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一类具有两种故障状态的M/M/1 可修排队系统的一个特征值及其应用

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摘 要: 本文研究了一类具有两种故障状态的M/M/1 可修排队系统时间依赖解的渐进性质问题. 利用概率母函数证明了 0 是该系统主算子及其共轭算子几何重数为 1 的特征值. 基于一定的约束条件下, 获得了系统的时间依赖解强收敛于该系统的稳态解. 推广了该排队系统动态分析的有关结论.

关键词: 具有两种故障状态的M/M/1 可修排队系统; 共轭算子; 几何重数; 特征值

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1 引言

可修排队系统是日常生活中最常见的排队模型之一. 近年来, 对可修排队系统研究被越来越多的学者所关注, 尤其对M/G/1 排队系统开展了大量的研究 [1−10]. 具有两种故障状态的M/G/1 可修排队系统是指服务台可能发生两种故障状态. 一种是由于服务台寿命终止而引发的故障. 另一种是由于服务员对服务台的操作失误等原因而引起的故障. 我们称第一种情况的故障为正常故障, 第二种故障为异常故障. 2002 年, 陈洋和朱翼隽等人 [11] 通过运用补充变量方法, 首次建立了描述具有两种故障状态的M/G/1 可修排队系统的数学模型, 并运用向量 Markov 过程理论和 Laplace 变换等方法, 得到了该模型的稳态可用度、故障频度以及更新频度等相关的排队指标和可靠性指标. 2005 年, 陈洋和朱翼隽等人 [12] 再次运用补充变量方法建立了描述具有三种状态可修排队系统的数学模型, 并给出了该系统稳态解存在的充分必要条件. 2024 年, 周学良和张庆红等人 [13] 运用 C_0- 半群理论证明了具有两种故障状态的M/G/1 可修排队系统存在唯一的非负时间依赖解。除此之外, 至今还没有发现关于该模型的动态分析方面的其它相关文献. 特别地, 当 $\mu(x)=\mu(\text{常数}), \beta_1(y)=\beta_1(\text{常数}), \beta_2(y)=\beta_2(\text{常数})$ 时, 具有两种故障状态的M/G/1 可修排队系统称为具有两种故障状态的M/M/1 可修排队系统.

本文在文献 [11] 和 [13] 研究结果的基础上, 研究具有两种故障状态的M/M/1 可修排队系统时间依赖解的渐近行为. 首先通过运用概率母函数, 证明 0 是一类具有两种故障状态的M/M/1 可修排队系统的主算子及其共轭算子几何重数为 1 的特征值. 其次结合文献 [11], 文献 [13] 的研究结果和文献 [14] 中的定理 14 , 并在一定的假设和约束条件下, 估计出该系

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统的时间依赖解的渐近性质: 即当时刻 t 趋向于无穷时, 该系统的时间依赖解强收敛于该系统的稳态解.

2 具有两种故障状态的 M/M/1 可修排队模型的转化

根据陈洋与朱翼隽等人[11], 当 $\mu(x)=\mu$(常数)$,\beta_1(y)=\beta_1$(常数)$,\beta_2(y)=\beta_2$(常数) 时, 具有两种故障状态的 M/M/1 可修排队系统可由以下偏微分方程组描述. 这里我们不妨设 $\alpha+\gamma+\lambda+\mu=\psi$,

$$\frac{dp_{1,0}(t)}{dt}+\lambda p_{1,0}(t)=\mu\int_0^\infty p_{1,1}(x,t)dx, \tag{2.1}$$

$$\begin{aligned}\frac{\partial p_{1,1}(x,t)}{\partial t}+\frac{\partial p_{1,1}(x,t)}{\partial x}=&-\psi p_{1,1}(x,t)+\beta_1\int_0^\infty p_{2,1}(x,y,t)dy\\&+\beta_2\int_0^\infty p_{3,1}(x,y,t)dy,\end{aligned} \tag{2.2}$$

$$\begin{aligned}\frac{\partial p_{1,n}(x,t)}{\partial t}+\frac{\partial p_{1,n}(x,t)}{\partial x}=&-\psi p_{1,n}(x,t)+\beta_1\int_0^\infty p_{2,n}(x,y,t)dy\\&+\beta_2\int_0^\infty p_{3,n}(x,y,t)dy+\lambda p_{1,n-1}(x,t),\ n\geqslant 2,\end{aligned} \tag{2.3}$$

$$\frac{\partial p_{2,1}(x,y,t)}{\partial t}+\frac{\partial p_{2,1}(x,y,t)}{\partial y}=-(\lambda+\beta_1)p_{2,1}(x,y,t), \tag{2.4}$$

$$\frac{\partial p_{2,n}(x,y,t)}{\partial t}+\frac{\partial p_{2,n}(x,y,t)}{\partial y}=-(\lambda+\beta_1)p_{2,n}(x,y,t)+\lambda p_{2,n-1}(x,y,t),\ n\geqslant 2, \tag{2.5}$$

$$\frac{\partial p_{3,1}(x,y,t)}{\partial t}+\frac{\partial p_{3,1}(x,y,t)}{\partial y}=-(\lambda+\beta_2)p_{3,1}(x,y,t), \tag{2.6}$$

$$\frac{\partial p_{3,n}(x,y,t)}{\partial t}+\frac{\partial p_{3,n}(x,y,t)}{\partial y}=-(\lambda+\beta_2)p_{3,n}(x,y,t)+\lambda p_{3,n-1}(x,y,t),\ n\geqslant 2, \tag{2.7}$$

$$p_{1,1}(0,t)=\lambda p_{1,0}(t)+\mu\int_0^\infty p_{1,2}(x,t)dx, \tag{2.8}$$

$$p_{1,n}(0,t)=\mu\int_0^\infty p_{1,n+1}(x,t)dx,\ n\geqslant 2, \tag{2.9}$$

$$p_{2,n}(x,0,t)=\gamma p_{1,n}(x,t),\ n\geqslant 1, \tag{2.10}$$

$$p_{3,n}(x,0,t)=\alpha p_{1,n}(x,t),\ n\geqslant 1, \tag{2.11}$$

$$p_{1,0}(0)=1,\ p_{1,j}(x,0)=0,\ p_{i,j}(x,y,0)=0\ i=2,3,\ j\geq 1. \tag{2.12}$$

其中 $(x,t)\in[0,\infty)\times[0,\infty)$, $(x,y,t)\in[0,\infty)\times[0,\infty)\times[0,\infty)$. $p_{1,0}(t)$ 表示在时刻 t 该系统中既没有顾客到达也没有服务的概率; $p_{1,n}(x,t)dx(n\geqslant 1)$ 表示在时刻 t 系统中有 n 个顾客, 服务台处于正常工作状态并且正在接受服务的顾客剩余服务时间在 $(x,x+dx]$ 内的概率; $p_{2,n}(x,y,t)dy(n\geqslant 1)$ 表示在时刻 t 系统中有 n 个顾客, 服务台处于异常故障状态并且正在被维修的服务台已经消耗掉的维修时间在 $(y,y+dy]$ 内的概率; $p_{3,n}(x,y,t)dy(n\geqslant 1)$ 表示在时刻 t 系统中有 n 个顾客, 服务台处于正常故障状态并且正在被维修的服务台已经消耗掉的维修时间在 $(y,y+dy]$ 内的概率. 顾客的输入是属于参数为 $\lambda(\lambda>0)$ 的 Poisson 过程, 服务台的寿命是服从参数为 α 的负指数分布. γ 是表示由于服务员操作不当等原因而引起

故障的失效率. μ 是表示服务台的服务率; β_1 是表示服务台处于异常故障状态的修复率; β_2 表示服务台处于正常故障状态的修复率.

