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涉及双周期 Fibonacci 数列和 Lucas 数列的 Dedekind 和

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摘 要: 本文研究了涉及双周期 Fibonacci $\{f_n\}$ 数列和双周期 Lucas $\{l_n\}$ 数列的 Dedekind 和的估计问题. 利用 Dedekind 和 $S(h,k)$ 的解析性质以及双周期 Fibonacci 数列和双周期 Lucas 数列的递推关系, 获得了 $\sum\limits_{n=1}^{m}\left(S\left(f_{2n},f_{2n+1}\right)+S\left(f_{2n+1},f_{2n+2}\right)\right)$ 与 $\sum\limits_{n=1}^{m}\left(S\left(l_{2n},l_{2n+1}\right)+S\left(l_{2n+1},l_{2n+2}\right)\right)$ 的估计式. 本文将涉及 Dedekind 和的线性递推数列的研究推广到了非线性.

关键词: 双周期 Fibonacci 数列; 双周期 Lucas 数列; Dedekind 和
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1 引言

Dedekind 和是 Dedekind 在研究 η 函数时引入的一种和式: 设 $k>0$, h 为任意整数, 经典的 Dedekind 和[1] $S\left(h,k\right)$ 被定义为:

$$S\left(h,k\right)=\sum_{a=1}^{k}\left(\left(\frac{a}{k}\right)\right)\left(\left(\frac{ah}{k}\right)\right),$$

$((x))$ 是分段函数:

$$((x))=\begin{cases}x-[x]-\frac{1}{2}, & x\notin Z,\\ 0, & x\in Z,\end{cases}$$

$[x]$ 是不超过 x 的最大整数, Z 是整数集. 许多学者对其进行了不同角度的研究, 见 [2–9]. 有关于 Dedekind 和最著名也是最重要的性质之一是互反公式, 即当 $h,k>0$ 且 $(h,k)=1$ 时有:

$$S\left(h,k\right)+S\left(k,h\right)=\frac{h^2+k^2+1}{12hk}-\frac{1}{4}.$$

当 $n\geq 2$ 时, 一般线性递推数列 Fibonacci$\{F_n\}$ 数列和 Lucas $\{L_n\}$ 数列, 由以下关系式给出:

$$F_0=0,\quad F_1=1,\quad F_n=F_{n-1}+F_{n-2},\quad n\geq 2,$$

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$$L_0=2,\quad L_1=1,\quad L_n=L_{n-1}+L_{n-2},\quad n\geq 2.$$

在 [10] 中, 学者推广线性递推数列为非线性递推数列, 即双周期 Fibonacci $\{f_n\}$ 数列, 其定义如下:

$$f_0=0,\quad f_1=1,\quad f_n=\begin{cases} af_{n-1}+f_{n-2}, & n\equiv 0 \pmod 2,\\ bf_{n-1}+f_{n-2}, & n\equiv 1 \pmod 2,\end{cases}\quad n\geq 2, \tag{1.1}$$

其中 a, b 为任意非负实数且 $a\geq 1$, $b\geq 1$. 显然当 $a=b=1$ 时, 双周期 Fibonacci$\{f_n\}$ 数列变为 Fibonacci $\{F_n\}$ 数列. 当 $a=b=k$ 时, 双周期 Fibonacci $\{f_n\}$ 数列变为 k-Fibonacci $\{Q_n\}$ 数列[11].

同样地, 在 [12] 中, 学者介绍了双周期 Lucas $\{l_n\}$ 数列, 定义如下:

$$l_0=2,\quad l_1=a,\quad l_n=\begin{cases} bl_{n-1}+l_{n-2}, & n\equiv 0 \pmod 2,\\ al_{n-1}+l_{n-2}, & n\equiv 1 \pmod 2,\end{cases}\quad n\geq 2, \tag{1.2}$$

其中 a, b 为任意非负实数且 $a\geq 1$, $b\geq 1$. 且当 $a=b=1$ 时, 双周期 Lucas$\{l_n\}$ 数列变为 Lucas $\{L_n\}$ 数列. 当 $a=b=k$ 时, 双周期 Lucas $\{l_n\}$ 数列变为 k-Lucas $\{P_n\}$ 数列[13].

