Vol. 45 (2025)
No. 6

数 学 杂 志
J. of Math. (PRC)

HOM-STRUCTURES ON A CLASS OF SOLVABLE LIE ALGEBRAS WITH FILIFORM NILRADICAL

ZHANG Ju-shuang [1], YUAN Ji-xia [1], ZHANG Shuang [2]

(*1. School of Mathematical Sciences, Heilongjiang University, Harbin 150080, China*)

(*2. School of Russian Studies, Heilongjiang University, Harbin 150080, China*)

Abstract: In this paper, we study the Hom-structures of a special class of solvable Lie algebras with naturally graded filiform nilradical $n_{n,1}$. Over an algebraically closed field $\mathbb{F}$ of zero characteristic, we calculate the Hom-structures of these solvable Lie algebras using the Hom-Jacobi identity, obtain the bases of these Hom-structures and observe that there are certain similarities among these bases.

Keywords: solvable Lie algebras; Hom-structures; filiform nilradical

2010 MR Subject Classification: 17B05; 17B30

Document code: A **Article ID: 0255-7797(2025)06-0478-07**

1 Introduction

Hom-Lie algebras are a new class of algebraic structures, which can be regarded as the deformation of Lie algebras. The study of Hom-Lie algebras originated from applications in Lie algebras and physics, especially in quantum field theory and string theory, where they provide new mathematical tools for the study of certain symmetries in these theories. Hom-Lie algebras are a class of nonassociative algebras satisfying antisymmetry and the Hom-Jacobi identity. In 2006, Hartwing, Larsson and Silvestrov introduced the structures of Hom-Lie algebras in order to study the deformable Witt algebras and Virasoro algebras [1]. In recent years, Hom-Lie algebras have received wide attention, for example, literature [2] describes cohomology and deformations on Hom-Lie algebras. The literature [3] provides the adjoint representation, trivial representation, derivations, deformations, central extensions

---

[*] **Received date:** 2025-03-28 **Accepted date:** 2025-06-23

Foundation item: Supported by National Natural Science Foundation of China(12271085); Supported by National Natural Science Foundation of Heilongjiang Province(LH2022A019); Basic Scientific Research Operating Funds for Provincial Universities in Heilongjiang Province(2020_KYYWF_1018); Heilongjiang University Outstanding Youth Science Foundation(JCL202103); Heilongjiang University Educational and Teaching Reform Research Project(2024C43); Heilongjiang University Postgraduate Education Reform Project(JGXM_YJS_2024010).

Biography: Zhang Jushuang(1998–), female, postgraduate, major in: Algebra.
E-mail: m15845319835@163.com

Corresponding author: Yuan Jixia(1982–), female, Professor, major in: Lietheory. E-mail: yuanjixia138@163.com

of Hom-Lie algebras, etc. The literature [4] provides a complete classification of Hom-Lie algebras on the Lie algebra $gl(2,C)$. The literature [5] extends the concept of Hom-Lie algebra to Hom-Lie superalgebra, and thus the study of Hom-Lie superalgebra is developed. Hom-structures have been widely studied by researchers, since Hom-algebras were proposed. For example, the literature [6] proves that the Hom-Lie algebra structures on finite-dimensional simple Lie algebras are trivial, and the literature [7] proves that the Hom-Lie superalgebra structures on finite-dimensional simple Lie superalgebras are also trivial.

By Levi's theorem, we know that any finite-dimensional Lie algebra is isomorphic to a direct sum of a semi-simple Lie algebra and a maximal solvable ideal [8]. In this paper, we will focus on studying a class of solvable Lie algebras. Since any solvable Lie algebra has a uniquely determined nilradical, the main work of studying solvable Lie algebras comes down to studying their nilradicals. The literature [9] describes $\frac{1}{2}$-derivations of finite-dimensional solvable Lie algebras with a filiform nilradical and the literature [10] describes local and 2-local $\frac{1}{2}$-derivation of these solvable Lie algebras. The literature [11] finds one-dimensional extensions of solvable Lie algebras with nilradical $n_{n,1}$. In this paper, we mainly study the Hom-structures on solvable Lie algebras with filiform nilradical $n_{n,1}$ over an algebraically closed field $\mathbb{F}$ of zero characteristic.

2 Preliminaries

In this section, we mainly introduce the definitions of the Lie algebra, Hom-Lie algebra and Hom-structure, then we recall the definitions of the solvable Lie algebra and nilpotent Lie algebra.

