}}\right|\right)\\ \leq&\left(\sup_{1>|\varphi(z)|>s}+\sup_{|\varphi(z)|\leq s}\right)(1-|z|^2)^\beta|\varphi'(z)|\left(\left|\frac{\partial f_j}{\partial z}(\varphi(z))\right|+\left|\frac{\partial f_j}{\partial \bar{z}}(\varphi(z))\right|\right)\\ \leq&\|f_j\|_{\mathcal{B}^{\alpha-1}_\mathcal{H}}\sup_{1>|\varphi(z)|>s}\frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}}+\frac{L}{2}\sup_{|\varphi(z)|\leq s}\left(\left|\frac{\partial f_j}{\partial z}(\varphi(z))\right|+\left|\frac{\partial f_j}{\partial \bar{z}}(\varphi(z))\right|\right)\\ \lesssim&\|f_j\|_{\mathcal{Z}^\alpha_\mathcal{H}}\cdot\varepsilon+\frac{L}{2}\cdot 2\varepsilon \leq (M+L)\varepsilon.\end{aligned}$$

这意味着,

$$\lim_{j\to\infty}\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\left(\left|\frac{\partial[C_\varphi f_j(z)]}{\partial z}\right|+\left|\frac{\partial[C_\varphi f_j(z)]}{\partial \bar{z}}\right|\right) = 0.$$

又 $\lim\limits_{j\to\infty}|C_\varphi f_j(0)| = \lim\limits_{j\to\infty}|f_j(\varphi(0))| = 0$, 于是 $\lim\limits_{j\to\infty}\|C_\varphi f_j\|_{\mathcal{B}^\beta_\mathcal{H}} = 0$. 由此证明了 (3) $\Rightarrow$ (1).

4 $C_\varphi - C_\psi : \mathcal{Z}_{\mathcal{H}}^{\alpha} \to \mathcal{B}_{\mathcal{H}}^{\beta}$ 的有界性与紧性

基于定理 3.1 和定理 3.2, 本节利用文献 [25] 中的类似方法研究调和空间上复合算子的差分 $C_\varphi - C_\psi : \mathcal{Z}_{\mathcal{H}}^{\alpha} \to \mathcal{B}_{\mathcal{H}}^{\beta}$ 的有界性和紧性.

定理 4.1 设 $\alpha > 1, \beta > 0$ 且 $\varphi, \psi \in S(\mathbb{D})$, 则下列条件等价:

(1)$C_\varphi - C_\psi : \mathcal{Z}_{\mathcal{H}}^{\alpha} \to \mathcal{B}_{\mathcal{H}}^{\beta}$ 有界.

(2)$C_\varphi - C_\psi : \mathcal{Z}_{\mathcal{H},0}^{\alpha} \to \mathcal{B}_{\mathcal{H}}^{\beta}$ 有界.

(3) 下列不等式成立:

$$\sup_{z\in\mathbb{D}} \left(\left|\varphi_{\#}^{\alpha,\beta}(z)\right| \rho(z) + \left|\varphi_{\#}^{\alpha,\beta}(z) - \psi_{\#}^{\alpha,\beta}(z)\right| \right) < \infty. \tag{4.1}$$

证 (1) $\Rightarrow$ (2) 是显然的. 现假设 (4.1) 成立. 设 $f \in \mathcal{Z}_{\mathcal{H}}^{\alpha}$ 且 $\|f\|_{\mathcal{Z}_{\mathcal{H}}^{\alpha}} \leqslant 1$. 令 $f = g + \bar{h}$, 其中 $g, h \in H(\mathbb{D}), h(0) = 0$, 则 $g, h \in \mathcal{Z}^{\alpha}$ 且 $\|g\|_{\mathcal{Z}^{\alpha}} \leqslant 1, \|h\|_{\mathcal{Z}^{\alpha}} \leqslant 1$. 于是

$$\begin{aligned}
& \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta \left(\left| \frac{\partial\left(\left(C_\varphi - C_\psi\right) f\right)}{\partial z}(z) \right| + \left| \frac{\partial\left(\left(C_\varphi - C_\psi\right) f\right)}{\partial \overline{z}}(z) \right| \right) \\
= \ & \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta \left(|g'(\varphi(z))\varphi'(z) - g'(\psi(z))\psi'(z)| + \left| \overline{h'(\varphi(z))\varphi'(z)} - \overline{h'(\psi(z))\psi'(z)} \right| \right) \\
\leq \ & \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta |g'(\varphi(z))\varphi'(z) - g'(\psi(z))\psi'(z)| \\
& + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta |h'(\varphi(z))\varphi'(z) - h'(\psi(z))\psi'(z)| \\
= \ & L_1 + L_2,
\end{aligned} \tag{4.2}$$

其中

$$\begin{aligned}
L_1 &:= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta |g'(\varphi(z))\varphi'(z) - g'(\psi(z))\psi'(z)|, \\
L_2 &:= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta |h'(\varphi(z))\varphi'(z) - h'(\psi(z))\psi'(z)|.
\end{aligned}$$

