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# MONTEL'S CRITERION AND BLOCH-VALIRON'S THEOREM CONCERNING WANDERING FUNCTIONS

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**Abstract:** In this paper, we study normal families of meromorphic functions. By using the idea in [16], we obtain some normality criteria for families of meromorphic functions concerning the wandering multiple functions, which extend and improve the well-known Montel's criterion, Bloch-Valiron's theorem, and the related results due to Carathéodory, and Grahl-Nevo et al..

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## 1 Introduction

Let $D$ be a domain in the complex plane $\mathbb{C}$, and $\mathcal{F}$ be a family of meromorphic functions in $D$. The family $\mathcal{F}$ is said to be normal in $D$, in the sense of Montel, if every sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence $\{f_{n_j}\}$ such that $\{f_{n_j}\}$ converges spherically locally uniformly on $D$ to a meromorphic function or $\infty$ [1, 2, 3].

The most celebrated theorem in the theory of normal families is Montel's criterion[4](cf. [1,2,3]), which is the local counterpart of Picard theorem and plays an important role in complex dynamics. It is also called Fundamental Normality Test(FNT)(see [2]).

**Theorem A** (Montel's criterion) Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D$, and $a_1, a_2, a_3$ are distinct complex numbers in the extended complex plane $\bar{\mathbb{C}}$. If, for each $f \in \mathcal{F}$, $f$ omits $a_1, a_2, a_3$ in $D$, then $\mathcal{F}$ is normal in $D$.

Montel's criterion has undergone various extensions and improvements(see [5-14], etc.). Chang, Fang and Zalcman[6](cf. [13]) replaced "three distincr values $a_1, a_2, a_3$" in Theorem A with "three distinct meromorphic functions $a_1(z), a_2(z), a_3(z)$", and obtained the following result.

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**Theorem B** Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D$, and let $a_1(z), a_2(z), a_3(z)$ be three meromorphic functions such that $a_i(z) \not\equiv a_j(z) (1 \leq i < j \leq 3)$ in $D$, one of which may be $\infty$ identically. If, for each $f \in \mathcal{F}$, $f(z) \neq a_i(z) (i = 1, 2, 3)$ in $D$, then $\mathcal{F}$ is normal in $D$.

Let $\sigma(z_1, z_2)$, $\chi(z_1, z_2)$ denote the spherical distance and the chordal distance between $z_1$ and $z_2$ in $\bar{\mathbb{C}}$ respectively. Since $\chi(z_1, z_2) \leq \sigma(z_1, z_2) \leq \frac{\pi}{2}\chi(z_1, z_2)$, then the metrics $\sigma$ and $\chi$ can be treated as one and the same from a topological point of view.

There is another splendid version of Montel's criterion due to Carathéodory[15].

**Theorem C** Let $\mathcal{F}$ be a family of functions meromorphic in a domain $D$ and $\varepsilon > 0$. Suppose that for each $f \in \mathcal{F}$, there exist distinct complex numbers $a_{1f}, a_{2f}, a_{3f} \in \bar{\mathbb{C}}$ such that $f$ omits $a_{1f}, a_{2f}, a_{3f}$ and

$$\sigma(a_{if}, a_{jf}) \geq \varepsilon, (1 \leq i < j \leq 3)$$

in $D$, then $\mathcal{F}$ is normal in $D$.

**Remark 1** Here, it should be noticed that the exception values $a_{1f}, a_{2f}, a_{3f}$ in Theorem C must be kind of "uniformly distinct" rather than "distinct".

Let $\mathcal{M}(D)$ denote the class of functions meromorphic in a domain $D$. Grahl and Nevo[16] combined the above two directions of extension by considering exceptional functions which depend on the respective function in the family under consideration, i.e. kind of "wandering" exceptional functions, and proved the following result.