本文仍沿用文献 [13] 中的符号. 记

$$X=\left\{(p_1,p_2,p_3)\left|\begin{array}{l} p_1\in X_1,p_2\in Y_2,p_3\in Y_3,\\ \|(p_1,p_2,p_3)\|=\|p_1\|_{X_1}+\|p_2\|_{Y_2}+\|p_3\|_{Y_3<\infty}\end{array}\right.\right\},$$

$$X_1=\left\{p_1\left|\begin{array}{l} p_1=(p_{1,0},p_{1,1},p_{1,2},p_{1,3},\cdots)\in R\times L^1[0,\infty)\times L^1[0,\infty)\times\cdots,\\ \|p_1\|=\|p_{1,0}\|+\sum\limits_{k=i}^{\infty}\|p_{1,i}\|_{L^1[0,\infty)}<\infty\end{array}\right.\right\},$$

$$Y_2=\left\{p_2\left|\begin{array}{l} p_2=(p_{2,1},p_{2,2},p_{2,3},p_{2,4},\cdots)\in L^1[0,\infty)\times L^1[0,\infty)\times L^1[0,\infty)\times\cdots,\\ \|p_2\|=\sum\limits_{k=i}^{\infty}\|p_{2,i}\|_{L^1[0,\infty)}<\infty\end{array}\right.\right\},$$

$$Y_3=\left\{p_3\left|\begin{array}{l} p_3=(p_{3,1},p_{3,2},p_{3,3},p_{3,4},\cdots)\in L^1[0,\infty)\times L^1[0,\infty)\times L^1[0,\infty)\times\cdots,\\ \|p_3\|=\sum\limits_{k=i}^{\infty}\|p_{3,i}\|_{L^1[0,\infty)}<\infty\end{array}\right.\right\},$$

容易验证, X 构成一个 Banach 空间.

为简便起见, 引入以下符号

$$\Psi_1=\begin{pmatrix} e^{-x} & 0 & 0 & 0 & \cdots\\ \lambda e^{-x} & 0 & \mu & 0 & \cdots\\ 0 & 0 & 0 & \mu & \cdots\\ 0 & 0 & 0 & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots\end{pmatrix},\ \Psi_2=\begin{pmatrix} 0 & \gamma & 0 & \cdots\\ 0 & 0 & \gamma & \cdots\\ 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots\end{pmatrix},\ \Psi_3=\begin{pmatrix} 0 & \alpha & 0 & \cdots\\ 0 & 0 & \alpha & \cdots\\ 0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots\end{pmatrix},$$

下面我们来定义算子及其定义域.

$$A(p_1,p_2,p_3)=\left(\begin{pmatrix} -\lambda & 0 & 0 & \cdots\\ 0 & -\dfrac{d}{dx} & 0 & \cdots\\ 0 & 0 & -\dfrac{d}{dx} & \cdots\\ \vdots & \vdots & \vdots & \vdots\end{pmatrix}\begin{pmatrix} p_{1,0}\\ p_{1,1}(x)\\ p_{1,2}(x)\\ p_{1,3}(x)\\ \vdots\end{pmatrix},\right.$$

$$\left.\begin{pmatrix} -\frac{\partial}{\partial y} & 0 & 0 & \cdots\\ 0 & -\frac{\partial}{\partial y} & 0 & \cdots\\ 0 & 0 & -\frac{\partial}{\partial y} & \cdots\\ \vdots & \vdots & \vdots & \vdots\end{pmatrix}\begin{pmatrix} p_{2,1}(x,y)\\ p_{2,2}(x,y)\\ p_{2,3}(x,y)\\ \vdots\end{pmatrix},\begin{pmatrix} -\frac{\partial}{\partial y} & 0 & 0 & \cdots\\ 0 & -\frac{\partial}{\partial y} & 0 & \cdots\\ 0 & 0 & -\frac{\partial}{\partial y} & \cdots\\ \vdots & \vdots & \vdots & \vdots\end{pmatrix}\begin{pmatrix} p_{3,1}(x,y)\\ p_{3,2}(x,y)\\ p_{3,3}(x,y)\\ \vdots\end{pmatrix}\right),$$

$$B(p_1,p_2,p_3)=\left(\begin{pmatrix} 0 & 0 & 0 & \cdots\\ 0 & -\psi & 0 & \cdots\\ 0 & \lambda & -\psi & \cdots\\ \vdots & \vdots & \vdots & \vdots\end{pmatrix}\begin{pmatrix} p_{1,0}\\ p_{1,1}(x)\\ p_{1,2}(x)\\ \vdots\end{pmatrix},\right.$$

$$\begin{pmatrix} -(\lambda+\beta_1) & 0 & 0 & \cdots\\ 0 & -(\lambda+\beta_1) & 0 & \cdots\\ 0 & \lambda & -(\lambda+\beta_1) & \cdots\\ \vdots & \vdots & \vdots & \vdots\end{pmatrix}\begin{pmatrix} p_{2,1}(x,y)\\ p_{2,2}(x,y)\\ p_{2,3}(x,y)\\ \vdots\end{pmatrix},$$

$$\begin{pmatrix} -(\lambda+\beta_2) & 0 & 0 & \cdots \\ \lambda & -(\lambda+\beta_2) & 0 & \cdots \\ 0 & \lambda & -(\lambda+\beta_2) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}\begin{pmatrix} p_{3,1}(x,y) \\ p_{3,2}(x,y) \\ p_{3,3}(x,y) \\ \vdots \end{pmatrix}\Bigg),$$

$$E(p_1,p_2,p_3) = \left(\begin{pmatrix} \mu\int_0^\infty p_{1,1}(x)dx \\ \beta_1\int_0^\infty p_{2,1}(x,y)dy + \beta_2\int_0^\infty p_{3,1}(x,y)dy \\ \beta_1\int_0^\infty p_{2,2}(x,y)dy + \beta_2\int_0^\infty p_{3,2}(x,y)dy \\ \beta_1\int_0^\infty p_{2,3}(x,y)dy + \beta_2\int_0^\infty p_{3,3}(x,y)dy \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \right),$$

$$D(A) = \left\{ (p_1,p_2,p_3)\in X \;\middle|\; \begin{array}{l} \frac{dp_{1,i}}{dx}\in L^1[0,\infty), i\geq 1, \\ \frac{\partial p_{2,i}}{\partial y}, \frac{\partial p_{3,i}}{\partial y}\in L^1([0,\infty)\times[0,\infty)), i\geq 1, \\ p_{1,k}(x), p_{2,i}(x,y), p_{3,i}(x,y) \text{ 是绝对连续并且满足} \\ p_1(0)=\int_0^\infty \Psi_1 p_1(x)dx, p_2(x,0)=\int_0^\infty \Psi_2 p_1(x)dx, \\ p_3(x,0)=\int_0^\infty \Psi_3 p_1(x)dx. \end{array} \right\},$$

$D(B)=D(E)=X$, 则上述偏微分方程组 (2.1) – (2.12) 可改写为 Banach 空间 X 中的抽象 Cauchy 问题:

$$\frac{d(p_1,p_2,p_3)(t)}{dt} = (A+B+E)(p_1,p_2,p_3)(t), \quad \forall t\in (0,\infty), \tag{2.13}$$

$$(p_1,p_2,p_3)(0) = (1,0,0,\cdots). \tag{2.14}$$

2024 年, 周学良和张庆红等人 [13] 证明了以下结果:

定理 2.1 $A+B+E$ 生成一个正压缩 C_0– 半群 $T(t)$. 系统 (2.13) – (2.14) 存在唯一的、正时间依赖解 $(p_1,p_2,p_3)(t)$ 并且满足

$$\|\big(p_{1,0}(t), p_{1,n}(\cdot,t), p_{2,n}(\cdot,\cdot,t), p_{3,n}(\cdot,\cdot,t)\big)\| = 1, \quad n\geq 1, \forall\, t\in[0,\infty).$$