根据引理 2.1 与引理 2.3, 有

$$\begin{aligned}
L_1 = & \sup_{z\in\mathbb{D}} \Big| \varphi_{\#}^{\alpha,\beta}(z) \left[\left(1-|\varphi(z)|^2\right)^{\alpha-1} g'(\varphi(z)) - \left(1-|\psi(z)|^2\right)^{\alpha-1} g'(\psi(z)) \right] \\
& + \left(1-|\psi(z)|^2\right)^{\alpha-1} g'(\psi(z)) \left(\varphi_{\#}^{\alpha,\beta}(z) - \psi_{\#}^{\alpha,\beta}(z)\right) \Big| \\
\lesssim & \sup_{z\in\mathbb{D}} \left(\left|\varphi_{\#}^{\alpha,\beta}(z)\right| \rho(z) + \left(1-|\psi(z)|^2\right)^{\alpha-1} |g'(\psi(z))| \left|\varphi_{\#}^{\alpha,\beta}(z) - \psi_{\#}^{\alpha,\beta}(z)\right| \right) \\
\lesssim & \sup_{z\in\mathbb{D}} \left(\left|\varphi_{\#}^{\alpha,\beta}(z)\right| \rho(z) + \left|\varphi_{\#}^{\alpha,\beta}(z) - \psi_{\#}^{\alpha,\beta}(z)\right| \right) < \infty.
\end{aligned}$$

同理可证

$$L_2 \lesssim \sup_{z\in\mathbb{D}} \left(\left|\varphi_{\#}^{\alpha,\beta}(z)\right| \rho(z) + \left|\varphi_{\#}^{\alpha,\beta}(z) - \psi_{\#}^{\alpha,\beta}(z)\right| \right) < \infty.$$

故对于任意的 $f \in \mathcal{Z}_{\mathcal{H}}^{\alpha}$, 有

$$\|\left(C_\varphi - C_\psi\right) f\|_{\mathcal{B}_{\mathcal{H}}^{\beta}} \lesssim |f(\varphi(0)) - f(\psi(0))| + \sup_{z\in\mathbb{D}} \left(\left|\varphi_{\#}^{\alpha,\beta}(z)\right| \rho(z) + \left|\varphi_{\#}^{\alpha,\beta}(z) - \psi_{\#}^{\alpha,\beta}(z)\right| \right) < \infty,$$

这意味着 $C_\varphi - C_\psi : \mathcal{Z}_{\mathcal{H}}^{\alpha} \to \mathcal{B}_{\mathcal{H}}^{\beta}$ 有界. 因此 (3) ⇒ (1) 成立.

接下来证明 (2) ⇒ (3). 假设 $C_\varphi - C_\psi : \mathcal{Z}_{\mathcal{H},0}^{\alpha} \to \mathcal{B}_{\mathcal{H}}^{\beta}$ 有界. 设 $\mathbb{D}_1 = \{w \in \mathbb{D} :\ \varphi(w) = 0\}$ 且 $\mathbb{D}_2 = \{w \in \mathbb{D} :\ \psi(w) = 0\}$. 对于固定的 $a \in \mathbb{D} \setminus \{0\}$, 定义

$$F_a(z) := f_a(z) + \overline{f_a(z)}, z \in \mathbb{D}, \tag{4.3}$$

其中

$$f_a(z) = \begin{cases} \frac{\left(1-|a|^2\right)^{\alpha-1}}{(2\alpha-3)\bar{a}(1-\bar{a}z)^{2\alpha-3}}, & \alpha \neq \frac{3}{2}, \\ -\frac{\left(1-|a|^2\right)^{\frac{1}{2}}\ln(1-\bar{a}z)}{\bar{a}}, & \alpha = \frac{3}{2}. \end{cases}$$

利用文献 [25] 中 $f_a \in \mathcal{Z}_0^{\alpha}$ 可得 $F_a \in \mathcal{Z}_{\mathcal{H},0}^{\alpha}$. 进而对于 $w \notin \mathbb{D}_1$ 成立

$$\begin{aligned} \infty &> \left\|(C_\varphi - C_\psi)\, F_{\varphi(w)}\right\|_{\mathcal{B}_{\mathcal{H}}^{\beta}} \\ &\geqslant \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\beta} \left(\left|\frac{\partial\left((C_\varphi - C_\psi)\, F_{\varphi(w)}\right)}{\partial z}(z)\right| + \left|\frac{\partial\left((C_\varphi - C_\psi)\, F_{\varphi(w)}\right)}{\partial \bar{z}}(z)\right| \right) \\ &= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\beta} \left(\left|f'_{\varphi(w)}(\varphi(z))\varphi'(z) - f'_{\varphi(w)}(\psi(z))\psi'(z)\right| + \left|\overline{f'_{\varphi(w)}(\varphi(z))\varphi'(z)} - \overline{f'_{\varphi(w)}(\psi(z))\psi'(z)}\right| \right) \\ &= 2\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\beta} \left|f'_{\varphi(w)}(\varphi(z))\varphi'(z) - f'_{\varphi(w)}(\psi(z))\psi'(z)\right|. \end{aligned}$$

类似于 [25, 定理 3.3] 的证明,

$$\sup_{w\in\mathbb{D}\setminus\mathbb{D}_1} \left| \varphi_{\#}^{\alpha,\beta}(w) - \psi_{\#}^{\alpha,\beta}(w) \left(\frac{(1-|\varphi(w)|^2)\,(1-|\psi(w)|^2)}{(1-\overline{\varphi(w)}\psi(w))^2} \right)^{\alpha-1} \right| < \infty. \tag{4.4}$$

进一步得到

$$\sup_{w\in\mathbb{D}\setminus\mathbb{D}_1} \left(\left|\varphi_{\#}^{\alpha,\beta}(w) - \psi_{\#}^{\alpha,\beta}(w)\right| - \left|\psi_{\#}^{\alpha,\beta}(w)\right| \rho(w) \right) < \infty. \tag{4.5}$$