我们给出 $\{f_n\}$ 和 $\{l_n\}$ 的负下标表达式如下:

$$f_{-n}=(-1)^{n+1}f_n,\quad l_{-n}=(-1)^n l_n. \tag{1.3}$$

另外定义分段函数 $\zeta(n)$ 如下:

$$\zeta(n)=\begin{cases} 0, & n\equiv 0 \pmod 2,\\ 1, & n\equiv 1 \pmod 2,\end{cases}\quad n\geq 2. \tag{1.4}$$

在 [10] 和 [12] 中, 学者们分别得到 $\{f_n\}$ 和 $\{l_n\}$ 的 Binet 公式如下:

$$f_n=\frac{a^{\zeta(n+1)}}{(ab)^{\left[\frac{n}{2}\right]}}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right), \tag{1.5}$$

$$l_n=\frac{a^{\zeta(n)}}{(ab)^{\left[\frac{n+1}{2}\right]}}\left(\alpha^n+\beta^n\right), \tag{1.6}$$

其中 α 和 β 是方程 $\omega^2-ab\omega-ab=0$ 的根, 即 $\alpha=\frac{ab+\sqrt{a^2b^2+4ab}}{2}$ 且 $\beta=\frac{ab-\sqrt{a^2b^2+4ab}}{2}$. 我们得到以下关于 α 和 β 的算术性质:

$$\alpha+\beta=ab,\quad \alpha-\beta=\sqrt{a^2b^2+4ab},\quad \alpha\beta=-ab.$$

显然, 当 $a\geq 1$ 及 $b\geq 1$ 时, $1<\alpha$ 且 $-1<\beta<0$. 更多有关双周期 Fibonacci 数列和双周期 Lucas 数列的讨论, 见 [14–20].

另外根据 Dedekind 和的互反公式以及 $\{f_n\}$ 与 $\{l_n\}$ 的递推关系, 我们可以得到如下等式:

$$S(f_0,f_1)=S(0,1)=0,\quad S(f_1,f_2)=S(1,a)=\frac{(a-1)(a-2)}{12a}.$$

当 a 是奇数时:

$$S\left(l_0, l_1\right)=S(2, a)=\frac{(a-1)(a-5)}{24 a},$$

$$S\left(l_1, l_2\right)=S(a, a b+2)=\frac{b\left(2 a b+3-a^2\right)}{24(a b+2)}.$$

近年来, 许多学者都研究了涉及二阶线性递推数列的 Dedekind 和, 比如: 1999 年, 张文鹏和易媛[21] 讨论了估计式 $\sum\limits_{n=1}^{m} S\left(F_n, F_{n+1}\right)$, 即当 m 是正整数时有:

$$\sum_{n=1}^{m} S\left(F_n, F_{n+1}\right)=-\frac{(\sqrt{5}-1)^2}{48} m+C(m)+\mathcal{O}\left(\frac{1}{\alpha^{2 m}}\right),$$

其中, $C(m)$ 是仅仅依赖于 m 的常数, “$\mathcal{O}$” 表示 Landau 符号 (对任意 $x \geq a$, $g(x)>0$, 若 $f(x) / g(x)$ 有界, 则我们记作 $f(x)=\mathcal{O}(g(x))$.

另外, 有学者[22] 得到了涉及 k-Fibonacci $\{Q_n\}$ 数列和 k-Lucas $\{P_n\}$ 数列的 Dedekind 和的估计式. 更多涉及线性递推数列 Dedekind 和的研究见 [23, 24].

本文研究了一类二阶非线性递推数列的 Dedekind 和, 即讨论了涉及双周期 Fibonacci $\{f_n\}$ 数列和双周期 Lucas $\{l_n\}$ 数列的 Dedekind 和的估计式. 本文是对前人研究成果的推广, 对 Dedekind 和与递推数列的研究有很大的帮助. 本文的主要结果如下:

定理 1.1 记 $\{f_n\}$ 是双周期 Fibonacci 数列, $\{l_n\}$ 是双周期 Lucas 数列, 令 m 是正整数, 则有:

$$\begin{aligned}\sum_{n=1}^{m}\left[S\left(f_{2n}, f_{2n+1}\right)+S\left(f_{2n+1}, f_{2n+2}\right)\right]= & \frac{(a \alpha-3 \alpha+a+b) m}{12 \alpha} \\ & +\frac{1}{12} \sum_{n=1}^{\infty}\left(\frac{\beta^{2n+2}}{(a b)^{n+1} f_{2n+2}}+\frac{a \beta^{2n+1}}{(a b)^{n+1} f_{2n+1}}+\frac{a b+4}{l_{4n+3}+a}\right)+\mathcal{O}\left(\frac{\alpha^{2m}}{\beta^{2m}}\right),\end{aligned} \tag{1.7}$$

且当 a 是奇数时,

$$\begin{aligned}\sum_{n=1}^{m}\left[S\left(l_{2n}, l_{2n+1}\right)+S\left(l_{2n+1}, l_{2n+2}\right)\right]= & \frac{(b \alpha-3 \alpha+a+b) m}{12 \alpha} \\ & +\frac{1}{12} \sum_{n=1}^{\infty}\left(\frac{(\alpha-\beta)}{\alpha}\left(\frac{a \beta^{2n+1}}{(a b)^{n+1} l_{2n+2}}+\frac{\beta^{2n}}{(a b)^{n} l_{2n+1}}\right)+\frac{1}{l_{4n+3}-a}\right)+\mathcal{O}\left(\frac{\alpha^{2m}}{\beta^{2m}}\right).\end{aligned} \tag{1.8}$$

2 引理

本节我们介绍几个引理, 以便定理的证明.

引理 2.1 记 $\{f_n\}$ 是双周期 Fibonacci 数列, $\{l_n\}$ 是双周期 Lucas 数列, 则

$$T\left(f_{2n}, f_{2n+1}, f_{2n+2}\right)=\left(\frac{a-3}{6}\right)\left(\zeta(n)+\frac{(-1)^n}{2}\right), \tag{2.1}$$

且当 a 是奇数时:

$$T\left(l_{2n},l_{2n+1},l_{2n+2}\right)=\left(\frac{b-3}{6}\right)\left(\zeta\left(n\right)+\frac{\left(-1\right)^{n}}{2}\right), \tag{2.2}$$

其中 $T\left(x,y,z\right)=S\left(x,y\right)+S\left(y,z\right)-\frac{x}{12y}-\frac{y}{12z}-\frac{1}{12yz}$.

证 在这里我们仅证明 (2.1), (2.2) 的证明类似. 根据 $\{f_n\}$ 的递推关系与 Dedekind 的性质, 有

$$\begin{aligned}
&S\left(f_{2n-1},f_{2n}\right)+S\left(f_{2n},f_{2n-1}\right)+S\left(f_{2n+1},f_{2n+2}\right)+S\left(f_{2n+2},f_{2n+1}\right)\\
=&S\left(f_{2n-1},f_{2n}\right)+S\left(af_{2n-1}+f_{2n-2},f_{2n-1}\right)+S\left(f_{2n+1},f_{2n+2}\right)+S\left(af_{2n+1}+f_{2n},f_{2n+1}\right)\\
=&S\left(f_{2n-1},f_{2n}\right)+S\left(f_{2n+1},f_{2n+2}\right)+S\left(f_{2n},f_{2n+1}\right)+S\left(f_{2n-2},f_{2n-1}\right).
\end{aligned} \tag{2.3}$$

又因为 $\left(f_{2n},f_{2n+1}\right)=1$, $\left(f_{2n+1},f_{2n+2}\right)=1$. 由 Dedekind 和的互反公式, 我们很容易得到:

$$\begin{aligned}
&S\left(f_{2n-1},f_{2n}\right)+S\left(f_{2n},f_{2n-1}\right)+S\left(f_{2n+1},f_{2n+2}\right)+S\left(f_{2n+2},f_{2n+1}\right)\\
=&\frac{1}{12f_{2n-1}f_{2n}}+\frac{1}{12f_{2n+1}f_{2n+2}}+\frac{f_{2n-1}}{12f_{2n}}+\frac{f_{2n}}{12f_{2n-1}}+\frac{f_{2n+1}}{12f_{2n+2}}+\frac{f_{2n+2}}{12f_{2n+1}}-\frac{1}{2}\\
=&\frac{1}{12f_{2n-1}f_{2n}}+\frac{1}{12f_{2n+1}f_{2n+2}}+\frac{f_{2n-1}}{12f_{2n}}+\frac{f_{2n-2}}{12f_{2n-1}}+\frac{f_{2n+1}}{12f_{2n+2}}+\frac{f_{2n}}{12f_{2n+1}}+\frac{a-3}{6}.
\end{aligned} \tag{2.4}$$

