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LOCALLY QUASIDIAGONAL EXTENSIONS OF C^*-ALGEBRAS

SHI Chang-li, YAO Hong-liang

(*School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing 210094, China*)

Abstract: This paper introduce the concept of locally quasidiagonal extension of C^*-algebras and give some basic properties. We use the method of analogy, based on some properties possessed by quasidiagonal extensions, we investigate whether local quasidiagonal extensions still retain these properties. We then show that an extension of a locally AF algebra by a locally AF algebra is a locally quasidiagonal extension.

Keywords: C^*-algebra; quasidiagonal C^*-algebra; locally quasidiagonal extension

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1 Introduction

Let I and B be two C^*-algebras. An extension of B by I is a short exact sequence $0 \to I \xrightarrow{\iota} E \xrightarrow{\pi} B \to 0$ of C^*-algebras. The extensions of C^*-algebras were first studied by Busby and the attension was not attracted until the development of BDF theory. The extension theory becomes more and more important since it describes how complicated C^*-algebras can be constructed. As the extension theory is concerned, there is a special case called the quasidiagonal extension and many results have been obtained up to now. Quasidiagonality plays a major role in C^*-algebras and Quasidiagonal(QD) C*-algebras have been studied for over 40 years. They are a large class of algebras which arise naturally in many contexts and include many of the basic examples of finite C^*-algebras[1]. Nathanial P. Brown gave a suitable definition of a quasidiagonal(QD) C^*-algebra and considered the extension of quasidiagonal(QD) C^*-algebras. A short exact sequence $0 \to I \xrightarrow{\iota} E \xrightarrow{\pi} B \to 0$ is called a quasidiagonal extension if there exists an approximate unit $(p_n)_n$ of I consisting of projections, which satisfies $\lim_{n\to\infty} \|p_n a - a p_n\| = 0$ for all $a \in E$. Since the theory of extensions is a very useful tool for studying the classification and the structure of C^*-algebras, the discussion of the quasidiagonal extension is very meaningful. This definition is invalid when I doesn't have an approximate unit consisting of projections. In this paper we

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Biography: Shi Changli(1999–), male, born in Yancheng, postgraduate, major in: operator algebras, E-mail: 321740632@qq.com.

Corresponding author: Yao Hongliang (1976–), male, associate professor, major in: functional analysis, operator algebras and C^*-algebra.

try to give a new definition on the basis of quasidiagonal extension and consider the relevant theorems derived from the new definition.

2 Locally Quasidiagonal Extension

Definition 2.1 [2] Let $0 \to I \xrightarrow{\iota} E \xrightarrow{\pi} B \to 0$ be a short exact sequence of C^*-algebras. Such a sequence is called a quasidiagonal extension if there exists an increasing approximate unit $(p_n)_n$ of I consisting of projections, which satisfies

$$\lim_{n\to\infty} \|p_n a - a p_n\| = 0$$

for all $a \in E$.

Definition 2.2 Let $0 \to I \xrightarrow{\iota} E \xrightarrow{\pi} B \to 0$ be a short exact sequence of C^*-algebras. Such a sequence is called a locally quasidiagonal(QD) extension, if for any $\varepsilon > 0$, finite sets $\mathcal{F} \subset E$ and $\mathcal{F}' \subset I$, there exists $i \in I$, satisfying $i \geq 0$ and $\|i\| \leq 1$, such that

$$\|if' - f'\| < \varepsilon, \quad \|if - i^2 f\| < \varepsilon, \text{ and } \|if - fi\| < \varepsilon$$

for all $f \in \mathcal{F}$ and $f' \in \mathcal{F}'$.

Theorem 2.3 Let $\{0 \to I_n \xrightarrow{\iota_n} E_n \xrightarrow{\pi_n} B_n \to 0\}$ be a sequence of short exact sequences of C^*-algebras, and let $\varphi_n : E_n \to E_{n+1}$ be a sequence of $*$-homomorphisms with $\varphi_n(I_n) \subset I_{n+1}$. Let $I = \lim\limits_{n\to\infty} \varphi_n(I_n)$, $E = \lim\limits_{n\to\infty} \varphi_n(E_n)$ and $B = \lim\limits_{n\to\infty} \varphi_n(B_n)$. If for any $n \in \mathbb{N}$, $0 \to I_n \xrightarrow{\iota_n} E_n \xrightarrow{\pi_n} B_n \to 0$ is locally quasidiagonal, then the extension $0 \to I \xrightarrow{\iota} E \xrightarrow{\pi} B \to 0$ is locally quasidiagonal.