另一方面, 对于固定的 $a \in \mathbb{D} \setminus \{0\}$, 定义

$$K_a(z) := k_a(z) + \overline{k_a(z)}, z \in \mathbb{D}, \tag{4.6}$$

其中

$$k_a(z) = \begin{cases} \frac{(a-z)\left(1-|a|^2\right)^{\alpha-1}}{(2\alpha-2)\bar{a}(1-\bar{a}z)^{2\alpha-2}} + \frac{\left(1-|a|^2\right)^{\alpha-1}}{(2\alpha-2)(2\alpha-3)\bar{a}^2(1-\bar{a}z)^{2\alpha-3}}, & \alpha \neq \frac{3}{2}, \\ \frac{(a-z)\left(1-|a|^2\right)^{\frac{1}{2}}}{\bar{a}(1-\bar{a}z)} - \frac{\left(1-|a|^2\right)^{\frac{1}{2}}\ln(1-\bar{a}z)}{\bar{a}^2}, & \alpha = \frac{3}{2}. \end{cases}$$

利用文献 [25] 中 $k_a \in \mathcal{Z}_0^{\alpha}$ 可得 $K_a \in \mathcal{Z}_{\mathcal{H},0}^{\alpha}$. 进而对于 $w \notin \mathbb{D}_1$, 有

$$\begin{aligned} \infty &> \left\|(C_\varphi - C_\psi)\, K_{\varphi(w)}\right\|_{\mathcal{B}_{\mathcal{H}}^{\beta}} \\ &\geqslant \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\beta} \left(\left|\frac{\partial\left((C_\varphi - C_\psi)\, K_{\varphi(w)}\right)}{\partial z}(z)\right| + \left|\frac{\partial\left((C_\varphi - C_\psi)\, K_{\varphi(w)}\right)}{\partial \bar{z}}(z)\right| \right) \\ &= 2\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\beta} \left|k'_{\varphi(w)}(\varphi(z))\varphi'(z) - k'_{\varphi(w)}(\psi(z))\psi'(z)\right| \\ &\geqslant 2\left|\psi_{\#}^{\alpha,\beta}(w) \left(\frac{\left(1-|\varphi(w)|^2\right)\left(1-|\psi(w)|^2\right)}{(1-\overline{\varphi(w)}\psi(w))^2} \right)^{\alpha-1}\right| \rho(w). \end{aligned}$$

这意味着

$$\sup_{w\in\mathbb{D}\setminus\mathbb{D}_1}\left|\psi_{\#}^{\alpha,\beta}(w)\left(\frac{(1-|\varphi(w)|^2)(1-|\psi(w)|^2)}{(1-\overline{\varphi(w)}\psi(w))^2}\right)^{\alpha-1}\right|\rho(w)<\infty. \tag{4.7}$$

将 (4.7) 代入 (4.4) 可得

$$\sup_{w\in\mathbb{D}\setminus\mathbb{D}_1}\left|\varphi_{\#}^{\alpha,\beta}(w)\right|\rho(w)<\infty. \tag{4.8}$$

同理可得

$$\sup_{w\in\mathbb{D}\setminus\mathbb{D}_2}\left|\psi_{\#}^{\alpha,\beta}(w)\right|\rho(w)<\infty. \tag{4.9}$$

结合 (4.5) 与 (4.9), 我们便可得出

$$\sup_{w\in\mathbb{D}\setminus(\mathbb{D}_1\cup\mathbb{D}_2)}\left|\varphi_{\#}^{\alpha,\beta}(w)-\psi_{\#}^{\alpha,\beta}(w)\right|<\infty. \tag{4.10}$$

故将 (4.8) 与 (4.10) 结合即可得到如下结论

$$\sup_{w\in\mathbb{D}\setminus(\mathbb{D}_1\cup\mathbb{D}_2)}\left(\left|\varphi_{\#}^{\alpha,\beta}(w)\right|\rho(w)+\left|\varphi_{\#}^{\alpha,\beta}(w)-\psi_{\#}^{\alpha,\beta}(w)\right|\right)<\infty. \tag{4.11}$$

若 $w\in\mathbb{D}_1\cap\mathbb{D}_2$, 则 $\rho(w)=0$. 我们取 $H(z):=z+\bar{z}\in\mathcal{Z}_{\mathcal{H},0}^{\alpha}$, 因此得到

$$\begin{aligned}\infty&>\|(C_\varphi-C_\psi)H\|_{\mathcal{B}_{\mathcal{H}}^{\beta}}\\&\geqslant\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\beta}\left(\left|\frac{\partial\left((C_\varphi-C_\psi)H\right)}{\partial z}(z)\right|+\left|\frac{\partial\left((C_\varphi-C_\psi)H\right)}{\partial\overline{z}}(z)\right|\right)\\&=\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\beta}\left(|\varphi'(z)-\psi'(z)|+\left|\overline{\varphi'(z)}-\overline{\psi'(z)}\right|\right)\\&=2\left|\left(1-|w|^2\right)^{\beta}\varphi'(w)-\left(1-|w|^2\right)^{\beta}\psi'(w)\right|\\&=\left|\varphi_{\#}^{\alpha,\beta}(w)\right|\cdot\rho(w)+\left|\varphi_{\#}^{\alpha,\beta}(w)-\psi_{\#}^{\alpha,\beta}(w)\right|,\end{aligned}$$