根据文献 [14] 的思想和方法, 不难证明 X 的共轭空间 X^* 为

$$X^* = \left\{ (q_1^*,q_2^*,q_3^*) \;\middle|\; \begin{array}{l} q_1^*\in X_1^*, q_2^*\in Y_2^*, q_3^*\in Y_3^*, \\ |||(q_1^*,q_2^*,q_3^*)||| = \sup\{\|q_1^*\|_{X_1^*}, \|q_2^*\|_{Y_2^*}, \|q_3^*\|_{Y_3^*}\} \end{array} \right\},$$

$$X_1^* = \left\{ q_1^* \;\middle|\; \begin{array}{l} q_1^* = (q_{1,0}^*, q_{1,1}^*, q_{1,2}^*, \cdots)\in l^\infty\times L^\infty[0,\infty)\times L^\infty[0,\infty)\times\cdots, \\ \|q_1^*\| = \sup\{\|q_{1,0}^*\|, \sup_{i\geq 1}\|q_{1,i}^*\|_{L^\infty[0,\infty)}\} < \infty \end{array} \right\},$$

$$Y_2^* = \left\{ q_2^* \;\middle|\; \begin{array}{l} q_2^* = (q_{2,1}^*, q_{2,2}^*, q_{2,3}^*, \cdots)\in L^\infty[0,\infty)\times L^\infty[0,\infty)\times L^\infty[0,\infty)\times\cdots, \\ \|q_2^*\| = \sup_{i\geq 1}\|q_{2,i}^*\|_{L^\infty([0,\infty)\times[0,\infty))} < \infty \end{array} \right\},$$

$$Y_3^* = \left\{ q_3^* \;\middle|\; \begin{array}{l} q_3^* = (q_{3,1}^*, q_{3,2}^*, q_{3,3}^*, \cdots)\in L^\infty[0,\infty)\times L^\infty[0,\infty)\times L^\infty[0,\infty)\times\cdots, \\ \|q_3^*\| = \sup_{i\geq 1}\|q_{3,i}^*\|_{L^\infty([0,\infty)\times[0,\infty))} < \infty \end{array} \right\},$$

显然, X^* 构成一个 Banach 空间.

根据文献 [11] 不难证明以下引理结果.

引理 2.1 系统达到稳态平衡的充分必要条件是 $\rho=\frac{\lambda}{\mu}(1+\frac{\gamma}{\beta_1}+\frac{\alpha}{\beta_2})<1$.

证 我们知道要使系统达到稳态平衡状态, 当且仅当 $\lim_{t\to\infty} p_0(t) > 0$. 即 $1 - \lim_{t\to\infty} p_0(t) = 1 - p_0 < 1$, 又由文献 [11] 中的引理 2 可知

$$1 - p_0 = \begin{cases} \rho, & \rho < 1, \\ 1, \rho \geq 1. \end{cases}$$

因此, 当系统达到稳态平衡的充分必要条件是 $\rho < 1$.

根据文献 [15] 和 [16] 的思想和方法, 由共轭算子的定义不难证明以下引理结果:

引理 2.2 $A + B + E$ 的共轭算子 $(A + B + E)^*$ 为

$$(A + B + E)^*(q_1^*, q_2^*, q_3^*) = (\mathcal{F} + \mathcal{G} + \mathcal{H} + \mathcal{I} + \mathcal{J})\,(q_1^*, q_2^*, q_3^*), \forall (q_1^*, q_2^*, q_3^*) \in D((A + B + E)^*) \tag{2.15}$$

其中

$$\mathcal{F}(q_1^*, q_2^*, q_3^*) = \left(\begin{pmatrix} -\lambda & 0 & 0 & \cdots \\ 0 & \mathcal{D}_0 & 0 & \cdots \\ 0 & 0 & \mathcal{D}_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} q_{1,0}^* \\ q_{1,1}^*(x) \\ q_{1,2}^*(x) \\ q_{1,3}^*(x) \\ \vdots \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} \mathcal{D}_1 & 0 & 0 & \cdots \\ 0 & \mathcal{D}_1 & 0 & \cdots \\ 0 & 0 & \mathcal{D}_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} q_{2,1}^*(x,y) \\ q_{2,2}^*(x,y) \\ q_{2,3}^*(x,y) \\ q_{2,4}^*(x,y) \\ \vdots \end{pmatrix}, \begin{pmatrix} \mathcal{D}_2 & 0 & 0 & \cdots \\ 0 & \mathcal{D}_2 & 0 & \cdots \\ 0 & 0 & \mathcal{D}_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} q_{3,1}^*(x,y) \\ q_{3,2}^*(x,y) \\ q_{3,3}^*(x,y) \\ q_{3,4}^*(x,y) \\ \vdots \end{pmatrix} \right),$$

$$\mathcal{G}(q_1^*, q_2^*, q_3^*) = \left(\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \lambda & 0 & \cdots \\ 0 & 0 & 0 & \lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} q_{1,0}^* \\ q_{1,1}^*(x) \\ q_{1,2}^*(x) \\ q_{1,3}^*(x) \\ \vdots \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & \lambda & 0 & 0 & \cdots \\ 0 & 0 & \lambda & 0 & \cdots \\ 0 & 0 & 0 & \lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} q_{2,1}^*(x,y) \\ q_{2,2}^*(x,y) \\ q_{2,3}^*(x,y) \\ q_{2,4}^*(x,y) \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 & \lambda & 0 & 0 & \cdots \\ 0 & 0 & \lambda & 0 & \cdots \\ 0 & 0 & 0 & \lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} q_{3,1}^*(x,y) \\ q_{3,2}^*(x,y) \\ q_{3,3}^*(x,y) \\ q_{3,4}^*(x,y) \\ \vdots \end{pmatrix} \right),$$

$$\mathcal{H}(q_1^*, q_2^*, q_3^*) = \left(\begin{pmatrix} 0 & \lambda & 0 & 0 & \cdots \\ \mu & 0 & 0 & 0 & \cdots \\ 0 & \mu & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} q_{1,0}^* \\ q_{1,1}^*(0) \\ q_{1,2}^*(0) \\ q_{1,3}^*(0) \\ \vdots \end{pmatrix}, \right.$$

$$\left.\begin{pmatrix} 0 & \beta_1 & 0 & 0 & \cdots \\ 0 & 0 & \beta_1 & 0 & \cdots \\ 0 & 0 & 0 & \beta_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}\begin{pmatrix} q_{1,0}^* \\ q_{1,1}^*(x) \\ q_{1,2}^*(x) \\ q_{1,3}^*(x) \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 & \beta_2 & 0 & 0 & \cdots \\ 0 & 0 & \beta_2 & 0 & \cdots \\ 0 & 0 & 0 & \beta_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}\begin{pmatrix} q_{1,0}^* \\ q_{1,1}^*(x) \\ q_{1,2}^*(x) \\ q_{1,3}^*(x) \\ \vdots \end{pmatrix}\right),$$

$$\mathcal{I}(q_1^*, q_2^*, q_3^*) = \left(\begin{pmatrix} 0 & 0 & 0 & \cdots \\ \gamma & 0 & 0 & \cdots \\ 0 & \gamma & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}\begin{pmatrix} q_{2,1}^*(x,0) \\ q_{2,2}^*(x,0) \\ q_{2,3}^*(x,0) \\ q_{2,4}^*(x,0) \\ \vdots \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}\begin{pmatrix} q_{2,1}^*(x,0) \\ q_{2,2}^*(x,0) \\ q_{2,3}^*(x,0) \\ q_{2,4}^*(x,0) \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}\begin{pmatrix} q_{3,1}^*(x,0) \\ q_{3,2}^*(x,0) \\ q_{3,3}^*(x,0) \\ q_{3,4}^*(x,0) \\ \vdots \end{pmatrix}\right),$$

$$\mathcal{J}(q_1^*, q_2^*, q_3^*) = \left(\begin{pmatrix} 0 & 0 & 0 & \cdots \\ \alpha & 0 & 0 & \cdots \\ 0 & \alpha & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}\begin{pmatrix} q_{3,1}^*(x,0) \\ q_{3,2}^*(x,0) \\ q_{3,3}^*(x,0) \\ q_{3,4}^*(x,0) \\ \vdots \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}\begin{pmatrix} q_{2,1}^*(x,0) \\ q_{2,2}^*(x,0) \\ q_{2,3}^*(x,0) \\ q_{2,4}^*(x,0) \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}\begin{pmatrix} q_{3,1}^*(x,0) \\ q_{3,2}^*(x,0) \\ q_{3,3}^*(x,0) \\ q_{3,4}^*(x,0) \\ \vdots \end{pmatrix}\right),$$