Proof For any $n \in \mathbb{N}$, there is a natural $*$-homomorphism $\varphi^{(n)} : E_n \to E$. Fix $\varepsilon > 0$, finite sets $\mathcal{F}' \subset I$ and $\mathcal{F} \subset E$. Then there exists a sufficiently large integer n, finite sets $\mathcal{F}'_n \subset I_n$ and $\mathcal{F}_n \subset E_n$ such that for any $f' \in \mathcal{F}'$ and $f \in \mathcal{F}$, there exist $\bar{f}' \in I_n$ and $\bar{f} \in E_n$ such that

$$\|\varphi^{(n)}(\bar{f}') - f'\| < \varepsilon, \quad \|\varphi^{(n)}(\bar{f}) - f\| < \varepsilon.$$

Since $0 \to I_n \xrightarrow{\iota_n} E_n \xrightarrow{\pi_n} B_n \to 0$ is loaclly quasidiagonal, for any $\varepsilon > 0$ there exists a positive element $\bar{i} \in I_n$ with $\|\bar{i}\| \leq 1$ such that

$$\|\bar{i}\bar{f}' - \bar{f}'\| < \varepsilon, \quad \|\bar{i}\bar{f} - \bar{f}\bar{i}\| < \varepsilon, \textit{ and } \|(\bar{i} - \bar{i}^2)\bar{f}\| < \varepsilon$$

for any $\bar{f}' \in \mathcal{F}'_n$, and $\bar{f} \in \mathcal{F}_n$.

Since $\varphi^{(n)}$ is $*$-homomorphism, we have $\|\varphi^{(n)}(\bar{i})\varphi^{(n)}(\bar{f}') - \varphi^{(n)}(\bar{f}')\| < \varepsilon$, $\|\varphi^{(n)}(\bar{i})\varphi^{(n)}(\bar{f}) - \varphi^{(n)}(\bar{f})\varphi^{(n)}(\bar{i})\| < \varepsilon$, and $\|(\varphi^{(n)}(\bar{i}) - \varphi^{(n)}(\bar{i})^2)\varphi^{(n)}(\bar{f})\| < \varepsilon$. Then one easily checks

$$\begin{aligned}
&\|\varphi^{(n)}(\bar{i})f' - f'\| \\
\leq&\|\varphi^{(n)}(\bar{i})f' - \varphi^{(n)}(\bar{i})\varphi^{(n)}(\bar{f}')\| + \|\varphi^{(n)}(\bar{i})\varphi^{(n)}(\bar{f}') - \varphi^{(n)}(\bar{f}')\| + \|\varphi^{(n)}(\bar{f}') - f'\| \\
\leq&\|\varphi^{(n)}(\bar{i})\|\|f' - \varphi^{(n)}(\bar{f}')\| + \|\varphi^{(n)}(\bar{i})\varphi^{(n)}(\bar{f}') - \varphi^{(n)}(\bar{f}')\| + \|\varphi^{(n)}(\bar{f}') - f'\| \\
<&\|\varphi^{(n)}(\bar{i})\|\varepsilon + 2\varepsilon,
\end{aligned}$$