所以

$$\begin{aligned}
&S\left(f_{2n},f_{2n+1}\right)+S\left(f_{2n+1},f_{2n+2}\right)-\frac{f_{2n}}{12f_{2n+1}}-\frac{f_{2n+1}}{12f_{2n+2}}-\frac{1}{12f_{2n+1}f_{2n+2}}\\
=&\frac{a-3}{6}-\left(S\left(f_{2n-2},f_{2n-1}\right)+S\left(f_{2n-1},f_{2n}\right)-\frac{f_{2n-2}}{12f_{2n-1}}-\frac{f_{2n-1}}{12f_{2n}}-\frac{1}{12f_{2n-1}f_{2n}}\right)\\
=&S\left(f_{2n-4},f_{2n-3}\right)+S\left(f_{2n-3},f_{2n-2}\right)-\frac{f_{2n-4}}{12f_{2n-3}}-\frac{f_{2n-3}}{12f_{2n-2}}-\frac{1}{12f_{2n-3}f_{2n-2}}\\
=&\cdots\\
=&\zeta\left(n\right)\times\left(\frac{a-3}{6}\right)+\left(-1\right)^{n}\times\left[S\left(f_0,f_1\right)+S\left(f_1,f_2\right)-\frac{f_0}{12f_1}-\frac{f_1}{12f_2}-\frac{1}{12f_1f_2}\right]\\
=&\zeta\left(n\right)\times\left(\frac{a-3}{6}\right)+\left(-1\right)^{n}\times\left(\frac{a-3}{12}\right).
\end{aligned}$$

引理 2.2 记 $\{f_n\}$ 是双周期 Fibonacci 数列, $\{l_n\}$ 是双周期 Lucas 数列, 对正整数 m 有:

$$\sum_{n=1}^{m}\left(\frac{f_{2n+1}}{f_{2n+2}}+\frac{f_{2n}}{f_{2n+1}}\right)=\frac{\left(a+b\right)m}{\alpha}+\sum_{n=1}^{\infty}\left(\frac{\beta^{2n+2}}{\left(ab\right)^{n+1}f_{2n+2}}+\frac{a\beta^{2n+1}}{\left(ab\right)^{n+1}f_{2n+1}}\right)+\mathcal{O}\left(\frac{\beta^{2m}}{\alpha^{2m}}\right), \tag{2.5}$$

$$\sum_{n=1}^{m}\left(\frac{l_{2n+1}}{l_{2n+2}}+\frac{l_{2n}}{l_{2n+1}}\right)=\frac{\left(a+b\right)m}{\alpha}+\sum_{n=1}^{\infty}\frac{\left(\alpha-\beta\right)}{\alpha}\left(\frac{a\beta^{2n+1}}{\left(ab\right)^{n+1}l_{2n+2}}+\frac{\beta^{2n}}{\left(ab\right)^{n}l_{2n+1}}\right)+\mathcal{O}\left(\frac{\beta^{2m}}{\alpha^{2m}}\right). \tag{2.6}$$

证 在这里我们仅证明 (2.5), (2.6) 的证明类似. 根据 $\{f_n\}$ 的 Binet 公式 (1.5), 有:

$$\begin{aligned}&\sum_{n=m+1}^{\infty}\frac{1}{\alpha}\left(\frac{\beta^{2n+1}}{(ab)^{n}f_{2n+2}}\right)=\sum_{n=m+1}^{\infty}\frac{b(\alpha-\beta)}{\alpha}\cdot\frac{\beta^{2n+1}}{(\alpha^{2n+2}-\beta^{2n+2})}\\&=\sum_{n=m+1}^{\infty}\frac{b(\alpha-\beta)}{\alpha}\cdot\frac{1}{\frac{\alpha^{2n+2}}{\beta^{2n+1}}\left(1-\frac{\beta^{2n+2}}{\alpha^{2n+2}}\right)},\end{aligned}$$