其意味着

$$\sup_{w\in\mathbb{D}_1\cap\mathbb{D}_2}\left(\left|\varphi_{\#}^{\alpha,\beta}(w)\right|\cdot\rho(w)+\left|\varphi_{\#}^{\alpha,\beta}(w)-\psi_{\#}^{\alpha,\beta}(w)\right|\right)<\infty. \tag{4.12}$$

若 $w\in\mathbb{D}_2\setminus\mathbb{D}_1$, 则 $\rho(w)=|\varphi(w)|$. 对于固定的 $a\in\mathbb{D}\setminus\{0\}$, 选取 $P_a(z):=p_a(z)+\overline{p_a(z)}$, 其中

$$p_a(z)=\begin{cases}-\frac{1}{\bar{a}^2}[\ln(1-\bar{a}z)+\bar{a}z], & \alpha=2,\\ \frac{1}{\bar{a}^2}\left[\frac{1}{1-\bar{a}z}+\ln(1-\bar{a}z)\right], & \alpha=3,\\ \frac{1}{\bar{a}^2}\left[\frac{1}{(\alpha-2)(1-\bar{a}z)^{\alpha-2}}-\frac{1}{(\alpha-3)(1-\bar{a}z)^{\alpha-3}}\right], & \alpha\neq2,\alpha\neq3.\end{cases}$$

利用文献 [25] 中 $p_a \in \mathcal{Z}_0^\alpha$ 可得 $P_a \in \mathcal{Z}_{\mathcal{H},0}^\alpha$. 由此可得

$$\begin{aligned}
\infty &> \left\|(C_\varphi - C_\psi) P_{\varphi(w)}\right\|_{\mathcal{B}_{\mathcal{H}}^\beta} \\
&\geqslant \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta \left(\left|\frac{\partial\left((C_\varphi - C_\psi) P_{\varphi(w)}\right)}{\partial z}(z)\right| + \left|\frac{\partial\left((C_\varphi - C_\psi) P_{\varphi(w)}\right)}{\partial \overline{z}}(z)\right|\right) \\
&\geqslant \left|\left(1-|w|^2\right)^\beta \varphi'(w) \frac{\varphi(w)}{\left(1-|\varphi(w)|^2\right)^{\alpha-1}} - 0\right| + \left|\left(1-|w|^2\right)^\beta \overline{\varphi'(w)} \frac{\overline{\varphi(w)}}{\left(1-|\varphi(w)|^2\right)^{\alpha-1}} - 0\right| \\
&= 2\frac{\left(1-|w|^2\right)^\beta |\varphi'(w)|}{\left(1-|\varphi(w)|^2\right)^{\alpha-1}} \rho(w) = 2\left|\varphi_\#^{\alpha,\beta}(w)\right| \rho(w).
\end{aligned}$$

这意味着 $\sup\limits_{w\in\mathbb{D}_2\setminus\mathbb{D}_1} \left|\varphi_\#^{\alpha,\beta}(w)\right| \rho(w) < \infty$.

类似地, 对于固定的 $a \in \mathbb{D}$, 我们定义 $q_a(z) = az - \frac{1}{2}z^2 \in \mathcal{Z}_0^\alpha$, 并取 $Q_a(z) := q_a(z) + \overline{q_a(z)} \in \mathcal{Z}_{\mathcal{H},0}^\alpha$. 于是有

$$\begin{aligned}
\infty &> \left\|(C_\varphi - C_\psi) Q_{\varphi(w)}\right\|_{\mathcal{B}_{\mathcal{H}}^\beta} \\
&\geqslant \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta \left(\left|\frac{\partial\left((C_\varphi - C_\psi) Q_{\varphi(w)}\right)}{\partial z}(z)\right| + \left|\frac{\partial\left((C_\varphi - C_\psi) Q_{\varphi(w)}\right)}{\partial \overline{z}}(z)\right|\right) \\
&\geqslant \left|0 - \left(1-|w|^2\right)^\beta \psi'(w)\varphi(w)\right| + \left|0 - \left(1-|w|^2\right)^\beta \overline{\psi'(w)\varphi(w)}\right| \\
&= 2\left|\psi_\#^{\alpha,\beta}(w)\right| \rho(w).
\end{aligned}$$

故

$$\sup_{w\in\mathbb{D}_2\setminus\mathbb{D}_1} \left|\psi_\#^{\alpha,\beta}(w)\right| \rho(w) < \infty. \tag{4.13}$$

将 (4.5) 与 (4.13) 结合便可得到

$$\sup_{w\in\mathbb{D}_2\setminus\mathbb{D}_1} \left|\varphi_\#^{\alpha,\beta}(w) - \psi_\#^{\alpha,\beta}(w)\right| < \infty.$$

因此推导出

$$\sup_{w\in\mathbb{D}_2\setminus\mathbb{D}_1} \left(\left|\varphi_\#^{\alpha,\beta}(w)\right| \cdot \rho(w) + \left|\varphi_\#^{\alpha,\beta}(w) - \psi_\#^{\alpha,\beta}(w)\right|\right) < \infty. \tag{4.14}$$

类似于 (4.14) 的推导步骤, 又可证

$$\sup_{w\in\mathbb{D}_1\setminus\mathbb{D}_2} \left(\left|\varphi_\#^{\alpha,\beta}(w)\right| \rho(w) + \left|\varphi_\#^{\alpha,\beta}(w) - \psi_\#^{\alpha,\beta}(w)\right|\right) < \infty. \tag{4.15}$$

综合 (4.11), (4.12), (4.14) 和 (4.15) 可证得 (4.1), 因此完成了 (2) $\Rightarrow$ (3) 的证明.