$$\begin{aligned}
&\|\varphi^{(n)}(\bar{i})f - f\varphi^{(n)}(\bar{i})\|\\
\le&\|\varphi^{(n)}(\bar{i})f - \varphi^{(n)}(\bar{i})\varphi^{(n)}(\bar{f})\| + \|\varphi^{(n)}(\bar{i})\varphi^{(n)}(\bar{f}) - \varphi^{(n)}(\bar{f})\varphi^{(n)}(\bar{i})\| + \|\varphi^{(n)}(\bar{f})\varphi^{(n)}(\bar{i}) - f\varphi^{(n)}(\bar{i})\|,\\
\le&\|\varphi^{(n)}(\bar{i})\|\|f - \varphi^{(n)}(\bar{f})\| + \|\varphi^{(n)}(\bar{i})\varphi^{(n)}(\bar{f}) - \varphi^{(n)}(\bar{f})\varphi^{(n)}(\bar{i})\| + \|\varphi^{(n)}(\bar{f}) - f\|\|\varphi^{(n)}(\bar{i})\|\\
<&2\|\varphi^{(n)}(\bar{i})\|\varepsilon + \varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
&\|(\varphi^{(n)}(\bar{i}) - \varphi^{(n)}(\bar{i})^2)f\|\\
\le&\|\varphi^{(n)}(\bar{i})f - \varphi^{(n)}(\bar{i})\varphi^{(n)}(\bar{f})\| + \|\varphi^{(n)}(\bar{i})\varphi^{(n)}(\bar{f}) - \varphi^{(n)}(\bar{i})^2\varphi^{(n)}(\bar{f})\| + \|\varphi^{(n)}(\bar{i})^2\varphi^{(n)}(\bar{f}) - \varphi^{(n)}(\bar{i})^2 f\|\\
\le&\|\varphi^{(n)}(\bar{i})\|\|f - \varphi^{(n)}(\bar{f})\| + \|\varphi^{(n)}(\bar{i})\varphi^{(n)}(\bar{f}) - \varphi^{(n)}(\bar{i})^2\varphi^{(n)}(\bar{f})\| + \|\varphi^{(n)}(\bar{i})^2\|\|\varphi^{(n)}(\bar{f}) - f\|\\
<&(2\|\varphi^{(n)}(\bar{i})\| + \|\varphi^{(n)}(\bar{i})^2\|)\varepsilon.
\end{aligned}$$

Thus $0 \to I \xrightarrow{\iota} E \xrightarrow{\pi} B \to 0$ is locally quasidiagonal.

Theorem 2.4 [3] Let $0 \to I \to E \to B \to 0$ be a locally quasidiagonal extension of C^*-algebras and A is a nuclear C^*-algebra, then $0 \to I \otimes A \to E \otimes A \to B \otimes A \to 0$ is a locally quasidiagonal extension.

Proof Fix finite sets $\mathcal{F}' \subset I$ and $\mathcal{F} \subset E$. Since $0 \to I \to E \to B \to 0$ is a locally quasidiagonal extension, for any $\varepsilon > 0$ there exists a positive element $i \in I$ with $\|i\| \le 1$ such that

$$\|if' - f'\| < \varepsilon, \quad \|if - fi\| < \varepsilon, \ \text{and} \ \|(i - i^2)f\| < \varepsilon$$

for any $f' \in \mathcal{F}'$ and $f \in \mathcal{F}$. Let $\{a_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit for A. Fix a finite set $\mathcal{A}' \subset A$. Then for any $\varepsilon > 0$, there exists $a_{\lambda_0} \in \{a_\lambda\}_{\lambda\in\Lambda}$ such that $\|a_{\lambda_0}a' - a'\| < \varepsilon$ for all $a' \in \mathcal{A}'$. And $\mathcal{F}' \otimes \mathcal{A}'$ is the finite set of $I \otimes A$, $\mathcal{F} \otimes \mathcal{A}'$ is the finite set of $E \otimes A$. And for any $f' \otimes a' \in \mathcal{F}' \otimes \mathcal{A}'$, $f \otimes a' \in \mathcal{F} \otimes \mathcal{A}'$ and for any $\varepsilon > 0$, we have

$$\begin{aligned}
&\|(i \otimes a_{\lambda_0})(f' \otimes a') - f' \otimes a'\|\\
=&\|if' \otimes a_{\lambda_0}a' - f' \otimes a'\|\\
=&\|if' \otimes a_{\lambda_0}a' - if' \otimes a' + if' \otimes a' - f' \otimes a'\|\\
\le&\|if'\|\|a_{\lambda_0}a' - a'\| + \|if' - f'\|\|a'\|\\
\le&\|if'\|\varepsilon + \varepsilon\|a'\|,
\end{aligned}$$