又因为当 $\epsilon\to 0$ 时, 有 $\frac{1}{1\pm\epsilon}=1+\mathcal{O}(\epsilon)$, 而 $1<\alpha$ 且 $-1<\beta<0$, 所以当 $n\to\infty$ 时, $\frac{\beta^{2n}}{\alpha^{2n}}\to 0$, 故 $\frac{1}{1-\frac{\beta^{2n+2}}{\alpha^{2n+2}}}=1+\mathcal{O}\left(\frac{\beta^{2n+2}}{\alpha^{2n+2}}\right)$, 即有

$$\begin{aligned}&\sum_{n=m+1}^{\infty}\frac{b(\alpha-\beta)}{\alpha}\cdot\frac{1}{\frac{\alpha^{2n+2}}{\beta^{2n+1}}\left(1-\frac{\beta^{2n+2}}{\alpha^{2n+2}}\right)}=\sum_{n=m+1}^{\infty}\frac{b(\alpha-\beta)}{\alpha}\cdot\left(\frac{\beta^{2n+1}}{\alpha^{2n+2}}\left(1+\mathcal{O}\left(\frac{\beta^{2n+2}}{\alpha^{2n+2}}\right)\right)\right)\\&=\sum_{n=m+1}^{\infty}\left(\frac{b(\alpha-\beta)}{\alpha}\cdot\frac{\beta^{2n+1}}{\alpha^{2n+2}}\right)+\mathcal{O}\left(\frac{\beta^{4m}}{\alpha^{4m}}\right)=\mathcal{O}\left(\frac{\beta^{4m}}{\alpha^{4m}}\right)+\mathcal{O}\left(\frac{\beta^{2m}}{\alpha^{2m}}\right)=\mathcal{O}\left(\frac{\beta^{2m}}{\alpha^{2m}}\right).\end{aligned}$$

同理,

$$\begin{aligned}&\sum_{n=m+1}^{\infty}\frac{a}{\alpha}\left(\frac{\beta^{2n}}{(ab)^{n}f_{2n+1}}\right)=\sum_{n=m+1}^{\infty}\frac{a(\alpha-\beta)}{\alpha}\cdot\frac{\beta^{2n}}{(\alpha^{2n+1}-\beta^{2n+1})}\\&=\sum_{n=m+1}^{\infty}\frac{a(\alpha-\beta)}{\alpha}\cdot\frac{1}{\frac{\alpha^{2n+1}}{\beta^{2n}}\left(1-\frac{\beta^{2n+1}}{\alpha^{2n+1}}\right)}=\sum_{n=m+1}^{\infty}\frac{a(\alpha-\beta)}{\alpha}\cdot\left(\frac{\beta^{2n}}{\alpha^{2n+1}}\left(1+\mathcal{O}\left(\frac{\beta^{2n+1}}{\alpha^{2n+1}}\right)\right)\right)\\&=\sum_{n=m+1}^{\infty}\left(\frac{a(\alpha-\beta)}{\alpha}\cdot\frac{\beta^{2n}}{\alpha^{2n+1}}\right)+\mathcal{O}\left(\frac{\beta^{4m}}{\alpha^{4m}}\right)=\mathcal{O}\left(\frac{\beta^{2m}}{\alpha^{2m}}\right).\end{aligned}$$

所以,

$$\begin{aligned}&\sum_{n=1}^{m}\left(\frac{f_{2n+1}}{f_{2n+2}}+\frac{f_{2n}}{f_{2n+1}}\right)=\frac{1}{\alpha}\sum_{n=1}^{m}\left(\frac{\alpha f_{2n+1}}{f_{2n+2}}+\frac{\alpha f_{2n}}{f_{2n+1}}\right)\\&=\frac{(a+b)m}{\alpha}-\sum_{n=1}^{m}\frac{1}{\alpha}\left(\frac{\beta^{2n+1}}{(ab)^{n}f_{2n+2}}+\frac{a\beta^{2n}}{(ab)^{n}f_{2n+1}}\right)\\&=\frac{(a+b)m}{\alpha}+\sum_{n=1}^{\infty}\left(\frac{\beta^{2n+2}}{(ab)^{n+1}f_{2n+2}}+\frac{a\beta^{2n+1}}{(ab)^{n+1}f_{2n+1}}\right)+\mathcal{O}\left(\frac{\beta^{2m}}{\alpha^{2m}}\right).\end{aligned}$$