接下来继续证明 $C_\varphi - C_\psi : \mathcal{Z}_{\mathcal{H}}^\alpha \to \mathcal{B}_{\mathcal{H}}^\beta$ 的紧性. 记

$$\begin{aligned}
&\Gamma(\varphi) := \{\{z_n\} \subset \mathbb{D} : |\varphi(z_n)| \to 1\}; \quad \Gamma(\psi) := \{\{z_n\} \subset \mathbb{D} : |\psi(z_n)| \to 1\}; \\
&D(\varphi) := \left\{\{z_n\} \subset \mathbb{D} : |\varphi(z_n)| \to 1, \left|\varphi_\#^{\alpha,\beta}(z_n)\right| \nrightarrow 0\right\}; \\
&D(\psi) := \left\{\{z_n\} \subset \mathbb{D} : |\psi(z_n)| \to 1, \left|\psi_\#^{\alpha,\beta}(z_n)\right| \nrightarrow 0\right\}.
\end{aligned}$$

定理 4.2 设 $\alpha>1,\beta>0$ 且 $\varphi,\psi\in S(\mathbb{D})$. 设 C_φ 与 C_ψ 均非紧但都有界, 则下列条件成立:

(1)$C_\varphi-C_\psi:\mathcal{Z}_{\mathcal{H}}^{\alpha}\to\mathcal{B}_{\mathcal{H}}^{\beta}$ 是紧的.

(2)$C_\varphi-C_\psi:\mathcal{Z}_{\mathcal{H},0}^{\alpha}\to\mathcal{B}_{\mathcal{H}}^{\beta}$ 是紧的.

(3) 下列条件成立:

$(i)D(\varphi)=D(\psi),D(\varphi)\subseteq\Gamma(\psi)$.

(ii) 对于 $\{z_n\}\in\Gamma(\varphi)\cap\Gamma(\psi)$,

$$\lim_{n\to\infty}\left[\left|\varphi_{\#}^{\alpha,\beta}(z_n)\right|\rho(z_n)+\left|\varphi_{\#}^{\alpha,\beta}(z_n)-\psi_{\#}^{\alpha,\beta}(z_n)\right|\right]=0.$$

证 (1) $\Rightarrow$ (2) 是显然的. 下证 (3) $\Rightarrow$ (1). 取 $\{f_n\}\subset\mathcal{Z}_{\mathcal{H}}^{\alpha}$ 使得 $\sup\limits_{n\in\mathbb{N}}\|f_n\|_{\mathcal{Z}_{\mathcal{H}}^{\alpha}}\leqslant 1$, 且在 $\mathbb{D}$ 上的紧子集上一致收敛到 0. 应用反证法, 我们假设 $\lim\limits_{n\to\infty}\|(C_\varphi-C_\psi)f_n\|_{\mathcal{B}_{\mathcal{H}}^{\beta}}\neq 0$, 则存在一个 $\varepsilon_0>0$ 与一个 $\{f_n\}$ 的子序列 $\{f_{n_k}\}$, 使得 $\|(C_\varphi-C_\psi)f_{n_k}\|_{\mathcal{B}_{\mathcal{H}}^{\beta}}>\varepsilon_0$. 为方便起见, 我们仍用 $\{f_n\}$ 来表示子序列 $\{f_{n_k}\}$. 因此对于 $\varepsilon_0>0$, 有 $\|(C_\varphi-C_\psi)f_n\|_{\mathcal{B}_{\mathcal{H}}^{\beta}}>\varepsilon_0$, 对于所有的 n 成立. 设 $f_n(z)=g_n(z)+\overline{h_n(z)}$, 其中 $g_n,h_n\in H(\mathbb{D}),h_n(0)=0$, 则 $g_n,h_n\in\mathcal{Z}^{\alpha}$. 由 $\{f_n\}$ 的性质知 $\{g_n\}$ 与 $\{h_n\}$ 满足 $\|g_n\|_{\mathcal{Z}^{\alpha}}\leqslant 1,\|h_n\|_{\mathcal{Z}^{\alpha}}\leqslant 1$ 且在 $\mathbb{D}$ 的紧子集上一致收敛到 0. 进而可得到 $g_n'(\varphi(z))=\frac{\partial f_n}{\partial z}(\varphi(z))$, $g_n'(\psi(z))=\frac{\partial f_n}{\partial z}(\psi(z))$, $\overline{h_n'(\varphi(z))}=\frac{\partial f_n}{\partial\bar z}(\varphi(z))$, $\overline{h_n'(\psi(z))}=\frac{\partial f_n}{\partial\bar z}(\psi(z))$, 于是

$$\begin{aligned}
&\|(C_\varphi-C_\psi)f_n\|_{\mathcal{B}_{\mathcal{H}}^{\beta}}\\
=&|f_n(\varphi(0))-f_n(\psi(0))|\\
&+\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\left(\left|\frac{\partial f_n}{\partial z}(\varphi(z))\varphi'(z)-\frac{\partial f_n}{\partial z}(\psi(z))\psi'(z)\right|+\left|\frac{\partial f_n}{\partial\bar z}(\varphi(z))\overline{\varphi'(z)}-\frac{\partial f_n}{\partial\bar z}(\psi(z))\overline{\psi'(z)}\right|\right)\\
=&|f_n(\varphi(0))-f_n(\psi(0))|\\
&+\sup_{z\in\mathbb{D}}\left(\left|\varphi_{\#}^{\alpha,\beta}(z)\left(1-|\varphi(z)|^2\right)^{\alpha-1}g_n'(\varphi(z))-\psi_{\#}^{\alpha,\beta}(z)\left(1-|\psi(z)|^2\right)^{\alpha-1}g_n'(\psi(z))\right|\right.\\
&\left.\quad+\left|\varphi_{\#}^{\alpha,\beta}(z)\left(1-|\varphi(z)|^2\right)^{\alpha-1}h_n'(\varphi(z))-\psi_{\#}^{\alpha,\beta}(z)\left(1-|\psi(z)|^2\right)^{\alpha-1}h_n'(\psi(z))\right|\right).
\end{aligned}$$