**Theorem D** Let $\mathcal{F}$ be a family of functions meromorphic in a domain $D$ and $\varepsilon > 0$. Suppose that for each $f \in \mathcal{F}$, there exist functions $a_{1f}, a_{2f}, a_{3f} \in \mathcal{M}(D) \cup \{\infty\}$ such that
(1) $f(z) \neq a_{1f}(z), a_{2f}(z), a_{3f}(z)$ in $D$;
(2) $\sigma(a_{if}(z), a_{jf}(z)) \geq \varepsilon$ for all $z \in D$ and $1 \leq i < j \leq 3$.
Then $\mathcal{F}$ is normal in $D$.

The next is due to Bloch-Valiron(see [3]), which is another extension of Montel's criterion.

**Theorem E** Let $a_1, a_2, \cdots, a_q (q \geq 3)$ be distinct complex numbers in $\bar{\mathbb{C}}$, and $l_1, l_2, \cdots, l_q$ be positive integers or $+\infty$, and let $\mathcal{F}$ be a family of meromorphic functions in a domain $D$. If, for each $f \in \mathcal{F}$, such that
(1) all zeros of $f - a_i$ have multiplicity at least $l_i (i = 1, 2 \cdots, q)$ in $D$;
(2) $\sum\limits_{i=1}^{q} \frac{1}{l_i} < q - 2$,
then $\mathcal{F}$ is normal in $D$.

It is natural to ask: does Theorem E still hold if the multiple values '$a_1, a_2, \cdots, a_q$' are replaced by the multiple wandering functions '$a_{1f}(z), a_{2f}(z), \cdots, a_{qf}(z)$'?

In this paper, we prove the following result.

**Theorem 1** Let $l_1, l_2, \cdots, l_q (q \geq 3)$ are positive integers or $+\infty$ and $\varepsilon > 0$, and let $\mathcal{F}$ be a family of meromorphic functions in a domain $D$. Suppose that for each $f \in \mathcal{F}$, there exist functions $a_{1f}, a_{2f}, \cdots, a_{qf} \in \mathcal{M}(D) \cup \{\infty\}$ such that
(1) all zeros of $f(z) - a_{if}(z)$ have multiplicity at least $l_i (i = 1, 2 \cdots, q)$ in $D$;

(2) $f$ and each $a_{if}(i=1,2\cdots,q)$ have no common poles;
(3) $\sum\limits_{i=1}^{q}\frac{1}{l_i}<q-2$;
(4) $\sigma(a_{if}(z),a_{jf}(z))\geq\varepsilon(1\leq i<j\leq q)$ in $D$.
Then $\mathcal{F}$ is normal in $D$.

**Remark 2** Obviously, Theorem 1 generalizes and improves Theorem E.

**Remark 3** The condition '$\sigma(a_{if}(z),a_{jf}(z))\geq\varepsilon(1\leq i<j\leq q)$' in Theorem 1 can not be replaced by '$|(a_{if}(z)-a_{jf}(z))|\geq\varepsilon(1\leq i<j\leq q)$', as is shown by the following example.

**Example 1** Let $\Delta=\{z:|z|<1\}$, $\mathcal{F}=\{f_n(z)=nz:n=1,2,\cdots\}$, $a_{1n}(z)=nz+n, a_{2n}(z)=nz+n+1,\cdots,a_{qn}(z)=nz+n+q-1$.

Clearly, for each $n$, $f_n(z)\neq a_{1n}(z),a_{2n}(z),\cdots,a_{qn}(z)$. This means that Conditions (1),(2) and (3) are satisfied( letting $l_1=l_2=\cdots=l_q=\infty$). For each $n$, we have that $|a_{in}(z)-a_{jn}(z)|\geq 1(1\leq i<j\leq q)$ in $\Delta$. However, $\mathcal{F}$ fails normal at $z=0$ since $f_n(0)=0$ and $f_n(1/n)=1$. Noting that

$$\begin{aligned}0<\sigma(a_{in}(z),a_{jn}(z))&\leq\frac{\pi}{2}\chi(a_{in}(z),a_{jn}(z))=\frac{\pi|a_{in}(z)-a_{jn}(z)|}{2\sqrt{1+|a_{in}(z)|^2}\sqrt{1+|a_{jn}(z)|^2}}\\&\leq\frac{(q-1)\pi}{2(1+(n+n|z|)^2)}\to 0\end{aligned}$$

for $1\leq i<j\leq q$ and all $z\in\Delta$ as $n\to\infty$.