这里

$$\mathcal{D}_0 = \frac{\mathrm{d}}{\mathrm{d}x} - \psi, \mathcal{D}_1 = \frac{\partial}{\partial y} - (\lambda + \beta_1), \mathcal{D}_2 = \frac{\partial}{\partial y} - (\lambda + \beta_2),$$

$$D((A+B+E)^*) = \left\{ (q_1^*, q_2^*, q_3^*) \in X^* \;\middle|\; \begin{array}{l} \frac{\mathrm{d}q_{1,n}(x)}{\mathrm{d}x}, \frac{\partial_{2,n}(x,y)}{\partial y} \text{ 和 } \frac{\partial_{3,n}(x,y)}{\partial y} \text{都存在, 并且} \\ q_{1,n}^*(\infty) = q_{2,n}^*(x,\infty) = q_{3,n}^*(x,\infty) = \mathcal{T}, n \geq 1 \end{array} \right\},$$

这里 $D((A+B+E)^*)$ 中的 $\mathcal{T}$ 是与 n 无关的常数.

3 系统 (1.13) − (1.14) 主算子的谱特征

引理 3.1 若 $\mu > \lambda$, $(\mu-\lambda)\beta_1\beta_2 - \lambda\gamma\beta_2 - \lambda\alpha\beta_1 > 0$, 那么 0 是系统主算子 $A+B+E$ 的几何重数为 1 的特征值.

证 考虑方程 $(A+B+E)(p_1,p_2,p_3)=0$, 这等价于下列微分方程组

$$\lambda p_{1,0} = \mu \int_0^\infty p_{1,1}(x)dx, \tag{3.1}$$

$$\frac{dp_{1,1}(x)}{dx} = -\psi p_{1,1}(x) + \beta_1 \int_0^\infty p_{2,1}(x,y)dy + \beta_2 \int_0^\infty p_{3,1}(x,y)dy, \tag{3.2}$$

$$\frac{dp_{1,n}(x)}{dx} = -\psi p_{1,n}(x) + \beta_1 \int_0^\infty p_{2,n}(x,y)dy + \beta_2 \int_0^\infty p_{3,n}(x,y)dy + \lambda p_{1,n-1}(x),\ n \geqslant 2, \tag{3.3}$$

$$\frac{\partial p_{2,1}(x,y)}{\partial y} = -(\lambda + \beta_1)p_{2,1}(x,y), \tag{3.4}$$

$$\frac{\partial p_{2,n}(x,y)}{\partial y} = -(\lambda + \beta_1)p_{2,n}(x,y) + \lambda p_{2,n-1}(x,y),\ n \geqslant 2, \tag{3.5}$$

$$\frac{\partial p_{3,1}(x,y)}{\partial y} = -(\lambda + \beta_2)p_{3,1}(x,y), \tag{3.6}$$

$$\frac{\partial p_{3,n}(x,y)}{\partial y} = -(\lambda + \beta_2)p_{3,n}(x,y) + \lambda p_{3,n-1}(x,y),\ n \geqslant 2, \tag{3.7}$$

$$p_{1,1}(0) = \lambda p_{1,0} + \mu \int_0^\infty p_{1,2}(x)dx, \tag{3.8}$$

$$p_{1,n}(0) = \mu \int_0^\infty p_{1,n+1}(x)dx,\ n \geqslant 2, \tag{3.9}$$

$$p_{2,n}(x,0) = \gamma p_{1,n}(x),\ n \geqslant 1, \tag{3.10}$$

$$p_{3,n}(x,0) = \alpha p_{1,n}(x),\ n \geqslant 1, \tag{3.11}$$

解 (3.4) – (3.7) 式得到

$$p_{2,1}(x,y) = a_{2,1}(x)e^{-(\lambda+\beta_1)y}, \tag{3.12}$$

$$\begin{aligned} p_{2,2}(x,y) &= a_{2,2}(x)e^{-(\lambda+\beta_1)y} + \lambda e^{-(\lambda+\beta_1)y}\int_0^y e^{(\lambda+\beta_1)\tau}p_{2,1}(x,\tau)d\tau \\ &= a_{2,2}(x)e^{-(\lambda+\beta_1)y} + \lambda e^{-(\lambda+\beta_1)y}\int_0^y e^{(\lambda+\beta_1)\tau}a_{2,1}(x)e^{-(\lambda+\beta_1)\tau}d\tau \\ &= \Big[a_{2,2}(x) + a_{2,1}(x)\lambda y\Big]e^{-(\lambda+\beta_1)y}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} p_{2,n}(x,y) &= a_{2,n}(x)e^{-(\lambda+\beta_1)y} + \lambda e^{-(\lambda+\beta_1)y}\int_0^y e^{(\lambda+\beta_1)\tau}p_{2,n-1}(x,\tau)d\tau \\ &= \Big[a_{2,n}(x) + a_{2,n-1}(x)\lambda y + a_{2,n-2}(x)\frac{(\lambda y)^2}{2!} + \cdots + a_{2,2}(x)\frac{(\lambda y)^{n-2}}{(n-2)!} \\ &\quad + a_{2,1}(x)\frac{(\lambda y)^{n-1}}{(n-1)!}\Big]e^{-(\lambda+\beta_1)y} = \sum_{i=1}^n a_{2,i}(x)\frac{(\lambda y)^{n-i}}{(n-i)!}e^{-(\lambda+\beta_1)y}, n \geq 1, \end{aligned} \tag{3.14}$$

$$p_{3,1}(x,y) = a_{3,1}(x)e^{-(\lambda+\beta_2)y}, \tag{3.15}$$

$$\begin{aligned} p_{3,2}(x,y) &= a_{3,2}(x)e^{-(\lambda+\beta_2)y} + \lambda e^{-(\lambda+\beta_2)y}\int_0^y e^{(\lambda+\beta_2)\tau}p_{3,1}(x,\tau)d\tau \\ &= a_{3,2}(x)e^{-(\lambda+\beta_2)y} + \lambda e^{-(\lambda+\beta_2)y}\int_0^y e^{(\lambda+\beta_2)\tau}a_{3,1}(x)e^{-(\lambda+\beta_2)\tau}d\tau \\ &= \Big[a_{3,2}(x) + a_{3,1(x)}\lambda y\Big]e^{-(\lambda+\beta_2)y}, \end{aligned} \tag{3.16}$$

$$p_{3,n}(x,y) = a_{3,n}(x)e^{-(\lambda+\beta_2)y} + \lambda e^{-(\lambda+\beta_2)y}\int_0^y e^{(\lambda+\beta_2)\tau}p_{3,n-1}(x,\tau)d\tau$$
$$= \bigg[a_{3,n}(x) + a_{3,n-1}(x)\lambda y + a_{3,n-2}(x)\frac{(\lambda y)^2}{2!} + \cdots + a_{3,2}(x)\frac{(\lambda y)^{n-2}}{(n-2)!}$$
$$+ a_{3,1}(x)\frac{(\lambda y)^{n-1}}{(n-1)!}\bigg]e^{-(\lambda+\beta_1)y} = \sum_{i=1}^{n} a_{3,i}(x)\frac{(\lambda y)^{n-i}}{(n-i)!}e^{-(\lambda+\beta_2)y}, n \geq 1. \tag{3.17}$$