根据 C_φ 与 C_ψ 的有界性, 则当 $n\to\infty$ 时 $f_n(\varphi(0))-f_n(\psi(0))\to 0$. 故存在 $\{z_n\}\subset\mathbb{D}$ 使得

$$\begin{aligned}
&\left|\varphi_{\#}^{\alpha,\beta}(z_n)\left(1-|\varphi(z_n)|^2\right)^{\alpha-1}g_n'(\varphi(z_n))-\psi_{\#}^{\alpha,\beta}(z_n)\left(1-|\psi(z_n)|^2\right)^{\alpha-1}g_n'(\psi(z_n))\right|\\
+&\left|\varphi_{\#}^{\alpha,\beta}(z_n)\left(1-|\varphi(z_n)|^2\right)^{\alpha-1}h_n'(\varphi(z_n))-\psi_{\#}^{\alpha,\beta}(z_n)\left(1-|\psi(z_n)|^2\right)^{\alpha-1}h_n'(\psi(z_n))\right|>\varepsilon_0.
\end{aligned}$$

上式表明, 至少有一项 $n\to\infty$ 时不趋于 0. 不妨设

$$\left|\varphi_{\#}^{\alpha,\beta}(z_n)\left(1-|\varphi(z_n)|^2\right)^{\alpha-1}g_n'(\varphi(z_n))-\psi_{\#}^{\alpha,\beta}(z_n)\left(1-|\psi(z_n)|^2\right)^{\alpha-1}g_n'(\psi(z_n))\right|>\tfrac{\varepsilon_0}{2}.\quad(4.16)$$

将 [25, 定理 2] 的证明中 (3) $\Rightarrow$ (1) 对于 f_n 的步骤应用于 g_n, 当 $n\to\infty$ 成立

$$\left|\varphi_{\#}^{\alpha,\beta}(z_n)\cdot\left(1-|\varphi(z_n)|^2\right)^{\alpha-1}g_n'(\varphi(z_n))-\psi_{\#}^{\alpha,\beta}(z_n)\cdot\left(1-|\psi(z_n)|^2\right)^{\alpha-1}g_n'(\psi(z_n))\right|\to 0.$$

与 (4.16) 矛盾. 综上所述, 即证 $C_\varphi-C_\psi:\mathcal{Z}_{\mathcal{H},0}^{\alpha}\to\mathcal{B}_{\mathcal{H}}^{\beta}$ 是紧的.

下面证明 (2) ⇒ (3). 假设 $C_\varphi - C_\psi : \mathcal{Z}^\alpha_{\mathcal{H},0} \to \mathcal{B}^\beta_{\mathcal{H}}$ 是紧的, 但 C_φ 与 C_ψ 都非紧. 定理 3.2 意味着此时存在序列 $\{z_n\} \in D(\varphi)$ 即 $|\varphi(z_n)| \to 1$ 时有 $\left|\varphi_{\#}^{\alpha,\beta}(z_n)\right| \nrightarrow 0$. 对于上述 $\{\varphi(z_n)\} \subset \mathbb{D}$ 我们参考 (4.3) 与 (4.6) 定义 $F_{\varphi(z_n)}$, $K_{\varphi(z_n)}$, 它们在 $\mathcal{Z}^\alpha_{\mathcal{H},0}$ 中都有界且当 $n \to \infty$ 时在 $\mathbb{D}$ 上的每个紧子集中一致收敛到 0. 根据引理 2.2, 当 $n \to \infty$ 时, 有

$$\begin{aligned}
0 \leftarrow &\left\|(C_\varphi - C_\psi) F_{\varphi(z_n)}\right\|_{\mathcal{B}^\beta_{\mathcal{H}}} \\
\geqslant &\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta \Big(\left|f'_{\varphi(z_n)}(\varphi(z))\varphi'(z) - f'_{\varphi(z_n)}(\psi(z))\psi'(z)\right| \\
&\qquad + \left|\overline{f'_{\varphi(z_n)}(\varphi(z))\varphi'(z)} - \overline{f'_{\varphi(z_n)}(\psi(z))\psi'(z)}\right|\Big) \\
\geqslant &\left|\varphi_{\#}^{\alpha,\beta}(z_n) - \psi_{\#}^{\alpha,\beta}(z_n)\left(\frac{\left(1-|\varphi(z_n)|^2\right)\left(1-|\psi(z_n)|^2\right)}{(1-\overline{\varphi(z_n)}\psi(z_n))^2}\right)^{\alpha-1}\right|,
\end{aligned} \tag{4.17}$$