$$\begin{aligned}
&\|(i \otimes a_{\lambda_0})(f \otimes a') - (f \otimes a')(i \otimes a_{\lambda_0})\|\\
=&\|if \otimes a_{\lambda_0}a' - fi \otimes a'a_{\lambda_0}\|\\
=&\|if \otimes a_{\lambda_0}a' - if \otimes a'a_{\lambda_0} + if \otimes a'a_{\lambda_0} - fi \otimes a'a_{\lambda_0}\|\\
\le&\|if\|\|a_{\lambda_0}a' - a'a_{\lambda_0}\| + \|if - fi\|\|a'a_{\lambda_0}\|\\
\le&\|if\|\varepsilon + \varepsilon\|a'a_{\lambda_0}\|,
\end{aligned}$$

and

$$\begin{aligned}&\|[i\otimes a_{\lambda_0}-(i\otimes a_{\lambda_0})^2](f\otimes a')\|\\=&\|[(i-i^2)\otimes a_{\lambda_0}](f\otimes a')+[i^2\otimes(a_{\lambda_0}-a_{\lambda_0}^2)](f\otimes a')\|\\=&\|(i-i^2)f\otimes a_{\lambda_0}a'+i^2f\otimes(a_{\lambda_0}-a_{\lambda_0}^2)a'\|\\\le&\|(i-i^2)f\|\|a_{\lambda_0}a'\|+\|i^2f\|\|a_{\lambda_0}\|\|a'-a_{\lambda_0}a'\|\\\le&\varepsilon\|a_{\lambda_0}a'\|+\|i^2f\|\|a_{\lambda_0}\|\varepsilon.\end{aligned}$$

This imples that $0\to I\otimes A\to E\otimes A\to B\otimes A\to 0$ is a locally quasidiagonal extension.

Theorem 2.5 Let $0\to I\xrightarrow{\iota}E\xrightarrow{\pi}B\to 0$ be a locally quasidiagonal extension of C^*-algebras. If I and B are stably finite, then E is stably finite.

Proof We just need to verify that for any partial isometry $v\in E$, if $vv^*\le v^*v$, then $vv^*=v^*v$. We put $\mathcal{F}=\{v,v^*,vv^*,v^*v\}$ and $\mathcal{F}'=\{v^*v-vv^*,vv^*-v^*v\}$. From the definition, for any $\varepsilon>0$ there exists $i\in I$ such that $i>0$, $\|i\|\le 1$ and

$$\|if'-f'\|<\varepsilon,\quad \|if-i^2f\|<\varepsilon,\ \text{and}\ \|if-fi\|<\varepsilon$$

for all $f'\in\mathcal{F}'$ and $f\in\mathcal{F}$. Then we have

$$\begin{aligned}&\|ivi+(1-i)v(1-i)-v\|=\|ivi-iv+ivi-vi\|\\=&\|ivi-i^2v+i^2v-iv+ivi-vi^2+vi^2-vi\|\\\le&\|i\|\|vi-iv\|+\|i^2v-iv\|+\|iv-vi\|\|i\|+\|vi^2-vi\|\\\le&\|i\|\varepsilon+\varepsilon+\varepsilon\|i\|+\varepsilon=2(\|i\|+1)\varepsilon.\end{aligned}$$

So

$$\|ivi+(1-i)v(1-i)-v\|\le 2(\|i\|+1)\varepsilon. \tag{2.1}$$

On the other hand, we have

$$\begin{aligned}&\|(ivi)^*(ivi)-iv^*vi\|=\|iv^*i^2vi-iv^*vi\|\\\le&\|iv^*i^2vi-i^2v^*ivi\|+\|i^2v^*ivi-i^2v^*vi^2\|+\|i^2v^*vi^2-i^2v^*vi\|+\|i^2v^*vi-iv^*vi\|\\\le&\|i\|\|v^*i-iv^*\|\|ivi\|+\|i^2v^*\|\|iv-vi\|\|i\|+\|i^2v^*\|\|v(i^2-i)\|+\|(i^2-i)v^*\|\|vi\|\\\le&\|i\|\varepsilon\|ivi\|+\|i^2v^*\|\varepsilon\|i\|+\|i^2v^*\|\varepsilon+\varepsilon\|vi\|.\end{aligned}$$

Since $ivi\in I$, for any $\varepsilon>0$ there exists a partial isometry $w\in I$ such that $ww^*\le w^*w$ and $\|w-ivi\|\le\varepsilon$.