解 (3.2) 和 (3.3) 式, 并将 (3.12) – (3.17) 式代入计算得 (令 $e^{-\psi x}\int_0^x e^{\psi\tau}f(\tau)d\tau = Ef(x)$)

$$p_{1,1}(x) = a_{1,1}e^{-\psi x} + e^{-\psi x}\int_0^x e^{\psi\tau}\bigg[\beta_1\int_0^\infty p_{2,1}(\tau,y)dy + \beta_2\int_0^\infty p_{3,1}(\tau,y)dy\bigg]d\tau$$
$$= a_{1,1}e^{-\psi x} + E\bigg[\beta_1\int_0^\infty p_{2,1}(\tau,y)dy + \beta_2\int_0^\infty p_{3,1}(\tau,y)y\bigg], \tag{3.18}$$
$$p_{1,2}(x) = a_{1,2}e^{-\psi x} + \lambda e^{-\psi x}\int_0^x e^{\psi\tau}p_{1,1}(\tau)d\tau + e^{-\psi x}\int_0^x e^{\psi\tau}\bigg[\beta_1\int_0^\infty p_{2,2}(\tau,y)dy$$
$$+ \beta_2\int_0^\infty p_{3,2}(\tau,y)dy\bigg]d\tau$$
$$= a_{1,2}e^{-\psi x} + a_{1,1}\lambda x e^{-\psi x} + \lambda E^2\bigg[\beta_1\int_0^\infty p_{2,1}(\tau,y)dy + \beta_2\int_0^\infty p_{3,1}(\tau,y)dy\bigg]$$
$$+ E\bigg[\beta_1\int_0^\infty p_{2,2}(\tau,y)dy + \beta_2\int_0^\infty p_{3,2}(\tau,y)dy\bigg], \tag{3.19}$$
$$p_{1,n}(x) = a_{1,n}e^{-\psi x} + \lambda e^{-\psi x}\int_0^x e^{\psi\tau}p_{1,n-1}(\tau)d\tau + e^{-\psi x}\int_0^x e^{\psi\tau}\bigg[\beta_1\int_0^\infty p_{2,n}(\tau,y)dy$$
$$+ \beta_2\int_0^\infty p_{3,n}(\tau,y)dy\bigg]d\tau$$
$$= e^{-\psi x}\sum_{i=1}^{n} a_{1,i}\frac{(\lambda x)^{n-i}}{(n-i)!} + \sum_{i=1}^{n}\lambda^{i-1}E^i\bigg[\beta_1\int_0^\infty p_{2,n-i+1}(\tau,y)dy$$
$$+ \beta_2\int_0^\infty p_{3,n-i+1}(\tau,y)dy\bigg],\ n \geqslant 2, \tag{3.20}$$

结合 (3.12) – (3.20) 与 (3.8) – (3.11) 式得到

$$a_{1,1} = p_{1,1}(0) = \lambda p_{1,0} + \mu\int_0^\infty p_{1,2}(x)dx$$
$$= \lambda p_{1,0} + \mu\int_0^\infty \bigg\{a_{1,2}e^{-\psi x} + a_{1,1}\lambda x e^{-\psi x} + \lambda E^2\bigg[\beta_1\int_0^\infty p_{2,1}(\tau,y)dy$$
$$+ \beta_2\int_0^\infty p_{3,1}(\tau,y)dy\bigg] + E\bigg[\beta_1\int_0^\infty p_{2,2}(\tau,y)dy + \beta_2\int_0^\infty p_{3,2}(\tau,y)dy\bigg]\bigg\}dx, \tag{3.21}$$
$$a_{1,n} = p_{1,n}(0) = \mu\int_0^\infty p_{1,n+1}(x)dx$$

$$
\begin{aligned}
&=\mu\int_0^\infty\left\{e^{-\psi x}\sum_{i=1}^{n+1}a_{1,i}\frac{(\lambda x)^{n-i+1}}{(n-i+1)!}+\sum_{i=1}^{n+1}\lambda^{i-1}E^i\left[\beta_1\int_0^\infty p_{2,n-i+2}(\tau,y)dy\right.\right.\\
&\quad\left.\left.+\beta_2\int_0^\infty p_{3,n-i+1}(\tau,y)dy\right]\right\}dx,\ n\geqslant 2, \qquad (3.22)
\end{aligned}
$$

$$
\begin{aligned}
a_{2,n}(x)=&p_{2,n}(x,0)=\gamma p_{1,n}(x)\\
=&\gamma\left\{e^{-\psi x}\sum_{i=1}^{n}a_{1,i}\frac{(\lambda x)^{n-i}}{(n-i)!}+\beta_1e^{-\psi x}\int_0^x e^{\psi\tau}\int_0^\infty\sum_{i=1}^{n}a_{2,i}(\tau)\frac{(\lambda y)^{n-i}}{(n-i)!}e^{-(\lambda+\beta_1)y}dyd\tau\right.\\
&\left.+\beta_2e^{-\psi x}\int_0^x e^{\psi\tau}\int_0^\infty\left[\sum_{i=1}^{n}a_{3,i}(\tau)\frac{(\lambda y)^{n-i}}{(n-i)!}e^{-(\lambda+\beta_2)y}\right]dyd\tau\right\}n\geqslant 1, \qquad (3.23)
\end{aligned}
$$

$$
\begin{aligned}
a_{3,n}(x)=&p_{3,n}(x,0)=\alpha p_{1,n}(x)\\
=&\alpha\left\{e^{-\psi x}\sum_{i=1}^{n}a_{1,i}\frac{(\lambda x)^{n-i}}{(n-i)!}+\beta_1e^{-\psi x}\int_0^x e^{\psi\tau}\int_0^\infty\sum_{i=1}^{n}a_{2,i}(\tau)\frac{(\lambda y)^{n-i}}{(n-i)!}e^{-(\lambda+\beta_1)y}dyd\tau\right.\\
&\left.+\beta_2e^{-\psi x}\int_0^x e^{\psi\tau}\int_0^\infty\left[\sum_{i=1}^{n}a_{3,i}(\tau)\frac{(\lambda y)^{n-i}}{(n-i)!}e^{-(\lambda+\beta_2)y}\right]dyd\tau\right\}n\geqslant 1, \qquad (3.24)
\end{aligned}
$$

由上述微分方程组容易看出直接求出每个 $p_{1,n}(x),p_{2,n}(x,y),p_{3,n}(x,y)$ 表达式, 并且证明 $p_{1,n}(x),p_{2,n}(x,y),p_{3,n}(x,y)\in D(A+B+E)$ 非常困难. 因此我们可运用文献 [15] 和 [16] 中的思想和方法, 首先引入下列概率母函数,

$$
P_1(x,z)=\sum_{n=1}^{\infty}p_{1,n}(x)z^n,P_2(x,y,z)=\sum_{n=1}^{\infty}p_{2,n}(x,y)z^n,P_3(x,y,z)=\sum_{n=1}^{\infty}p_{3,n}(x,y)z^n,|z|<1.
$$