引理 2.3 记 $\{f_n\}$ 是双周期 Fibonacci 数列, $\{l_n\}$ 是双周期 Lucas 数列, 对正整数 m 有:

$$\sum_{n=1}^{m}\frac{1}{f_{2n+1}f_{2n+2}}=\sum_{n=1}^{\infty}\frac{ab+4}{l_{4n+3}+a}+\mathcal{O}\left(\frac{\beta^{2m}}{\alpha^{2m}}\right),\tag{2.7}$$

$$\sum_{n=1}^{m}\frac{1}{l_{2n+1}l_{2n+2}}=\sum_{n=1}^{\infty}\frac{1}{l_{4n+3}-a}+\mathcal{O}\left(\frac{\beta^{2m}}{\alpha^{2m}}\right).\tag{2.8}$$

证 在这里我们仅证明 (2.7), (2.8) 的证明类似. 根据 $\{f_n\}$ 的 Binet 公式 (1.5), 有:

$$f_{2n+1}f_{2n+2}=\frac{a\left(\alpha^{4n+3}+\beta^{4n+3}+(ab)^{2n+2}\right)}{(ab)^{2n+1}(\alpha-\beta)^2}=\frac{abl_{4n+3}+a^2b}{a^2b^2+4ab}=\frac{l_{4n+3}+a}{ab+4},$$

且

$$\begin{aligned}\sum_{n=m+1}^{\infty}\frac{ab+4}{l_{4n+3}+a}&=\frac{(ab+4)}{a}\sum_{n=m+1}^{\infty}\frac{(ab)^{2n+2}}{\alpha^{4n+3}+\beta^{4n+3}+(ab)^{2n+2}}\\&=\frac{(ab+4)}{a}\sum_{n=m+1}^{\infty}\frac{1}{\left(\frac{\alpha^{4n+3}}{(ab)^{2n+2}}\left(1+\frac{\beta^{4n+3}}{\alpha^{4n+3}}+\frac{\beta^{2n+2}}{\alpha^{2n+1}}\right)\right)}\\&=\frac{(ab+4)}{a}\sum_{n=m+1}^{\infty}\frac{\beta^{2n+2}}{\alpha^{2n+1}}\left(1+\mathcal{O}\left(\frac{\beta^{2m}}{\alpha^{2m}}\right)\right)\\&=\mathcal{O}\left(\frac{\beta^{2m}}{\alpha^{2m}}\right).\end{aligned}$$

所以,

$$\sum_{n=1}^{m}\frac{1}{f_{2n+1}f_{2n+2}}=\sum_{n=1}^{m}\frac{ab+4}{l_{4n+3}+a}=\sum_{n=1}^{\infty}\frac{ab+4}{l_{4n+3}+a}+\mathcal{O}\left(\frac{\beta^{2m}}{\alpha^{2m}}\right).$$

3 定理的证明

在这里我们仅证明 (1.7), (1.8) 的证明类似. 首先由 (2.3) 可得

$$\begin{aligned}&\sum_{n=1}^{m}\left[S(f_{2n-1},f_{2n})+S(f_{2n},f_{2n-1})+S(f_{2n+1},f_{2n+2})+S(f_{2n+2},f_{2n+1})\right]\\=&\sum_{n=1}^{m}\left[S(f_{2n-1},f_{2n})+S(f_{2n+1},f_{2n+2})+S(f_{2n},f_{2n+1})+S(f_{2n-2},f_{2n-1})\right]\\=&2\sum_{n=1}^{m}\left[S(f_{2n},f_{2n+1})+S(f_{2n+1},f_{2n+2})\right]-S(f_{2m+1},f_{2m+2})-S(f_{2m},f_{2m+1})\\&+S(f_1,f_2)+S(f_0,f_1).\end{aligned}$$