And since $\|w-ivi\|\le\varepsilon$, we have $\|(ww^*-w^*w)-(iv^*vi-ivv^*i)\|\le\varepsilon$. Note that $ww^*=w^*w$ since I is stably finite. We have

$$\|iv^*vi-ivv^*i\|\le\varepsilon. \tag{2.2}$$

Note that

$$\begin{aligned}&\|(1-i)v^*v(1-i)-(1-i)vv^*(1-i)\|\\=&\|(v^*v-vv^*)-(v^*v-vv^*)i+i(v^*v-vv^*)i-i(v^*v-vv^*)\|\\\leq&\|(v^*v-vv^*)-(v^*v-vv^*)i\|+\|i\|\|(v^*v-vv^*)i-(v^*v-vv^*)\|\\\leq&\varepsilon+\|i\|\varepsilon.\end{aligned}$$

Since

$$\|(ivi)(1-i)v(1-i)\|\leq\|iv\|\|(i-i^2)v\|\|1-i\|\leq\|iv\|\varepsilon\|1-i\|,$$

from (2.1) and (2.2) we obtain that $\|v^*v-vv^*\|$ is sufficiently small. Since v^*v-vv^* is a projection, $v^*v=vv^*$, this means that E is stably finite.

Definition 2.6 [2] A subset $\Omega\subset B(H)$ is called a quasidiagonal set of operators if for each finite set $\omega\subset\Omega$, finite set $\chi\subset H$ and $\varepsilon>0$ there exists a finite rank projection $P\in B(H)$ such that $\|TP-PT\|\leq\varepsilon$ and $\|P(x)-x\|\leq\varepsilon$ for all $T\in\omega$ and $x\in\chi$.

Definition 2.7 [2] Let A be a C^*-algebra. Then A is called quasidiagonal(QD) if there exists a faithful representation $\pi:A\to B(H)$ such that $\pi(A)$ is a quasidiagonal set of operators.

Theorem 2.8 Let $0\to I\xrightarrow{\iota}E\xrightarrow{\pi}B\to 0$ be a locally quasidiagonal extension of C^*-algebras. If both I and B are quasidiagonal, then E is quasidiagonal.

Proof To ease notation somewhat, we identify I with $\iota(I)$. For any finite subsets $\mathcal{F}\subset E$ and $\mathcal{F}'\subset I$, there exists $i\in I$ which satisfies the conditions of Definition 2.2. Now consider the contractive completely positive map $\varphi:E\to I\oplus B$, $\varphi(a)=iai\oplus\pi(a)$ where $a\in E$. Evidently these maps are asymptotically multiplicative, and the proof is as follows. For any $\varepsilon>0$ and all $a,b\in\mathcal{F}\subset E$, we have

$$\begin{aligned}&\|\varphi(ab)-\varphi(a)\varphi(b)\|=\|iabi\oplus\pi(ab)-(iai\oplus\pi(a))(ibi\oplus\pi(b))\|\\=&\|(iabi-iaiibi)\oplus(\pi(ab)-\pi(a)\pi(b))\|=\max\{\|iabi-iai^2bi\|,\|\pi(ab)-\pi(a)\pi(b)\|\}\\=&\|iabi-iai^2bi\|\leq\|iabi-i^2abi\|+\|i^2abi-iai^2bi\|\leq\|(i-i^2)ab\|\|i\|+\|i^2abi-iaibi\|\\&+\|iaibi-iai^2bi\|\\\leq&\|(i-i^2)ab\|\|i\|+\|i\|\|ia-ai\|\|bi\|+\|ia\|\|(i-i^2)b\|\|i\|\\\leq&(\|i\|+\|i\|\|bi\|+\|ia\|\|i\|)\varepsilon,\end{aligned}$$

and $\|\varphi(a)\|>\|a\|-\varepsilon$. Since I and B are quasidiagonal, by Lemma 4.1 of [2], there exists a contractive completely positive map $\psi:I\oplus B\to M_n(\mathbb{C})$ such that

$$\|\psi(\varphi(a)\varphi(b))-\psi(\varphi(a))\psi(\varphi(b))\|<\varepsilon,\ \text{and}\ \|\psi(\varphi(a))\|>\|\varphi(a)\|-\varepsilon.$$

