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THE U-CLEAN GRAPHS OF FINITE COMMUTATIVE RINGS

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Abstract: Let R be a finite commutative ring with identity 1. The U-clean graph of R, denoted by $U\text{-}Cl(R)$, is a graph with vertices in form (e, u), where e is a nonzero idempotent of R and u is a unit of R. In this paper, some basic properties of $U\text{-}Cl(R)$ and the explicit structures of $U\text{-}Cl(\mathbb{Z}_p \times \mathbb{Z}_q)$ are given, where p, q are primes. We prove that $U\text{-}Cl(\mathbb{Z}_p \times \mathbb{Z}_q)$ is Eulerian if and only if $p = 2$, $q = 2$. Moreover, the clique number, the chromatic number of the U-clean graph for some classes of rings are given in this paper.

Keywords: U-clean graph; Eulerian; unit

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1 Introduction

The zero-divisor graph of a commutative ring was first introduced by Beck in [1]. He showed some conclusions about coloring. In [2], Anderson and Livingston studied a zero-divisor graph of a ring R, denoted by $\Gamma(R)$, whose vertices are nonzero divisors of R, and two distinct vertices x and y are adjacent if and only if $xy = 0$. In [3], Akbari and Habibi introduced idempotent graph, denoted by $I(R)$, whose vertices are all nontrivial idempotent elements of a ring R, and two distinct vertices e and f are adjacent if and only if $ef = fe = 0$. Habibi [4] and Celikel introduced clean graph of a ring R, denoted by $Cl(R)$, which is a simple graph with vertices in form (e, u), where e is an idempotent and u is a unit of R, and two distinct vertices (e, u) and (f, v) are adjacent if and only if $ef = fe = 0$ or $uv = vu = 1$. They investigated the clique number, chromatic number, independence number and domination number of the subgraph $Cl_2(R)$ of $Cl(R)$, where R is a commutative Artin ring with identity, see [4]. In [5], Boonsawang discussed the clean graph of the finite ring $\mathbb{F}_{p^n} \times \mathbb{F}_{q^m}$, where p and q are primes, n and m are positive integers, and he gave two explicit structures of $Cl_2(\mathbb{F}_{p^n} \times \mathbb{F}_{q^m})$. In recent years, graph theory associated with rings has attracted many

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researches and scholars, see for instance [6-8].

In this paper, R always denotes a finite commutative ring with identity. An element e of R is called an idempotent if $e^2 = e$, and u of R is called a unit, if there exists v of R such that $uv = 1$. By $Id(R)$, $Id^*(R)$ and $U(R)$, we mean the sets of idempotents, nonzero idempotents and units of R, respectively. If an element a of R can be written by $e+u$, where $e \in Id(R)$, $u \in U(R)$, then a is called clean. The ring R is clean if every element of R is clean. For any undefined notation or terminology in ring theory, we refer the reader to [9].

Let us briefly mention some other notations about graph theory which will be used in this paper. Let G denote a graph. Then the vertex set of G is referred to as $V(G)$, its edge set as $E(G)$. Two adjacent vertices a and b in G are denoted as $a \sim b$. G is called connected if any two of its vertices are linked by a path in G. The distance $d_G(x, y)$ in G of two vertices is the length of a shortest $x - y$ path in G. The greatest distance between any two vertices in G is the diameter of G, denoted by $diam(G)$. If all the vertices of G are pairwise adjacent, then G is complete. A complete graph on n vertices is a K_n, a K_3 is called a triangle. The degree $d_G(v) = d(v)$ of a vertex v is the number $|E(v)|$ of edges at v, the number $\delta(G) = min\{d(v)|v \in V\}$ is the minimum degree of G, the number $\Delta(G) = max\{d(v)|v \in V\}$ is maximum degree. If all the vertices of G have the same degree k, then G is k-regular, or simply regular. Call a closed walk in a graph an Euler tour if it traverses every edge of the graph exactly once. A graph is Eulerian if it admits an Euler tour. The greatest integer r such that $K_r \subseteq G$ is the clique number $\omega(G)$ of G. A vertex colouring of a graph $G = (V, E)$ is a map $C : V \to S$ with $C(v) \neq C(w)$ whenever v and w are adjacent. The elements of the set S are called the available colours. All that interests us about S is its size: typically, we shall be asking for the smallest integer k such that G has a k-colouring, a vertex colouring $C : V \to \{1, \cdots, k\}$. This k is the (vertex-) chromatic number of G, it is called by $\chi(G)$. We refer the reader to [10] and [11] for general background on graph theory and for all undefined notions used in the text.