定理2.1 保证 $P_1(x,z)$, $P_2(x,y,z)$ 和 $P_3(x,y,z)$ 都具有意义. 由幂级数的基本知识, Fubini 定理以及 (3.2) 和 (3.3) 式计算得

$$
\begin{aligned}
\frac{\partial P_1(x,z)}{\partial x}&=-\psi\sum_{n=1}^{\infty}p_{1,n}(x)z^n+\beta_1\sum_{n=1}^{\infty}\int_0^\infty p_{2,n}(x,y)z^ndy+\beta_2\sum_{n=1}^{\infty}\int_0^\infty p_{3,n}(x,y)z^ndy\\
&\quad+\lambda\sum_{n=2}^{\infty}p_{1,n-1}(x)z^n\\
&=-\psi P_1(x,z)+\beta_1\int_0^\infty P_2(x,y,z)dy+\beta_2\int_0^\infty P_3(x,y,z)dy+\lambda zP_1(x,z)\\
&=(\lambda z-\psi)P_1(x,z)+\beta_1\int_0^\infty P_2(x,y,z)dy+\beta_2\int_0^\infty P_3(x,y,z)dy, \qquad (3.25)
\end{aligned}
$$

由 (3.4) 和 (3.5) 式得到

$$
\begin{aligned}
\frac{\partial P_2(x,y,z)}{\partial y}&=-(\lambda+\beta_1)\sum_{n=1}^{\infty}p_{2,n}(x,y)z^n+\lambda\sum_{n=2}^{\infty}p_{2,n-1}(x,y)z^n\\
&=-(\lambda+\beta_1)P_2(x,y,z)+\lambda zP_2(x,y,z)=(\lambda z-\lambda-\beta_1)P_2(x,y,z),\\
&\Longrightarrow\\
P_2(x,y,z)&=P_2(x,0,z)e^{(\lambda z-\lambda-\beta_1)y}, \qquad (3.26)
\end{aligned}
$$

由 (3.6) 和 (3.7) 式得到

$$\begin{aligned}\frac{\partial P_3(x,y,z)}{\partial y} &= -(\lambda+\beta_2)\sum_{n=1}^{\infty}p_{3,n}(x,y)z^n+\lambda\sum_{n=2}^{\infty}p_{3,n-1}(x,y)z^n\\ &= -(\lambda+\beta_1)P_3(x,y,z)+\lambda zP_3(x,y,z)=(\lambda z-\lambda-\beta_2)P_3(x,y,z),\\ &\Longrightarrow\\ P_3(x,y,z) &= P_3(x,0,z)e^{(\lambda z-\lambda-\beta_2)y}, \end{aligned} \tag{3.27}$$

由 (3.8) – (3.11) 和 (3.1) 式, 计算得到

$$\begin{aligned}P_1(0,z)=\sum_{n=1}^{\infty}P_1(0)z^n &= \lambda zp_{1,0}+\mu\int_0^{\infty}\sum_{n=1}^{\infty}p_{1,n+1}(x)z^ndx\\ &=\lambda zp_{1,0}+\frac{\mu}{z}\int_0^{\infty}\left(\sum_{n=1}^{\infty}p_{1,n}(x)z^n-p_{1,1}(x)z\right)dx\\ &=\lambda zp_{1,0}+\frac{\mu}{z}\int_0^{\infty}P_1(x,z)dx-\mu\int_0^{\infty}p_{1,1}(x)dx\\ &=\frac{\mu}{z}\int_0^{\infty}P_1(x,z)dx+\lambda(z-1)p_{1,0},\end{aligned} \tag{3.28}$$

$$P_2(x,0,z)=\sum_{n=1}^{\infty}P_2(x,0)z^n=\gamma\sum_{n=1}^{\infty}p_{1,n}(x)z^n=\gamma P_1(x,z), \tag{3.29}$$

$$P_3(x,0,z)=\sum_{n=1}^{\infty}P_3(x,0)z^n=\alpha\sum_{n=1}^{\infty}p_{1,n}(x)z^n=\alpha P_1(x,z), \tag{3.30}$$

将 (3.29) 和 (3.30) 分别代入 (3.26) 和 (3.27) 得到

$$P_2(x,y,z)=P_2(x,0,z)e^{(\lambda z-\lambda-\beta_1)y}=\gamma P_1(x,z)e^{(\lambda z-\lambda-\beta_1)y}, \tag{3.31}$$

$$P_3(x,y,z)=P_3(x,0,z)e^{(\lambda z-\lambda-\beta_2)y}=\alpha P_1(x,z)e^{(\lambda z-\lambda-\beta_2)y}, \tag{3.32}$$

将 (3.31) 和 (3.32) 代入 (3.25) 式, 计算得到

$$\begin{aligned}\frac{\partial P_1(x,z)}{\partial x} &= (\lambda z-\psi)P_1(x,z)+\beta_1\int_0^{\infty}P_2(x,y,z)dy+\beta_2\int_0^{\infty}P_3(x,y,z)dy\\ &= (\lambda z-\psi)P_1(x,z)+\gamma\beta_1\int_0^{\infty}P_1(x,z)e^{(\lambda z-\lambda-\beta_1)y}dy\\ &\quad+\alpha\beta_2\int_0^{\infty}P_1(x,z)e^{(\lambda z-\lambda-\beta_2)y}dy\\ &= (\lambda z-\psi)P_1(x,z)-\frac{\beta_1\gamma}{\lambda z-\lambda-\beta_1}P_1(x,z)-\frac{\beta_2\alpha}{\lambda z-\lambda-\beta_2}P_1(x,z)\\ &= \left[(\lambda z-\psi)-\frac{\beta_1\gamma}{\lambda z-\lambda-\beta_1}-\frac{\beta_2\alpha}{\lambda z-\lambda-\beta_2}\right]P_1(x,z),\\ &\Longrightarrow\\ P_1(x,z) &= P_1(0,z)e^{[(\lambda z-\psi)-\frac{\beta_1\gamma}{\lambda z-\lambda-\beta_1}-\frac{\beta_2\alpha}{\lambda z-\lambda-\beta_2}]x},\end{aligned} \tag{3.33}$$

不妨设函数 $\omega(z)=(\lambda z-\psi)-\frac{\beta_1\gamma}{\lambda z-\lambda-\beta_1}-\frac{\beta_2\alpha}{\lambda z-\lambda-\beta_2}$, 将 (3.33) 式代入 (3.28) 式, 计算得到

$$\begin{aligned}
P_1(0,z)=&\frac{\mu}{z}\int_0^\infty P_1(x,z)dx+\lambda(z-1)p_{1,0}\\
=&\frac{\mu}{z}\int_0^\infty P_1(0,z)e^{[(\lambda z-\psi)-\frac{\beta_1\gamma}{\lambda z-\lambda-\beta_1}-\frac{\beta_2\alpha}{\lambda z-\lambda-\beta_2}]x}dx+\lambda(z-1)p_{1,0}\\
=&\frac{\mu}{z}\int_0^\infty P_1(0,z)e^{\omega(z)x}dx+\lambda(z-1)p_{1,0}\\
=&\frac{-\mu}{z\omega(z)}P_1(0,z)+\lambda(z-1)p_{1,0},\\
\Longrightarrow&\\
\left[1+\frac{\mu}{z\omega(z)}\right]P_1(0,z)=&\lambda(z-1)p_{1,0},\\
\Longrightarrow&\\
P_1(0,z)=&\frac{\lambda(z-1)}{1+\frac{\mu}{z\omega(z)}}p_{1,0}, && (3.34)\\
\frac{d\omega(z)}{dz}=&\omega'(z)=\lambda+\frac{\lambda\gamma\beta_1}{(\lambda z-\lambda-\beta_1)^2}+\frac{\lambda\alpha\beta_2}{(\lambda z-\lambda-\beta_2)^2}, && (3.35)\\
\omega'(1)=&\lambda+\frac{\lambda\gamma\beta_1}{(-\beta_1)^2}+\frac{\lambda\alpha\beta_2}{(-\beta_2)^2}=\frac{\lambda(\beta_1\beta_2+\gamma\beta_2+\alpha\beta_1)}{\beta_1\beta_2}, && (3.36)\\
\omega(1)=&(\lambda-\psi)-\frac{\beta_1\gamma}{-\beta_1}-\frac{\beta_2\alpha}{-\beta_2}=(\lambda-\psi)+\gamma+\alpha=-\mu, && (3.37)
\end{aligned}$$