Let $\rho=\psi\circ\varphi$. We compute that

$$\begin{aligned}&\|\rho(ab)-\rho(a)\rho(b)\|\leq\|\psi(\varphi(ab))-\psi(\varphi(a)\varphi(b))\|+\|\psi(\varphi(a)\varphi(b))-\psi(\varphi(a))\psi(\varphi(b))\|\\\leq&(\|i\|+\|i\|\|bi\|+\|ia\|\|i\|+1)\varepsilon\end{aligned}$$

and

$$\|\rho(a)\| > \|\varphi(a)\| - \varepsilon > \|a\| - 2\varepsilon$$

for all $a, b \in \mathcal{F}$. Consequently, by Lemma 4.1 of [2], E is quasidiagonal.

Definition 2.9 Let $0 \to I \xrightarrow{\iota} E \xrightarrow{\pi} B \to 0$ be a short exact sequence of C^*-algebras. We call this a generalized quasidiagonal extension if there exists an approximate unit $(e_\lambda)_\lambda$ of I such that

$$\lim_\lambda \|e_\lambda a - a e_\lambda\| = 0 \textit{ and } \lim_\lambda \|(e_\lambda - {e_\lambda}^2)a\| = 0$$

for all $a \in E$.

Theorem 2.10 Let $0 \to I \xrightarrow{\iota} E \xrightarrow{\pi} B \to 0$ be a locally quasidiagonal extension of C^*-algebras. If E is separable, then $0 \to I \xrightarrow{\iota} E \xrightarrow{\pi} B \to 0$ is generalized quasidiagonal.

Proof Since E is separable, I is separable. Then there exist countable sets $T \subset E$ and $T' \subset I$ such that $\overline{T} = E$ and $\overline{T'} = I$. We may assume that $T = \{x_1, \ldots, x_n, \ldots | x_i \in E\}$ and $T' = \{a_1, \ldots, a_n, \ldots | a_i \in I\}$. Put $T_n = \{x_1, \ldots, x_n | x_i \in E\}$, $T'_n = \{a_1, \ldots, a_n | a_i \in I\}$, $\mathcal{F}_1 = \{x_1\}$ and $\mathcal{F}'_1 = \{a_1\}$. Then for any $\varepsilon > 0$ and for $\mathcal{F}_1$ and $\mathcal{F}'_1$, there exists a positive element $e_1 \in I$ with $\|e_1\| \leq 1$ such that

$$\|e_1 a - a\| < \varepsilon, \quad \|e_1 f - e_1^2 f\| < \varepsilon, \text{ and } \|e_1 f - f e_1\| < \varepsilon$$

for any $f \in \mathcal{F}_1$ and $a \in \mathcal{F}'_1$. Set $\mathcal{F}_n = \mathcal{F}_{n-1} \cup \{x_n, e_{n-1}\}$, $\mathcal{F}'_n = \mathcal{F}'_{n-1} \cup \{a_n, e_{n-1}\}$, $(n \in \mathbb{N})$. Then for any $n \in \mathbb{N}$, there exists a positive element $e_n \in I$ with $\|e_n\| \leq 1$ such that

$$\|e_n a - a\| < \frac{1}{2^n 2n(2n+1)}, \quad \|e_n f - e_n^2 f\| < \frac{1}{2^n 2n(2n+1)}, \text{ and } \|e_n f - f e_n\| < \frac{1}{2^n 2n(2n+1)}$$

for any $a \in \mathcal{F}'_n$ and $f \in \mathcal{F}_n$.