Definition 2.1 [12] A Lie algebra over the field $\mathbb{F}$ is a vector space $\mathcal{G}$ together with a bilinear map $[\cdot,\cdot]:\mathcal{G}\times\mathcal{G}\to\mathcal{G}$, called the Lie bracket of $\mathcal{G}$, which is skew symmetric, i.e.

$$[a,b]=-[b,a],\forall a,b\in\mathcal{G},$$

and satisfies the Jacobi identity, i.e.

$$[a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0,\forall a,b,c\in\mathcal{G}.$$

The Hom-Lie algebra can be regarded as a deformation of the Lie algebra. Next, we will introduce the definition of the Hom-Lie algebra.

Definition 2.2 [13] A Hom-Lie algebra $(\mathcal{G},\ \varphi\ )$ is a non-associative algebra $\mathcal{G}$ together with an algebra homomorphism $\varphi:\mathcal{G}\to\mathcal{G}$, such that

$$[a,b]=-[b,a],\forall a,b\in\mathcal{G},$$

$$[\varphi(a),[b,c]]+[\varphi(b),[c,a]]+[\varphi(c),[a,b]]=0,\forall a,b,c\in\mathcal{G},\tag{2.1}$$

where $[\cdot,\cdot]$ denotes the product in $\mathcal{G}$. We call the equation (2.1) the Hom-Jacobi identity. Let $(\mathcal{G},\ [\cdot,\cdot]\ )$ be a Lie algebra and φ be a linear map satisfying the Hom-Jacobi identity, then φ is a Hom-structure on $(\mathcal{G},\ [\cdot,\cdot]\ )$.

Obviously, all the Hom-structures of Lie algebra $\mathcal{G}$ constitute a vector space, denoted $\mathrm{HS}\,(\mathcal{G})$.

For a Lie algebra $\mathcal{G}$, consider the following lower central and derived sequences:

$$\mathcal{G}^1 = \mathcal{G}, \mathcal{G}^{k+1} = \left[\mathcal{G}, \mathcal{G}^k\right], k \geq 1,$$

$$\mathcal{G}^{(0)} = \mathcal{G}, \mathcal{G}^{(1)} = \left[\mathcal{G}, \mathcal{G}\right], \mathcal{G}^{(s+1)} = \left[\mathcal{G}^{(s)}, \mathcal{G}^{(s)}\right], s \geq 1.$$

Definition 2.3 [12] A Lie algebra $\mathcal{G}$ is called nilpotent (respectively, solvable), if there exists $p \in \mathbb{N}$ ($q \in \mathbb{N}$) such that $\mathcal{G}^p = 0$ (respectively, $\mathcal{G}^{(q)} = 0$).

Any solvable Lie algebra $\mathcal{G}$ contains a unique maximal nilpotent ideal, called the nilradical of this Lie algebra $\mathcal{G}$. Thus, we can consider a given nilpotent Lie algebra as a nilradical, and then find all of its extensions to solvable Lie algebras.

It is well known that there are two types of naturally graded filiform Lie algebras, the second type only occurs if the dimension of the algebra is even, and any naturally graded filiform Lie algebra is isomorphic to one of the following non-isomorphic algebras [14].

$$n_{n,1} : [e_i, e_1] = -[e_i, e_1] = e_{i+1}, \quad 2 \leq i \leq n-1.$$

$$Q_{2n} : \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, 2 \leq i \leq 2n-2, \\ [e_i, e_{2n+1-i}] = -[e_{2n+1-i}, e_i] = (-1)^i e_{2n}, 2 \leq i \leq n. \end{cases}$$

All solvable Lie algebras whose nilradical are naturally graded filiform Lie algebras $n_{n,1}$ are classified in [15]. Here we introduce the solvable Lie algebras with nilradical $n_{n,1}$:

$$S_{n+1,1}(\beta) : \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, 2 \leq i \leq n-1, \\ [e_i, x] = -[x, e_i] = (i-2+\beta)e_i, 2 \leq i \leq n, \\ [e_1, x] = -[x, e_1] = e_1; \end{cases}$$

$$S_{n+1,2}(\alpha_3, \cdots, \alpha_{n-1}) : \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, 2 \leq i \leq n-1, \\ [e_i, x] = -[x, e_i] = e_i + \displaystyle\sum_{l=i+2}^{n} \alpha_{l+1-i} e_l, 2 \leq i \leq n, \end{cases}$$

if the first non-vanishing parameter $\{\alpha_3, \cdots, \alpha_{n-1}\}$ exists, it can be assumed to be equal to 1.