又由 (2.4) 可得,

$$\begin{aligned}&\sum_{n=1}^{m}\left[S(f_{2n-1},f_{2n})+S(f_{2n},f_{2n-1})+S(f_{2n+1},f_{2n+2})+S(f_{2n+2},f_{2n+1})\right]\\=&\frac{1}{12}\sum_{n=1}^{m}\left(\frac{1}{f_{2n-1}f_{2n}}+\frac{1}{f_{2n+1}f_{2n+2}}+\frac{f_{2n-1}}{f_{2n}}+\frac{f_{2n-2}}{f_{2n-1}}+\frac{f_{2n+1}}{f_{2n+2}}+\frac{f_{2n}}{f_{2n+1}}\right)+\frac{m(a-3)}{6}\\=&\frac{1}{6}\sum_{n=1}^{m}\left(\frac{1}{f_{2n+1}f_{2n+2}}+\frac{f_{2n+1}}{f_{2n+2}}+\frac{f_{2n}}{f_{2n+1}}\right)-\frac{1}{12f_{2m+1}f_{2m+2}}+\frac{1}{12f_1f_2}-\frac{f_{2m+1}}{12f_{2m+2}}\\&-\frac{f_{2m}}{12f_{2m+1}}+\frac{f_0}{12f_1}+\frac{f_1}{12f_2}+\frac{m(a-3)}{6}.\end{aligned}$$

因此, 由引理 2.1, 引理 2.2 和引理 2.3 可得到

$$
\begin{aligned}
&2\sum_{n=1}^{m}\left[S\left(f_{2n},f_{2n+1}\right)+S\left(f_{2n+1},f_{2n+2}\right)\right]\\
=&\frac{1}{6}\sum_{n=1}^{m}\left(\frac{1}{f_{2n+1}f_{2n+2}}+\frac{f_{2n+1}}{f_{2n+2}}+\frac{f_{2n}}{f_{2n+1}}\right)+S\left(f_{2m+1},f_{2m+2}\right)+S\left(f_{2m},f_{2m+1}\right)\\
&-\frac{1}{12f_{2m+1}f_{2m+2}}-\frac{f_{2m+1}}{12f_{2m+2}}-\frac{f_{2m}}{12f_{2m+1}}+\frac{(2m-1)(a-3)}{12}\\
=&\frac{1}{6}\sum_{n=1}^{m}\left(\frac{1}{f_{2n+1}f_{2n+2}}+\frac{f_{2n+1}}{f_{2n+2}}+\frac{f_{2n}}{f_{2n+1}}\right)+\zeta(m)\times\left(\frac{a-3}{6}\right)\\
&+(-1)^{m}\times\left(\frac{a-3}{12}\right)+\frac{(2m-1)(a-3)}{12}\\
=&\frac{1}{6}\sum_{n=1}^{m}\left(\frac{1}{f_{2n+1}f_{2n+2}}+\frac{f_{2n+1}}{f_{2n+2}}+\frac{f_{2n}}{f_{2n+1}}\right)+\frac{(a-3)m}{6}\\
=&\frac{m(a\alpha-3\alpha+a+b)}{6\alpha}\\
&+\frac{1}{6}\sum_{n=1}^{\infty}\left(\frac{\beta^{2n+2}}{(ab)^{n+1}f_{2n+2}}+\frac{a\beta^{2n+1}}{(ab)^{n+1}f_{2n+1}}+\frac{ab+4}{l_{4n+3}+a}\right)+\mathcal{O}\left(\frac{\alpha^{2m}}{\beta^{2m}}\right).
\end{aligned}
$$

其中, 因 $a\geq 1$, $b\geq 1$ 故 $-1<\beta<0$ 且 $1<\alpha$.

参考文献

[1] Apostol T M. Modular functions and Dirichlet series in number theory [M]. New York: Springer-Verlag, 1990:41.

[2] Carlitz L. The reciprocity theorem for Dedekind sums [J]. Pacific J.Math, 1953, 3(3): 523–527.

[3] Mordell L J. The reciprocity formula for Dedekind sums [J]. Amer. J. Math, 1951, 73(3): 593–598.

[4] Myerson G. Dedekind sums and uniform distribution [J]. J. Number Theory, 1988, 28(3): 233–239.

[5] Zheng Z Y. Dedekind sums and uniform distribution (mod1) [J]. Acta Mathematica Sinica, 1995, 11(1): 62–67.

[6] Conrey J B, Fransen E, Klein R, et al. Mean values of Dedekind sums [J]. J. Number Theory, 1996, 56(2): 214–226.