Let $q=3$ in Theorem 1, we have

**Corollary 1** Let $l_1,l_2,l_3$ are positive integers or $+\infty$ and $\varepsilon>0$, and let $\mathcal{F}$ be a family of meromorphic functions in a domain $D$. Suppose that for each $f\in\mathcal{F}$, there exist functions $a_{1f},a_{2f},a_{3f}\in\mathcal{M}(D)\cup\{\infty\}$ such that
(1) all zeros of $f(z)-a_{if}(z)$ have multiplicity at least $l_i(i=1,2,3)$ in $D$;
(2) $f$ and each $a_{if}(i=1,2,3)$ have no common poles;
(3) $\frac{1}{l_1}+\frac{1}{l_2}+\frac{1}{l_3}<1$;
(4) $\sigma(a_{if}(z),a_{jf}(z))\geq\varepsilon(1\leq i<j\leq 3)$ in $D$.
Then $\mathcal{F}$ is normal in $D$.

**Remark 4** Corollary 1 improves and generalizes Montel's criterion and Theorems C and D.

## 2 Some Lemmas

To prove our results, we need some lemmas.

For $p\in\mathbb{N}$ and $j=1,2,\cdots,p$, we introduce the projections

$$\pi_j:(\mathcal{M}(D))^p\longrightarrow\mathcal{M}(D)$$

as

$$\pi_j(f_1,\cdots,f_p):=f_j$$

for $(f_1,\cdots,f_p)\in(\mathcal{M}(D))^p$. The next is called "simultaneous rescaling version of Zalcman's lemma", which is due to Grahl and Nevo.

**Lemma 1** [16] Let $p \in \mathbb{N}$ and $\mathcal{F} \subseteq (\mathcal{M}(D))^p$. Assume that there exists $j_0 \in \{1, \cdots, p\}$ such that the family $\pi_{j_0}(\mathcal{F})$ is not normal at $z_0 \in D$. Then there exist $(f_{1n}, \cdots, f_{pn}) \in \mathcal{F}, z_n \in D$ with $z_n \to z_0$, and positive numbers $\rho_n \to 0$ such that

$$g_{jn}(\zeta) = f_{jn}(z_n + \rho_n \zeta) \to g_j(\zeta), (j = 1, \cdots, p)$$

converges spherically and locally uniformly on $\mathbb{C}$, where $g_1, \cdots, g_p \in \mathcal{M}(\mathbb{C}) \bigcup \{\infty\}$, are not all constants.

**Lemma 2** [16] Let $\varepsilon > 0$, and $\mathcal{F} \subseteq (\mathcal{M}(D))^2$ be a family of pairs of meromorphic functions in a domain $D$. If for all $(a, b) \in \mathcal{F}$,

$$\sigma(a(z), b(z)) \geq \varepsilon$$

in $D$, then the families $\{a : (a, b) \in \mathcal{F}\}$ and $\{b : (a, b) \in \mathcal{F}\}$ are normal in $D$.

**Remark 5** Set $\mathcal{M}_1(D) = \mathcal{M}(D) \cup \{\infty\}$, it is easy to see that Lemmas 1 and 2 are still valid if $\mathcal{M}(D)$ is replaced by $\mathcal{M}_1(D)$(for details, see [16, Lemma 6 and Theorem 2]).