由 (3.34) 式和罗必达法则计算得

$$\begin{aligned}
\lim_{z\to1}P_1(0,z)=&\lim_{z\to1}\frac{\lambda(z-1)}{1+\frac{\mu}{z\omega(z)}}p_{1,0}=\lim_{z\to1}\frac{\lambda}{\frac{-\mu[\omega(z)+z\omega'(z)]}{(z\omega(z))^2}}p_{1,0}=\lim_{z\to1}\frac{-\lambda(z\omega(z))^2}{\mu[\omega(z)+z\omega'(z)]}p_{1,0}\\
=&\frac{\lambda\mu\beta_1\beta_2}{(\mu-\lambda)\beta_1\beta_2-\lambda\gamma\beta_2-\lambda\alpha\beta_1}p_{1,0}<\infty, && (3.38)
\end{aligned}$$

联立 (3.33) 和 (3.38) 式得到

$$\begin{aligned}
\sum_{n=1}^\infty p_{1,n}(x)=&\lim_{z\to1}P_1(x,z)=\lim_{z\to1}P_1(0,z)e^{[(\lambda z-\psi)-\frac{\beta_1\gamma}{\lambda z-\lambda-\beta_1}-\frac{\beta_2\alpha}{\lambda z-\lambda-\beta_2}]x},\\
=&\frac{(\lambda\mu\beta_1\beta_2)p_{1,0}}{(\mu-\lambda)\beta_1\beta_2-\lambda\gamma\beta_2-\lambda\alpha\beta_1}e^{-\mu x}, && (3.39)\\
\Longrightarrow&\\
\sum_{n=1}^\infty\int_0^\infty p_{1,n}(x)dx=&\frac{1}{\mu}\times\frac{(\lambda\mu\beta_1\beta_2)p_{1,0}}{(\mu-\lambda)\beta_1\beta_2-\lambda\gamma\beta_2-\lambda\alpha\beta_1}\\
=&\frac{(\lambda\beta_1\beta_2)p_{1,0}}{(\mu-\lambda)\beta_1\beta_2-\lambda\gamma\beta_2-\lambda\alpha\beta_1}<\infty, && (3.40)
\end{aligned}$$

联合 (3.26), (3.27) 和 (3.39) 式计算得

$$\begin{aligned}\sum_{n=1}^{\infty} p_{2,n}(x,y) &= \lim_{z\to 1} P_2(x,y,z) = \lim_{z\to 1} P_2(x,0,z)e^{(\lambda z-\lambda-\beta_1)y} = \lim_{z\to 1} \gamma P_1(x,z)e^{(\lambda z-\lambda-\beta_1)y} \\ &= \frac{(\gamma\lambda\mu\beta_1\beta_2)p_{1,0}}{(\mu-\lambda)\beta_1\beta_2-\lambda\gamma\beta_2-\lambda\alpha\beta_1}e^{-\beta_1 y}, \end{aligned} \tag{3.41}$$

$$\Longrightarrow$$

$$\begin{aligned}\sum_{n=1}^{\infty}\int_0^{\infty} p_{2,n}(x,y)dy &= \frac{1}{\beta_1}\times\frac{(\gamma\lambda\mu\beta_1\beta_2)p_{1,0}}{(\mu-\lambda)\beta_1\beta_2-\lambda\gamma\beta_2-\lambda\alpha\beta_1} \\ &= \frac{(\gamma\lambda\mu\beta_2)p_{1,0}}{(\mu-\lambda)\beta_1\beta_2-\lambda\gamma\beta_2-\lambda\alpha\beta_1} < \infty, \end{aligned} \tag{3.42}$$

$$\begin{aligned}\sum_{n=1}^{\infty} p_{3,n}(x,y) &= \lim_{z\to 1} P_3(x,y,z) = \lim_{z\to 1} P_3(x,0,z)e^{(\lambda z-\lambda-\beta_2)y} = \lim_{z\to 1} \alpha P_1(x,z)e^{(\lambda z-\lambda-\beta_2)y} \\ &= \frac{(\alpha\lambda\mu\beta_1\beta_2)p_{1,0}}{(\mu-\lambda)\beta_1\beta_2-\lambda\gamma\beta_2-\lambda\alpha\beta_1}e^{-\beta_2 y}, \end{aligned} \tag{3.43}$$

$$\Longrightarrow$$

$$\begin{aligned}\sum_{n=1}^{\infty}\int_0^{\infty} p_{3,n}(x,y)dy &= \frac{1}{\beta_2}\times\frac{(\alpha\lambda\mu\beta_1\beta_2)p_{1,0}}{(\mu-\lambda)\beta_1\beta_2-\lambda\gamma\beta_2-\lambda\alpha\beta_1} \\ &= \frac{(\alpha\lambda\mu\beta_1)p_{1,0}}{(\mu-\lambda)\beta_1\beta_2-\lambda\gamma\beta_2-\lambda\alpha\beta_1} < \infty, \end{aligned} \tag{3.44}$$

由 (3.40), (3.42) 和 (3.44) 式估计出

$$\|(p_1,p_2,p_3)\| = \|p_1\|+\|p_2\|+\|p_3\| = \frac{\lambda(\beta_1\beta_2+\gamma\mu\beta_2+\alpha\mu\beta_1)p_{1,0}}{(\mu-\lambda)\beta_1\beta_2-\lambda\gamma\beta_2-\lambda\alpha\beta_1} < \infty. \tag{3.45}$$

(3.45) 式说明 $(p_1,p_2,p_3)\in X$, 即 0 是 $A+B+E$ 的特征值. 此外, (3.1), (3.12) − (3.24) 容易看出对应于主算子特征值 0 的特征向量子空间的维数是 1, 即特征值 0 的几何重数为 1.

注 在一个排队系统中, 因为 μ 表示服务率, λ 表示顾客到达率, 所以可推出 $\mu>\lambda$. 否则, 随着时间的延长, 排队系统的队长将达到无限长.

根据引理 2.1 $\rho=\frac{\lambda}{\mu}(1+\frac{\gamma}{\beta_1}+\frac{\alpha}{\beta_2})<1$ 可推得

$$\begin{aligned}&\lambda(1+\frac{\gamma}{\beta_1}+\frac{\alpha}{\beta_2})<\mu \\ &\Longrightarrow \lambda\beta_1\beta_2+\lambda\gamma\beta_2+\lambda\alpha\beta_1<\mu\beta_1\beta_2 \\ &\Longrightarrow \mu\beta_1\beta_2-\lambda\beta_1\beta_2-\lambda\gamma\beta_2-\lambda\alpha\beta_1>0 \\ &\Longrightarrow (\mu-\lambda)\beta_1\beta_2-\lambda\gamma\beta_2-\lambda\alpha\beta_1>0\end{aligned}$$

上式说明引理的条件具有实际的物理背景, 符合实际应用的合理性. 引理 3.1 证毕.

引理 3.2 如果 $\mu>\lambda$, $(\mu-\lambda)\beta_1\beta_2-\lambda\gamma\beta_2-\lambda\alpha\beta_1>0$, 那么 0 是共轭算子 $(A+B+E)^*$ 的几何重数为 1 的特征值.