且

$$\begin{aligned}
0 \leftarrow &\left\|(C_\varphi - C_\psi) K_{\varphi(z_n)}\right\|_{\mathcal{B}^\beta_{\mathcal{H}}} \\
\geqslant &\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta \left(\left|k'_{\varphi(z_n)}(\varphi(z))\varphi'(z) - k'_{\varphi(z_n)}(\psi(z))\psi'(z)\right|\right. \\
= &\left|\psi_{\#}^{\alpha,\beta}(z_n)\left(\frac{\left(1-|\varphi(z_n)|^2\right)\left(1-|\psi(z_n)|^2\right)}{(1-\overline{\varphi(z_n)}\psi(z_n))^2}\right)^{\alpha-1}\right|\rho(z_n).
\end{aligned} \tag{4.18}$$

将 (4.17) 与 (4.18) 相结合, 我们得到

$$\lim_{n\to\infty} \left|\varphi_{\#}^{\alpha,\beta}(z_n)\right| \rho(z_n) = 0. \tag{4.19}$$

由于 $\{z_n\} \in D(\varphi)$, 我们有 $\left|\varphi_{\#}^{\alpha,\beta}(z_n)\right| \nrightarrow 0, n \to \infty$, 又根据 (4.19) 可得

$$\lim_{n\to\infty} \left|\frac{\varphi(z_n) - \psi(z_n)}{1-\overline{\varphi(z_n)}\psi(z_n)}\right| = \lim_{n\to\infty} \rho(z_n) = 0.$$

因此, 对于任意的 $\{z_n\} \in D(\varphi)$, 有

$$\lim_{n\to\infty} |\varphi(z_n) - \psi(z_n)| = 0, \tag{4.20}$$

且 $\lim_{n\to\infty} \left||\varphi(z_n)| - |\psi(z_n)|\right| \leqslant \lim_{n\to\infty} |\varphi(z_n) - \psi(z_n)| = 0$. 故 $|\psi(z_n)| \to 1$, $n \to \infty$. 因此, 对于任意的 $\{z_n\} \in D(\varphi)$, 有 $\{z_n\} \in \Gamma(\psi)$, 则 $D(\varphi) \subseteq \Gamma(\psi)$. 进一步得到 $D(\varphi) \subseteq \Gamma(\varphi) \cap \Gamma(\psi)$. 又对于任意的 $\{z_n\} \in \Gamma(\varphi) \cap \Gamma(\psi)$, 由于 (4.17), 当 $n \to \infty$ 时, 有

$$\begin{aligned}
0 \leftarrow &\left\|(C_\varphi - C_\psi) F_{\varphi(z_n)}\right\|_{\mathcal{B}^\beta_{\mathcal{H}}} \\
\geqslant &\left|\varphi_{\#}^{\alpha,\beta}(z_n) - \psi_{\#}^{\alpha,\beta}(z_n)\left(\frac{\left(1-|\varphi(z_n)|^2\right)\left(1-|\psi(z_n)|^2\right)}{(1-\overline{\varphi(z_n)}\psi(z_n))^2}\right)^{\alpha-1}\right| \\
\geqslant &\left|\varphi_{\#}^{\alpha,\beta}(z_n) - \psi_{\#}^{\alpha,\beta}(z_n)\right| \\
&- \left|\psi_{\#}^{\alpha,\beta}(z_n)\left[\left(1-|\varphi(z_n)|^2\right)^{\alpha-1} f'_{\varphi(z_n)}(\varphi(z_n)) - \left(1-|\psi(z_n)|^2\right)^{\alpha-1} f'_{\varphi(z_n)}(\psi(z_n))\right]\right| \\
\gtrsim &\left|\varphi_{\#}^{\alpha,\beta}(z_n) - \psi_{\#}^{\alpha,\beta}(z_n)\right| - \left|\psi_{\#}^{\alpha,\beta}(z_n)\right| \rho(z_n),
\end{aligned}$$

则根据 C_ψ 的有界性与 $\rho(z_n)\to 0$, $n\to\infty$, 可以推得

$$\lim_{n\to\infty}\left|\varphi_{\#}^{\alpha,\beta}(z_n)-\psi_{\#}^{\alpha,\beta}(z_n)\right|=0. \tag{4.21}$$

因此对于任意的 $\{z_n\}\in\Gamma(\varphi)\cap\Gamma(\psi)$, 将 (4.19) 与 (4.21) 相结合, 便可得到

$$\lim_{n\to\infty}\left(\left|\varphi_{\#}^{\alpha,\beta}(z_n)\right|\rho(z_n)+\left|\varphi_{\#}^{\alpha,\beta}(z_n)-\psi_{\#}^{\alpha,\beta}(z_n)\right|\right)=0.$$

对于 $\{z_n\}\in D(\varphi)$, 根据 (4.20) 与 (4.21)

$$\lim_{n\to\infty}|\psi(z_n)|=\lim_{n\to\infty}|\varphi(z_n)|=1,$$
$$\lim_{n\to\infty}\left|\psi_{\#}^{\alpha,\beta}(z_n)\right|=\lim_{n\to\infty}\left|\varphi_{\#}^{\alpha,\beta}(z_n)\right|\neq 0.$$

于是 $D(\varphi)\subseteq D(\psi)$. 同理易证 $D(\psi)\subseteq D(\varphi)$. 因此 $D(\varphi)=D(\psi)$. 故 (2) $\Rightarrow$ (3) 成立.

参 考 文 献

[1] Bonet J, Lindström M, Wolf E. Differences of composition operators between weighted Banach spaces of holomorphic functions[J]. Journal of the Australian Mathematical Society, 2008, 84(1): 9–20.