Let $f$ be a nonconstant meromorphic function on the complex plane $\mathbb{C}$. We now recall some standard notations of Nevanlinna value distribution theory (see [1,3]). Set $\log^+ x = \max\{\log x, 0\}(\log^+ 0 = 0)$. We define the proximity function of $f$ by

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta.$$

The function

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r$$

is called the counting function of poles of $f$, where $n(t, f)$ denotes the number of poles of $f$ on $|z| \leq t$. We also denote by

$$\overline{N}(r, f) = \int_0^r \frac{\overline{n}(t, f) - \overline{n}(0, f)}{t} dt + \overline{n}(0, f) \log r$$

the counting function of poles, where $\overline{n}(t, f)$ counts each pole only once despite its multiplicity. The sum

$$T(r, f) = m(r, f) + N(r, f)$$

is called the Nevanlinna characteristic function of $f$. We denote by $S(r, f)$ any function satisfying

$$S(r, f) = o\{T(r, f)\},$$

as $r \to \infty$, possibly outside a set with finite measure, not always the same at every occurrence. The following is the well-known Nevanlinna second fundamental theorem.

**Lemma 3** (see [1,3]) Let $f$ be a nonconstant meromorphic function in $\mathbb{C}$, $a_1, a_2, \cdots, a_q (q \geq 3)$ are distinct complex numbers in $\bar{\mathbb{C}}$. Then

$$(q-2)T(r, f) \leq \sum_{i=1}^{q} \bar{N}(r, \frac{1}{f - a_i}) + S(r, f).$$

## 3 Proof of Theorem 1

By the assumptions of the theorem, for each $f \in \mathcal{F}$, there exist functions

$$a_{1f}, a_{2f}, \cdots, a_{qf} \in \mathcal{M}_1(D)$$

such that all zeros of $f - a_{if}$ have multiplicity at least $l_i (i = 1, 2, \cdots, q)$, $f$ and $a_{if} (i = 1, 2 \cdots, q)$ have no common poles, and $\sigma(a_{if}(z), a_{jf}(z)) \geq \varepsilon (1 \leq i < j \leq q)$ in $D$.

Suppose, on the contrary, that $\mathcal{F}$ is not normal at $z_0 \in D$. Then $\pi_1(\mathcal{F}_1)$ is not normal at $z_0$, where

$$\mathcal{F}_1 = \{(f, a_{1f}, \cdots, a_{qf})\} \subseteq (\mathcal{M}_1(D))^{q+1}.$$

By Lemma 1, there exists a sequence $\{(f_n, a_{1f_n}, \cdots, a_{qf_n})\} \subseteq \mathcal{F}_1$, points $z_n \in D$, and positive numbers $\rho_n$ such that $z_n \to z_0$, $\rho_n \to 0$,

$$g_n(\zeta) = f_n(z_n + \rho_n \zeta) \to g(\zeta), \tag{3.1}$$

and

$$h_{in}(\zeta) = a_{if_n}(z_n + \rho_n \zeta) \to h_i(\zeta), (i = 1, 2, \cdots, q) \tag{3.2}$$

locally uniformly with respect to the spherical metric on $\mathbb{C}$, where $g, h_1, \cdots, h_p \in \mathcal{M}_1(\mathbb{C})$, are not all constants.

Noting that $\sigma(a_{1f_n}(z), a_{jf_n}(z)) \geq \varepsilon( 2 \leq j \leq q)$ for each $n$ and $z \in D$, Lemma 2 implies that the families $\{a_{if_n}\} (i = 1, 2, \cdots, q)$ are normal in $D$. Then, there exist subsequences, (without loss of generality) again denoted by $\{a_{if_n}\} (i = 1, 2, \cdots, q)$ such that

$$a_{if_n}(z) \to A_i(z), (i = 1, 2, \cdots, q) \tag{3.3}$$

converges spherically and locally uniformly on $D$, where $A_i(z) \in \mathcal{M}_1(D) (i = 1, 2, \cdots, q)$. It follows that $a_{if_n}(z_n + \rho_n \zeta) \to A_i(z_0)$. This, together with (3.2), gives that $h_i(\zeta) = A_i(z_0)$ $(i = 1, 2, \cdots, q)$. Hence, $g$ must be noconstant.