证 讨论方程 $(A+B+E)^*(q_1^*,q_2^*,q_3^*)=0$, 即

$$-\lambda q_{1,0}^*+\lambda q_{1,1}^*(0)=0, \tag{3.46}$$

$$\frac{dq_{1,1}^*(x)}{dx}=\psi q_{1,1}^*(x)-\mu q_{1,0}^*-\gamma q_{2,1}^*(x,0)-\alpha q_{3,1}^*(x,0)-\lambda q_{1,2}^*(x), \tag{3.47}$$

$$\frac{dq_{1,n}^*(x)}{dx}=\psi q_{1,n}^*(x)-\mu q_{1,n-1}^*(0)-\gamma q_{2,n}^*(x,0)-\alpha q_{3,n}^*(x,0)-\lambda q_{1,n+1}^*(x), n\geqslant 2, \tag{3.48}$$

$$\frac{dq_{2,n}^*(x,y)}{dy}=(\widetilde{\gamma}+\lambda+\beta_1)q_{2,n}^*(x,y)-\beta_1 q_{1,n}^*(x)-\lambda q_{2,n+1}^*(x,y),\ n\geqslant 1, \tag{3.49}$$

$$\frac{dq_{3,n}^*(x,y)}{dy}=(\widetilde{\gamma}+\lambda+\beta_2)q_{3,n}^*(x,y)-\beta_2 q_{1,n}^*(x)-\lambda q_{3,n+1}^*(x,y),\ n\geqslant 1, \tag{3.50}$$

$$q_{1,n}^*(\infty)=q_{2,n}^*(x,\infty)=q_{3,n}^*(x,\infty)=\mathcal{T},\ n\geqslant 1. \tag{3.51}$$

容易验证

$$(q_1^*,q_2^*,q_3^*)=\left(\begin{pmatrix}\mathcal{T}\\ \mathcal{T}\\ \mathcal{T}\\ \vdots\end{pmatrix},\begin{pmatrix}\mathcal{T}\\ \mathcal{T}\\ \mathcal{T}\\ \vdots\end{pmatrix},\begin{pmatrix}\mathcal{T}\\ \mathcal{T}\\ \mathcal{T}\\ \vdots\end{pmatrix}\right)\in D((A+B+E)^*)$$

是方程组 (3.46)−(3.51) 的一个非零解 (注意到 $\mathcal{T}\neq 0$), 即 0 是该系统共轭算子 $(A+B+E)^*$ 的特征值. 此外, (3.46) − (3.50) 等价于

$$q_{1,0}^*=q_{1,1}^*(0), \tag{3.52}$$

$$q_{1,2}^*(x)=\frac{1}{\lambda}\left[-\frac{dq_{1,1}^*(x)}{dx}+\psi q_{1,1}^*(x)-\mu q_{1,0}^*-\gamma q_{2,1}^*(x,0)-\alpha q_{3,1}^*(x,0)\right], \tag{3.53}$$

$$\begin{aligned}q_{1,n+1}^*(x)=&\frac{1}{\lambda}\left[-\frac{dq_{1,n}^*(x)}{dx}+\psi q_{1,n}^*(x)-\mu q_{1,n-1}^*(0)-\gamma q_{2,n}^*(x,0)\right.\\&\left.-\alpha q_{3,n}^*(x,0)\right], n\geqslant 2,\end{aligned} \tag{3.54}$$

$$q_{2,n+1}^*(x,y)=\frac{1}{\lambda}\left[-\frac{dq_{2,n}^*(x,y)}{dy}+(\widetilde{\gamma}+\lambda+\beta_1)q_{2,n}^*(x,y)-\beta_1 q_{1,n}^*(x)\right],\ n\geqslant 1, \tag{3.55}$$

$$q_{3,n+1}^*(x,y)=\frac{1}{\lambda}\left[-\frac{dq_{3,n}^*(x,y)}{dy}+(\widetilde{\gamma}+\lambda+\beta_2)q_{3,n}^*(x,y)-\beta_2 q_{1,n}^*(x)\right],\ n\geqslant 1, \tag{3.56}$$

由 (3.52) − (3.56) 可知对应于特征值 0 的特征向量生成一维的线性子空间, 所以特征值 0 的几何重数为 1.

4 系统 (1.13) − (1.14) 时间依赖解的渐进性质

由文献 [14] 中的定理 14 可以看出, 系统 (2.13) − (2.14) 的时间依赖解的渐进性质是由主算子 $(A+B+E)$ 在虚轴上的谱分布决定. 如果我们能得到在虚轴上除了 0 点外其它所有点都属于该系统主算子 $(A+B+E)$ 的豫解集, 那么由此结果并结合定理 2.1, 引理 2.2, 引理

3.1, 引理 3.2 和文献 [14] 中的定理 14 得到系统 (2.13) – (2.14) 的时间依赖解的渐近行为: 即系统 (2.13) – (2.14) 的时间依赖解强收敛于该系统的稳态解.

定理 4.1 若 $\mu > \lambda$, $(\mu - \lambda)\beta_1\beta_2 - \lambda\gamma\beta_2 - \lambda\alpha\beta_1 > 0$, 则当时刻 t 趋向于无穷时, 系统 (2.13)-(2.14) 的时间依赖解强收敛于其稳态解, 即

$$\lim_{t\to\infty}\|(p_1, p_2, p_3)(\cdot, \cdot, t) - \mathcal{K}(p_1, p_2, p_3)(\cdot)\| = 0.$$

其中 (p_1, p_2, p_3) 是在引理 3.1 中的特征向量, 并且 $\mathcal{K}$ 是由引理 3.2 中的特征向量和初值 $(p_1, p_2, p_3)(0)$ 确定.

5 结论

本文运用文献 [15] 和 [16] 中的研究思想和方法, 并在文献 [11] 和 [13] 的研究基础上, 当 $\mu(x) = \mu$(常数), $\beta_1(y) = \beta_1$(常数), $\beta_2(y) = \beta_2$(常数) 时, 对具有两种故障状态的 M/M/1 可修排队系统时间依赖解的渐近性质进行了研究. 首先是通过运用概率母函数, 证明了 0 是该系统的主算子及其共轭算子几何重数为 1 的特征值. 其次是在一定的假设和约束条件下, 并结合定理 2.1, 引理 2.2, 引理 3.1, 引理 3.2 和文献 [14] 中的定理 14 可推出该系统 (2.13) – (2.14) 的时间依赖解的渐近性质. 根据文献 [14] 定理 14 可以看出, 系统 (2.13) – (2.14) 的时间依赖解的渐进性质是由主算子 $(A + B + E)$ 在虚轴上的谱分布决定. 因此, 我们非常有必要进一步研究在虚轴上除了 0 点外其它所有点是否都属于该模型主算子的豫解集. 根据查阅相关文献所知, 至今还没有发现这方面的研究结果. 接下来可以利用文献 [7], [9], [17], [18] 中的方法和理论, 运用边界扰动思想把原来的系统进行简化, 然后讨论相应的边界算子得到所需要的主算子的豫解集. 从而再结合相关结果得到该系统 (2.13) – (2.14) 的时间依赖解强收敛于其稳态解, 这也是我们下一步即将开展的工作.

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AN EIGENVALUE OF THE REPAIRABLE M/M/1 QUEUEING SYSTEM WITH TWO KINDS OF BREAKDOWN STATES AND ITS APPLICATION

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Abstract: The asymptotic property of the time-dependent solution corresponding to a repairable M/M/1 queueing system with two kinds of breakdown states has been studied. We prove that 0 is an eigenvalue of the main operator and its conjugate operator with geometric multiplicity one corresponding to the queueing system by using the probability generating function. Based on certain constraints, the time-dependent solution of the system strongly converges to the steady-state solution of the system is obtained. Some conclusions of the dynamic analysis of the queuing system are extended.

Keywords: the repairable queueing system with two kinds of breakdown states; adjoint operator; geometric multiplicity; eigenvalue

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