[2] Choe B R, Koo H, Wang Maofa. Compact linear combination of composition operators on Bergman spaces[J]. Journal of Functional Analysis, 2020, 278(5): 108393.

[3] Colonna F. Characterisation of the isometric composition operators on the Bloch space[J]. Bulletin of the Australian Mathematical Society, 2005, 72(2): 283–290.

[4] Choe B R, Choi K, Koo H, et al. Difference of weighted composition operators[J]. Journal of Functional Analysis, 2020, 278(5): 108401.

[5] Dai Jineng, Ouyang C. Differences of weighted composition operators on $H_\alpha^\infty(B_N)$[J]. Journal of Inequalities and Applications, 2009, 2009: 1–19.

[6] Fan Junmei, Lu Yufeng, Yang Yixin. Difference of composition operators on some analytic function spaces[J]. Acta Mathematica Sinica, English Series, 2021, 37(9): 1384–1400.

[7] Hosokawa T, Ohno S. Differences of composition operators on the Bloch spaces[J]. Journal of Operator Theory, 2007, 57(2): 229–242.

[8] Liang Yuxia. New characterizations for differences of weighted differentiation composition operators from a Bloch-type space to a weighted-type space[J]. Periodica Mathematica Hungarica, 2018, 77: 119–138.

[9] Liang Yuxia, Chen Cui. New characterizations for differences of Volterra-type operators from α-weighted-type space to β-Bloch-Orlicz space[J]. Mathematische Nachrichten, 2018, 291(14-15): 2298–2317.

[10] Liu Xiaosong, Li Songxiao. Differences of generalized weighted composition operators from the Bloch space into Bers-type spaces[J]. Filomat, 2017, 31(6): 1671–1680.

[11] Moorhouse J. Compact differences of composition operators[J]. Journal of Functional Analysis, 2005, 219(1): 70–92.

[12] Madigan K, Matheson A. Compact composition operators on the Bloch space[J]. Transactions of the American Mathematical Society, 1995, 347(7): 2679–2687.

[13] Sharma A K, Ueki S I. Differences of composition operators from analytic Besov spaces into little Bloch type spaces[J]. Filomat, 2020, 1: 713–722.
[14] Shi Yecheng, Li Songxiao. Differences of composition operators on Bloch type spaces[J]. Complex Analysis and Operator Theory, 2017, 11: 227–242.
[15] Moorhouse J. Compact differences of composition operators[J]. Journal of Functional Analysis, 2005, 219(1): 70–92.
[16] Bakhit M A, Dahshan N M, Tahier R, et al. Composition operators from harmonic Lipschitz space into weighted harmonic Zygmund space[J]. International Journal of Analysis and Applications, 2023, 21: 125–125.
[17] Cowen C C, MacCluer B D. Composition operators on spaces of analytic functions[M]. Boca Raton: CRC press, 1995.
[18] Guo Xincui, Zhou Zehua. New characterizations for weighted composition operator from Zygmund type spaces to Bloch type spaces[J]. Czechoslovak Mathematical Journal, 2015, 65(2): 331–346.
[19] Ohno S, Stroethoff K, Zhao Ruhan. Weighted composition operators between Bloch-type spaces[J]. The Rocky Mountain Journal of Mathematics, 2003: 191–215.
[20] Stević S. On a product-type operator from Bloch spaces to weighted-type spaces on the unit ball[J]. Applied Mathematics and Computation, 2011, 217(12): 5930–5935.
[21] Zhu Kehe. Operator theory in function spaces[M]. New York and Basel: American Mathematical Soc., 2007.
[22] Zhu Kehe. Spaces of holomorphic functions in the unit ball[M]. New York: Springer, 2005.
[23] Esmaeili K, Lindström M. Weighted composition operators between Zygmund type spaces and their essential norms[J]. Integral Equations and Operator Theory, 2013, 75(4): 473–490.
[24] Krantz S G, Stević S. On the iterated logarithmic Bloch space on the unit ball[J]. Nonlinear Analysis: Theory, Methods & Applications, 2009, 71(5-6): 1772–1795.
[25] 刘金昊, 梁玉霞, 周航. 从 Zygmund 型空间到 Bloch 型空间的复合算子的差分 [J]. 天津师范大学学报 (自然科学版), 2024, 44(03): 1–6+44.

DIFFERENCES OF COMPOSITION OPERATORS FROM HARMONIC ZYGMUND-TYPE SPACES TO HARMONIC BLOCH-TYPE SPACES

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Abstract: This paper investigates the properties of the difference of composition operators from harmonic Zygmund-type spaces $\mathcal{Z}^{\alpha}_{\mathcal{H}}(\alpha > 1)$ into harmonic Bloch-type spaces $\mathcal{B}^{\beta}_{\mathcal{H}}(0 < \beta < \infty)$ on the unit disk. Using the properties of harmonic function spaces, Stirling formula and the test functions to obtain a necessary and sufficient condition for the bounded and compact composition operator $C_\varphi : \mathcal{Z}^{\alpha}_{\mathcal{H}} \to \mathcal{B}^{\beta}_{\mathcal{H}}$. And equivalent conditions for the boundedness and compactness of the difference of composition operators $C_\varphi - C_\psi : \mathcal{Z}^{\alpha}_{\mathcal{H}} \to \mathcal{B}^{\beta}_{\mathcal{H}}$ are presented.

Keywords: difference; composition operator; harmonic Zygmund-type space; harmonic Bloch-type space

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