$$S_{n+1,3} : \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, 2 \leq i \leq n-1, \\ [e_i, x] = -[x, e_i] = (i-1)e_i, 2 \leq i \leq n, \\ [e_1, x] = -[x, e_1] = e_1 + e_2; \end{cases}$$

$$S_{n+2} : \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, 2 \leq i \leq n-1, \\ [e_i, x_1] = -[x_1, e_i] = (i-2)e_i, 3 \leq i \leq n, \\ [e_i, x_2] = -[x_2, e_i] = e_i, 2 \leq i \leq n, \\ [e_1, x_1] = -[x_1, e_1] = e_1, \end{cases}$$

where $\{e_1, e_2, \cdots, e_n, x\}$ is a standard basis of $S_{n+1,1}(\beta)$, $S_{n+1,2}(\alpha_3, \cdots, \alpha_{n-1})$, $S_{n+1,3}$, and $\{e_1, e_2, \cdots, e_n, x_1, x_2\}$ is a standard basis of S_{n+2}.

3 Hom-structures

In this section, we will calculate the Hom-structures on solvable Lie algebras with nilradical $n_{n,1}$.

Theorem 3.1 The bases of HS $(S_{n+1,1}(\beta))$, HS $(S_{n+1,2}(\alpha_3, \cdots, \alpha_{n-1}))$, HS $(S_{n+1,3})$ and HS (S_{n+2}) are as follows:

Table 1 Hom-structures of A Class of Solvable Lie Algebras

Hom-structures	The bases
HS $(S_{n+1,1}(\beta))$	$\sum\limits_{i=2}^{n-2} E_{i,i+1} + E_{n+1,1},\ E_{11} - \sum\limits_{i=2}^{n-1} (i-2+\beta)E_{ii},$ $\sum\limits_{i=2}^{n-1} (i-1+\beta)E_{ii} + E_{n+1,n+1},$ $E_{in}, E_{n+1,j}, E_{1k}, 2 \le i,j,k \le n.$
HS $(S_{n+1,2}(\alpha_3, \cdots, \alpha_{n-1}))$	$E_{ij}, 2 \le i,j \le n,\ E_{11} + E_{n+1,n+1},$ $E_{n+1,t}, E_{1k}, 2 \le t,k \le n.$
HS $(S_{n+1,3})$	$E_{n+1,1} + \sum\limits_{i=2}^{n-2} E_{i,i+1},\ E_{11} - \sum\limits_{i=2}^{n-1} (i-1) E_{ii},$ $E_{n+1,n+1} + \sum\limits_{i=2}^{n-1} iE_{ii},\ E_{in}, E_{n+1,j}, E_{1k}, 2 \le i,j,k \le n.$
HS (S_{n+2})	$E_{in},\ E_{n+1,j},\ E_{1k}, 2 \le i,j,k \le n,$ $\sum\limits_{i=1,i\ne n}^{n+2} E_{ii},\ E_{n+2,n}.$

In this paper, we provide the proof only for HS $(S_{n+1,1}(\beta))$.

Proof Suppose

$$\Gamma = \left\{ \sum_{i=2}^{n-1} (i-1+\beta)E_{ii} + E_{n+1,n+1} \right\} \cup \{E_{in}, E_{n+1,j}, E_{1k} \,|2 \le i,j,k \le n\}$$
$$\cup \left\{ \sum_{i=2}^{n-2} E_{i,i+1} + E_{n+1,1}, E_{11} - \sum_{i=2}^{n-1} (i-2+\beta)E_{ii} \right\}.$$

Obviously, Γ is a linearly independent set and $\Gamma \subseteq$ HS $(S_{n+1,1}(\beta))$. For all $\varphi \in$ HS $(S_{n+1,1}(\beta))$, suppose that the matrix of φ in the standard basis is $\mathrm{A} = (a_{ij})_{(n+1)\times(n+1)}$, where $a_{ij} \in \mathbb{F}$, that is

$$\begin{pmatrix} \varphi(e_1) & \varphi(e_2) & \cdots & \varphi(e_n) & \varphi(x) \end{pmatrix}^{\mathrm{T}} = \mathrm{A} \begin{pmatrix} e_1 & e_2 & \cdots & e_n & x \end{pmatrix}^{\mathrm{T}}.$$

Putting $a = e_k, b = e_l, c = x$ in equation (2.1), where $k \ne l, k, l = 1, 2, \cdots, n$.