By (3.3), we have

$$\varepsilon \leq \sigma(a_{if_n}(z), a_{jf_n}(z)) \to \sigma(A_i(z), A_j(z)), (1 \leq i < j \leq q)$$

in $D$ as $n \to \infty$. This means that $A_1(z_0), A_2(z_0), \cdots, A_q(z_0) (\in \bar{\mathbb{C}})$ are distinct.

We distinguish two cases.

**Case 1.** $A_1(z_0), A_2(z_0), \cdots, A_q(z_0) \in \mathbb{C}$.

By (3.1), we have

$$f_n(z_n + \rho_n \zeta) - a_{if_n}(z_n + \rho_n \zeta) \to g(\zeta) - A_i(z_0) \tag{3.4}$$

spherically and locally uniformly on $\mathbb{C}$. Since all zeros of $f_n - a_{if_n}$ have multiplicity at least $l_i$, Hurwitz's theorem and (3.4) implies that all zeros of $g(\zeta) - A_i(z_0)$ have multiplicity at least $l_i(i=1,2,\cdots,q)$. By Nevanlinna first fundamental theorem and Lemma 3, we have

$$\begin{aligned}(q-2)T(r,g) &\leq \sum_{i=1}^{q}\bar{N}(r,\frac{1}{g-A_i(z_0)})+S(r,g)\\ &\leq \sum_{i=1}^{q}\frac{1}{l_i}N(r,\frac{1}{g-A_i(z_0)})+S(r,g)\\ &\leq \sum_{i=1}^{q}\frac{1}{l_i}T(r,g)+S(r,g),\end{aligned}$$

that is,

$$(q-2-\sum_{i=1}^{q}\frac{1}{l_i})T(r,g)\leq S(r,g).$$

This yields that $g$ is a constant, a contradiction.

**Case 2.** One of $A_1(z_0),A_2(z_0),\cdots,A_q(z_0)$ is infinite, say, $A_q(z_0)=\infty$.

Then $A_1(z_0),A_2(z_0),\cdots,A_{q-1}(z_0)$ are finite. As above, all zeros of $g(\zeta)-A_i(z_0)$ have multiplicity at least $l_i(i=1,2,\cdots,q-1)$. We have

$$\frac{a_{qf_n}(z_n+\rho_n\zeta)-f_n(z_n+\rho_n\zeta)}{f_n(z_n+\rho_n\zeta)a_{qf_n}(z_n+\rho_n\zeta)}=\frac{1}{f_n(z_n+\rho_n\zeta)}-\frac{1}{a_{qf_n}(z_n+\rho_n\zeta)}\to\frac{1}{g(\zeta)}$$

spherically and locally uniformly on $\mathbb{C}$. Since $f_n$ and $a_{qf_n}$ have no common poles, we conclude that all zeros of $1/f_n(z_n+\rho_n\zeta)-1/a_{qf_n}(z_n+\rho_n\zeta)$ arise only from the zeros of $f_n(z_n+\rho_n\zeta)-a_{qf_n}(z_n+\rho_n\zeta)$. By Hurwitz's theorem, $1/g$ has only zeros with multiplicity at least $l_q$, and hence all poles of $g$ have multiplicity at least $l_q$. Again by Nevanlinna first fundamental theorem and Lemma 3, we arrive at a contradiction.

Theorem 1 is thus proved.

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# 关于Wandering函数的Montel定则与Bloch-Valiron定理

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**摘要**: 本文研究了亚纯函数正规族, 利用文献[16]中的思想, 得到了关于wandering重函数的亚纯函数族正规定则, 由此推广与改进了著名的Montel正规定则、Bloch-Valiron定理、及Carathéodory、Grahl-Nevo等人的相关结果.

**关键词**: 亚纯函数; 正规族; Montel正规定则; Bloch-Valiron定理; wandering重函数

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