Now we set $e_i^{(n)} = e_n \ldots e_1 \ldots e_n (i < n)$. If $m < n$, then

$$\begin{aligned} e_i^{(n)} - e_i^{(m)} =& e_n \ldots e_1 \ldots e_n - e_m \ldots e_1 \ldots e_m \\ =& e_n \ldots e_1 \ldots e_n - e_n \ldots e_1 \ldots e_{n-1} + e_n \ldots e_1 \ldots e_{n-1} - e_{n-1} \ldots e_1 \ldots e_{n-1} \\ & + \ldots + e_{m+1} \ldots e_1 \ldots e_m - e_m \ldots e_1 \ldots e_m \\ =& e_n \ldots e_1 \ldots e_{n-2}(e_{n-1}e_n - e_{n-1}) + (e_n e_{n-1} - e_{n-1})e_{n-2} \ldots e_1 \ldots e_{n-1} \\ & + \ldots + (e_{m+1}e_m - e_m)e_{m-1} \ldots e_1 \ldots e_m \\ <& 2(n-m)\frac{1}{2^n 2n(2n+1)} < \frac{1}{2^n(2n+1)}. \end{aligned}$$

So there exists α_i such that

$$\lim_{n \to \infty} e_i^{(n)} = \alpha_i.$$

Note that $\|e_i^{(n)} - \alpha_i\| \leq \frac{1}{2^n(2n+1)}$ and

$$\|\alpha_i - e_i\| \leq \|\alpha_i - e_i^{(n)}\| + \|e_i^{(n)} - e_i\| < \frac{1}{2^n(2n+1)} + 2n\frac{1}{2^n 2n(2n+1)} < \frac{1}{2^n}.$$

Thus $\{\alpha_i\}$ is the approximate unit of I. Since $\|\alpha_i - e_i\| \leq \frac{1}{2^n}$, we have

$$\|\alpha_i f - f\alpha_i\| \leq \frac{1}{2^n 2n(2n+1)}, \ \textit{and} \ \|(\alpha_i - \alpha_i^2)f\| \leq \frac{1}{2^n 2n(2n+1)}.$$

Therefore $0 \to I \xrightarrow{\iota} E \xrightarrow{\pi} B \to 0$ is generalized quasidiagonal by the definition.

3 Locally AF Algebra

Theorem 3.1 [5] If A is a locally AF algebra, then there exists a net $(p_\lambda)_{\lambda\in\Lambda}$ of projections in A such that $a = \lim_\lambda ap_\lambda$ for all $a \in A$.

Proof Let $\mathcal{S}$ be the set of all finite subsets of the closed unit ball of A. Put $\Lambda = \{(S, \frac{1}{n}) : s \in \mathcal{S}, n \in \mathbb{N}\}$. Define a partial ordering relation in Λ by $(S_1, \frac{1}{n_1}) \leq (S_2, \frac{1}{n_2})$ if and only if $S_1 \subseteq S_2$ and $\frac{1}{n_1} \geq \frac{1}{n_2}$. It is easily checked that Λ is a directed set. For each $\lambda = (S, \frac{1}{n})$ in Λ, there exists a finite dimensional C^*-subalgebra A_λ of A such that $S_0 \subset_{1/n} A_0$. Denote by p_λ the identity of A_λ.

For any $a \in A$ and $\varepsilon > 0$, take $\lambda_0 = (\{a\}, 1/N)$, where N is a positive integer with $1/N \leq \varepsilon$. It is easy to see that if $\lambda \geq \lambda_0$, then $\|p_\lambda a - a\| < \varepsilon$.

The following two lemmas are used to prove Theorem 3.4, but we refer to the proofs of {[4], Lemma III.6.1 and III.6.2} for details.

Lemma 3.2 (see [4]) Suppose that J is an locally AF ideal of a C^*-algebra A. Then for each projection p in A/J, there exists a projection P in A such that $P + J = p$.

Lemma 3.3 (see [4]) Suppose that J is a locally AF ideal of a C^*-algebra A and that B is a finite dimensional subalgebra of A/J. Then there is a (not necessarily unital)$*$-monomorphism ρ of B into A such that $\tau\rho = id_B$.

Theorem 3.4 Suppose that $0 \to J \xrightarrow{j} A \xrightarrow{\tau} B \to 0$ is an exact sequence of C^*-algebra and that J and B are locally AF algebras. Then $0 \to J \xrightarrow{j} A \xrightarrow{\tau} B \to 0$ is locally quasidiagonal.