Case 1 If $2 \le k, l \le n-1$, we can obtain

$$(k-2+\beta)^2 a_{l,n+1} e_k + (k-2+\beta)\, a_{l,1} e_{k+1} - (l-2+\beta)\, a_{k,1} e_{l+1}$$
$$-\,(l-2+\beta)^2 a_{k,n+1} e_l = 0,$$

then

$$a_{l,n+1} = a_{l,1} = 0, \quad 2 \le l \le n-1. \tag{3.1}$$

Case 2 If $k = 1, 2 \le l \le n-2$, we can obtain

$$\begin{aligned} & a_{l,n+1}e_1 - (l-2+\beta)^2 a_{1,n+1}e_l + [(l-1+\beta)\, a_{n+1,n+1} - (l-2+\beta)\, a_{11}]\, e_{l+1} \\ & + a_{n+1,1}e_{l+2} - \sum_{i=3}^{n} a_{l,i-1}e_i = 0, \end{aligned}$$

then

$$a_{l,2} = a_{l,3} = \cdots = a_{l,l-2} = 0, \quad 4 \le l \le n-2; \tag{3.2}$$

$$a_{l,l-1} = -(l-2+\beta)^2 a_{1,n+1}, \quad 3 \le l \le n-2;$$

$$a_{l,l} = (l-1+\beta)\, a_{n+1,n+1} - (l-2+\beta)\, a_{11}, \quad 2 \le l \le n-2; \tag{3.3}$$

$$a_{l,l+1} = a_{n+1,1}, 2 \le l \le n-2; a_{l,l+2} = a_{l,l+3} = \cdots = a_{l,n-1} = 0, 2 \le l \le n-3. \tag{3.4}$$

In particular, when $l = 2$, we can obtain $a_{1,n+1} = 0$, then

$$a_{l,l-1} = 0, \quad 3 \le l \le n-2. \tag{3.5}$$

Case 3 If $k = 1, l = n-1$, we can obtain

$$\begin{aligned} & a_{n-1,n+1}e_1 - \sum_{i=3}^{n-2} a_{n-1,i-1}e_i - \left[a_{n-1,n-2} + (n-3+\beta)^2 a_{1,n+1}\right] e_{n-1} \\ & + [(n-2+\beta)\, a_{n+1,n+1} - a_{n-1,n-1} - (n-3+\beta)\, a_{11}]\, e_n = 0, \end{aligned}$$

then

$$a_{n-1,2} = a_{n-1,3} = \cdots = a_{n-1,n-3} = 0; \tag{3.6}$$

$$a_{n-1,n-2} = -(n-3+\beta)^2 a_{1,n+1} = 0; \tag{3.7}$$

$$a_{n-1,n-1} = (n-2+\beta)\, a_{n+1,n+1} - (n-3+\beta)\, a_{11}. \tag{3.8}$$

Case 4 If $k = n, 2 \le l \le n-1$, we can obtain

$$-(l-2+\beta)^2 a_{n,n+1}e_l - (l-2+\beta)\, a_{n1}e_{l+1} + (n-2+\beta)^2 a_{l,n+1}e_n = 0,$$

then

$$a_{n1} = a_{n,n+1} = 0. \tag{3.9}$$

Case 5 If $k = n, l = 1$, we can obtain

$$-a_{n,n+1}e_1 + \sum_{i=3}^{n-1} a_{n,i-1}e_i + \left[a_{n,n-1} + (n-2+\beta)^2 a_{1,n+1}\right] e_n = 0,$$

then

$$a_{n,2} = a_{n,3} = \cdots = a_{n,n-2} = 0; \tag{3.10}$$

$$a_{n,n-1} = -(n-2+\beta)^2 a_{1,n+1} = 0. \tag{3.11}$$

By (3.1)–(3.11) and $a_{1,n+1} = 0$, we have

$$\mathrm{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} & 0 \\ 0 & a_{22} & a_{n+1,1} & \cdots & 0 & a_{2n} & 0 \\ 0 & 0 & a_{33} & \cdots & 0 & a_{3n} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_{nn} & 0 \\ a_{n+1,1} & a_{n+1,2} & a_{n+1,3} & \cdots & a_{n+1,n-1} & a_{n+1,n} & a_{n+1,n+1} \end{pmatrix},$$

where $a_{ii} = (i-1+\beta)a_{n+1,n+1} - (i-2+\beta)a_{11}, 2 \le i \le n-1$. Then the matrix A can be expressed linearly by Γ.