[7] Zhang W P. A note on the mean square value of the Dedekind sums [J]. Acta Mathematica Hungarica, 2000, 86(4): 275–289.

[8] Zhang W P. On the mean values of Dedekind sums [J]. Journal de Théorie des Nombres de Bordeaux, 1996, 8(2): 429–442.

[9] Zhang W P. A sum analogous to Dedekind sums and its hybrid mean value formula [J]. Acta Arithmetica, 2003, 107(1): 1–8.

[10] Edson M, Yayenie O. A new generalization of Fibonacci sequence and extended Binet's formula [J]. Integers, 2009, 9(6): 639–654.

[11] Falcón S, Plaza Á. On the Fibonacci k-numbers[J]. Chaos Solitons and Fractals, 2007, 32(5): 1615–1624.

[12] Bilgici G. Two generalizations of Lucas sequence[J]. Appl. Math. Comput., 2014, 245: 526–538.

[13] Falcon S. On the k-Lucas numbers [J]. Int. J. Contemp. Math. Sci., 2011, 6(21): 1039–1050.

[14] Tan E, Leung H H. Some basic properties of the generalized bi-periodic Fibonacci and Lucas sequences [J]. Adv. Diff. Equ., 2020, DOI: 10.1186/s13662-020-2507-4.

[15] Du T T, Wu Z G. Some identities involving the bi-periodic Fibonacci and Lucas polynomials [J], AIMS Math., 2023, 8(3): 5838–5846.

[16] Guo B N, Polatl E, Qi F. Determinantal formulas and recurrent relations for bi-periodic Fibonacci and Lucas polynomials [M]. New Trends in Applied Analysis and Computational Mathematics, 2021, 263–276.

[17] Du T T, Wu Z G. Some identities of the generalized bi-periodic Fibonacci and Lucas polynomials [J]. AIMS Math., 2024, 9(3): 7492–7510.

[18] Du T T, Wang L. On the power sums problem of bi-periodic Fibonacci and Lucas polynomials [J]. AIMS Math., 2024, 9(4): 7810–7818.

[19] Du T T, Wu Z G. On the reciprocal products of generalized Fibonacci sequences [J]. J. Inequal. Appl., 2022, DOI: 10.1186/s13660-022-02889-8.

[20] Choi G, Choo Y. On the reciprocal sums of square of generalized bi-periodic Fibonacci numbers [J]. Miskolc. Math. Notes., 2018, 19(1): 201–209.

[21] Zhang W P, Yi Y. On the Fibonacci numbers and the Dedekind sums [J]. Fibonacci Quart, 2000, 40(2): 223–226.

[22] Zhao F Z, Wang T M. Some results on generalized Fibonacci and Lucas numbers and Dedekind sums [J]. Fibonacci Quart, 2004, 42(3): 250–255.

[23] 关文吉, 李小雪. Dedekind 和及第一类 Chebyshev 多项式 [J]. 数学学报: 中文版, 2019, 62(2): 219–224.

[24] 胡宏. 涉及 Fibonacci 序列的 Dedekind 和 [J]. 东南大学学报 (自然科学版), 2004, 34(6): 869–871.

THE DEDEKIND SUMS INVOLVING BI-PERIODIC FIBONACCI AND LUCAS SEQUENCES

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Abstract: This paper investigates the estimation problem of Dedekind sums involving the bi-periodic Fibonacci $\{f_n\}$ and Lucas $\{l_n\}$ sequence. By utilizing the analytic properties of the Dedekind sum $S(h,k)$, and the recurrence relations of bi-periodic Fibonacci $\{f_n\}$ and Lucas $\{l_n\}$ sequence, we derive asymptotic estimates for the sums $\sum\limits_{n=1}^{m}\left(S\left(f_{2n},f_{2n+1}\right)+S\left(f_{2n+1},f_{2n+2}\right)\right)$ and $\sum\limits_{n=1}^{m}\left(S\left(l_{2n},l_{2n+1}\right)+S\left(l_{2n+1},l_{2n+2}\right)\right)$. This work extends the study of Dedekind sums from linear recurrence sequences to the nonlinear recurrence sequences.

Keywords: Bi-periodic Fibonacci sequence; Bi-periodic Lucas sequence; Dedekind sums

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