Proof Take finite sets $\mathcal{F} \subseteq A$, $\mathcal{F}' \subseteq J$ and $\varepsilon > 0$. Since $\tau(\mathcal{F}) = \{\tau(f) | f \in \mathcal{F}\}$ lies in the locally AF algebra B, there exists a finite dimensional C^*-subalgebra B_0 of B such that, for any $f \in \mathcal{F}$, there exists b in B_0 such that $\|\tau(f) - b\| < \varepsilon/3$. Put $A_0 = \rho(B_0)$. By the last lemma, there exists a $*$-monomorphism $\rho : B \to A$ such that $\tau\rho = \mathrm{id}_B$. Since $\tau(f - \rho \circ \tau(f)) = 0$ for all $f \in \mathcal{F}$, there exists a finite dimensional C^*-subalgebra J_0 of J such that, for each $f_i \in \mathcal{F}$, there exists c_i in J_0 such that $\|f_i - \rho \circ \tau(f_i) - c_i\| < \varepsilon/3$.

Let $e_{ij}^{(s)}$ for $1 \leq s \leq k, 1 \leq i, j \leq n_s$ be a set of matrix units for A_0; and let p be the unit of A_0. For each s, $e_{11}^{(s)} J e_{11}^{(s)}$ is loaclly AF. And thus there exists a net $(p_\lambda^{(s)})_{\lambda\in\Lambda}$ of projections in $e_{11}^{(s)} J e_{11}^{(s)}$ such that $a = \lim_\lambda ap_\lambda^{(s)}$ for all $a \in e_{11}^{(s)} J e_{11}^{(s)}$. Similarly, there exists a net $(q_\theta)_{\theta\in\Theta}$ of projections in $p^\perp J p^\perp$ such that $a = \lim_\theta aq_\theta$ for all $a \in p^\perp J p^\perp$. It is clear that $q_\theta a = aq_\theta = 0$ for all $a \in A_0$. Define

$$p_\lambda = \sum_{s=1}^{k} \sum_{i=1}^{n_s} e_{i1}^{(s)} p_\lambda^{(s)} e_{1i}^{(s)}.$$

In addition,

$$e_{ij}^{(s)}p_\lambda = e_{ij}^{(s)}e_{j1}^{(s)}p_\lambda^{(s)}e_{1j}^{(s)} = e_{i1}^{(s)}p_\lambda^{(s)}e_{1j}^{(s)} = e_{i1}^{(s)}p_\lambda^{(s)}e_{1i}^{(s)}e_{ij}^{(s)} = p_\lambda e_{ij}^{(s)}.$$

Hence each p_λ commutes with A_0.

For each $a \in A_0$ and c in J, we have

$$\lim_{\lambda,\theta}(p_\lambda + q_\theta)(a+c)(p_\lambda + q_\theta) + (p_\lambda + q_\theta)^\perp a(p_\lambda + q_\theta)^\perp = a + \lim_{\lambda,\theta}(p_\lambda + q_\theta)c(p_\lambda + q_\theta) = a + c.$$

So there exist sufficiently large λ and θ such that

$$\|\rho\circ\tau(f_i) + c_i - (p_\lambda + q_\theta)(\rho\circ\tau(f_i) + c_i)(p_\lambda + q_\theta) - (p_\lambda + q_\theta)^\perp(\rho\circ\tau(f_i))(p_\lambda + q_\theta)^\perp\| < \varepsilon$$

and $\|(p_\lambda + q_\theta)f' - f'\| < \varepsilon$ for all $f' \in \mathcal{F}'$. This implies that $0 \to J \xrightarrow{j} A \xrightarrow{\tau} B \to 0$ is locally quasidiagonal.

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C^*-代数的局部拟对角扩张

侍昌礼，姚洪亮

(南京理工大学数学与统计学院，南京 210094)

摘要: 本文给出了局部拟对角扩张的定义并研究了它的性质. 利用类比的方法, 根据拟对角扩张所具有的一些性质, 研究局部拟对角扩张是否仍具有这些性质. 获得了局部拟对角扩张仍然可以保持拟对角扩张的一些性质的结果, 同时还证明了局部AF代数的扩张是局部拟对角扩张.

关键词: C^*-代数, 拟对角扩张C^*-代数, 拟对角扩张

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