References

[1] Makhlouf A, Silvestrov S. Hom-Lie admissible Hom-coalgebras and Hom-Hopf algebras[J]. Noncommutative Geometry, 2007, 81(1): 189–206.

[2] Ammar F, Ejbehi Z, Makhlouf A. Cohomology and deformations of Hom-algebras[J]. Journal of Lie theory, 2011, 21(4): 813–836.

[3] Sheng Yunhe. Representations of Hom-Lie algebras[J]. Algebras and Representation Theory, 2012, 15(6): 1081–1098.

[4] Wu Haiyan, Tang Xiaomin, Sheng Yuqiu. Hom-Lie algebra structures on the Lie algebra $gl(2, C)$[J]. Journal of Natural Science of Heilongjiang University, 2014, 23(1): 180–197.

[5] Ammar F, Makhlouf A. Hom-Lie superalgebras and Hom-Lie admissible superalgebras[J]. Journal of Algebra, 2010, 324(7): 1513–1528.

[6] Jin Quanqin, Li Xiaochao. Hom-Lie algebra structures on semi-simple Lie algebras[J]. Journal of Algebra, 2008, 319(4): 1398–1408.

[7] Cao Bintao, Luo Li. Hom-Lie superalgebra structures on finite-dimensional simple Lie superalgebras[J]. Journal of Lie theory, 2013, 23(4): 1115–1128.

[8] Li Xiaochao, Li Dongya, Jin Quanqin. A class of solvable Lie algebras and their Hom-Lie algebra structures[J]. Chinese Quarterly Journal of Mathematics, 2014, 29(2): 231–237.

[9] Abdurasulov K, Adashev J, Eshmeteva S. Transposed poisson structures on solvable Lie algebras with filiform nilradical[J]. Communications in Mathematics, 2024, 32(3): 441–483.

[10] Khudoyberdiyev A, Yusupov B. Local and 2-local $\frac{1}{2}$-derivations on finite-dimensional Lie algebras[J]. Results in Mathematics, 2024, 79(5): 210–232.

[11] Khudoyberdiyev A K, Sheraliyeva S A. Extensions of solvable Lie algebras with naturally graded filiform nilradical[J]. Journal of Algebra and Its Applications, 2024, 23(10): 2450161.

[12] Humphreys J E. Introduction to Lie algebras and representation theory[J]. Graduate Texts in Mathematics, 1980, 9(74): 31–45.

[13] Hartwig J, Larsson D, Silvestrov S. Deformations of Lie algebras using σ-derivations[J]. Journal of Algebra, 2006, 295(2): 314–361.

[14] Vergne M. Cohomologie des algèbres de Lie nilpotentes. Application à létude de la variété des algèbres de Lie nilpotentes[J]. Bulletin de la Société Mathématique de France, 1970, 98: 81–116.

[15] Casas J M, Ladra M, Omirov B A, et al. Classification of solvable Leibniz algebras with naturally graded filiform nilradical[J]. Linear Algebra and its Applications, 2013, 438(7): 2973–3000.

一类具有Filiform幂零根基的可解李代数的Hom-结构

张菊双 [1], 远继霞 [1], 张爽 [2]

(1. 黑龙江大学数学科学学院, 黑龙江 哈尔滨 150080)

(2. 黑龙江大学俄语学院, 黑龙江 哈尔滨 150080)

摘要: 本文研究了一类特殊的可解李代数的Hom-结构, 此类李代数是一类具有自然阶化Filiform幂零根基$n_{n,1}$的可解李代数. 在特征零代数闭域$\mathbb{F}$上, 本文利用Hom-Jacobi等式计算了此类可解李代数的Hom-结构, 得到了这些Hom-结构的基底, 并且发现这些基底存在一定的相似之处.

关键词: 可解李代数; Hom-结构; Filiform 幂零根基

MR(2010)**主题分类号:** 17B05; 17B30 **中图分类号:** O152.5