We define U-clean graph of a ring R, denoted by U-$Cl(R)$, whose vertices are related to clean elements of R. U-$Cl(R)$ is a graph with vertices $\{(e, u){:}e \in Id^*(R),\, u \in U(R)\}$ and two distinct vertices (e, u) and (f, v) are adjacent if and only if $ef = 1$ or $uv = 1$. Let U-$Cl_1(R)$ be the subgraph of U-$Cl(R)$ induced by $\{(1, u){:}u \in U(R)\}$. Let U-$Cl_2(R)$ be the subgraph of U-$Cl(R)$ induced by $\{(e, u){:}1 \neq e \in Id^*(R),\, u \in U(R)\}$. In section 2, we give some examples for U-clean graph to illustrate our definitions. We also discuss the graph structures such as connectivity, diameter, degree, regularity, etc. In section 3, we focus on the U-clean graph of the finite ring $\mathbb{Z}_p \times \mathbb{Z}_q$, where p and q are primes. Among many results in this section, we determine the degree of U-$Cl(\mathbb{Z}_p \times \mathbb{Z}_q)$, and prove U-$Cl(\mathbb{Z}_p \times \mathbb{Z}_q)$ is Eulerlian if and only if $p = 2$, $q = 2$. In section 4, we discuss the parameters $(\omega(G), \chi(G))$ of U-clean graph of a finite commutative ring R, and we obtain the exact value of these parameters.

2 Basic properties of U-clean graphs

Definition 2.1 Let R be a finite commutative ring with nonzero identity 1. The U-clean graph, denoted by $U\text{-}Cl(R)$ of R, which is a graph with vertices $\{(e,u){:}e \in Id^*(R), u \in U(R)\}$. Two distinct vertices (e,u) and (f,v) are adjacent if and only if $ef = 1$ or $uv = 1$.

Let $U\text{-}Cl_1(R)$ be the subgraph of $U\text{-}Cl(R)$ induced by $\{(1,u){:}u \in U(R)\}$. Let $U\text{-}Cl_2(R)$ be the subgraph of $U\text{-}Cl(R)$ induced by $\{(e,u){:}1 \neq e \in Id^*(R),\ u \in U(R)\}$.

Definition 2.2 Let R be a finite commutative ring. Let $U_1(R)$ and $U_2(R)$ be the subsets of $U(R)$, denote by $U_1(R)$=$\{u \in U(R){:}u^{-1} = u\}$ and $U_2(R)$=$\{u \in U(R){:}u^{-1} \neq u\}$, respectively.

Remark 2.3 $U\text{-}Cl_1(R)$ is a complete subgraph of $U\text{-}Cl(R)$, $U\text{-}Cl(R)$ is the union of $U\text{-}Cl_1(R)$ and $U\text{-}Cl_2(R)$. If R has no nontrivial idempotent and $|U(R)| = n$, then $U\text{-}Cl(R)$ is a complete graph, K_n. If R has a nontrivial idempotent and $U_2(R) = \varnothing$, then there is a subgraph of $U\text{-}Cl(R)$, $K_{|U_1(R)|}$.

Example 2.4 (1) $Id^*(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{(0,1),(1,0),(1,1)\}$, $U(R) = \{(1,1)\}$. It is easy to check that $V(U\text{-}Cl(\mathbb{Z}_2 \times \mathbb{Z}_2))$=$\{((0,1),(1,1)),((1,0),(1,1)),((1,1),(1,1))\}$.

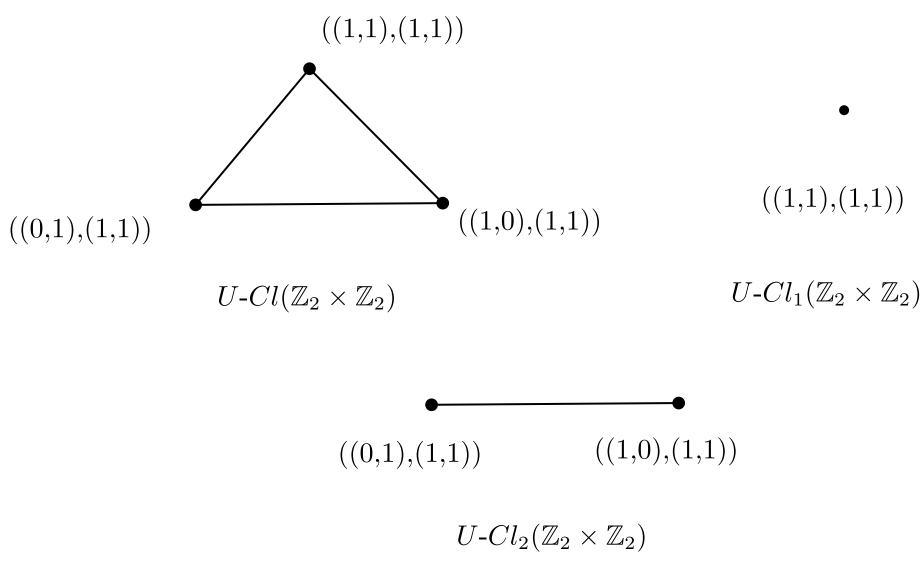


Figure 1: The U-clean graph of $\mathbb{Z}_2 \times \mathbb{Z}_2$

(2) For the ring $\mathbb{Z}_2 \times \mathbb{Z}_3$, $V(U\text{-}Cl(\mathbb{Z}_2 \times \mathbb{Z}_3))$=$\{$((0,1),(1,1)),((0,1),(1,2)),((1,0),(1,1)), ((1,0),(1,2)),((1,1),(1,1)),((1,1),(1,2))$\}$.

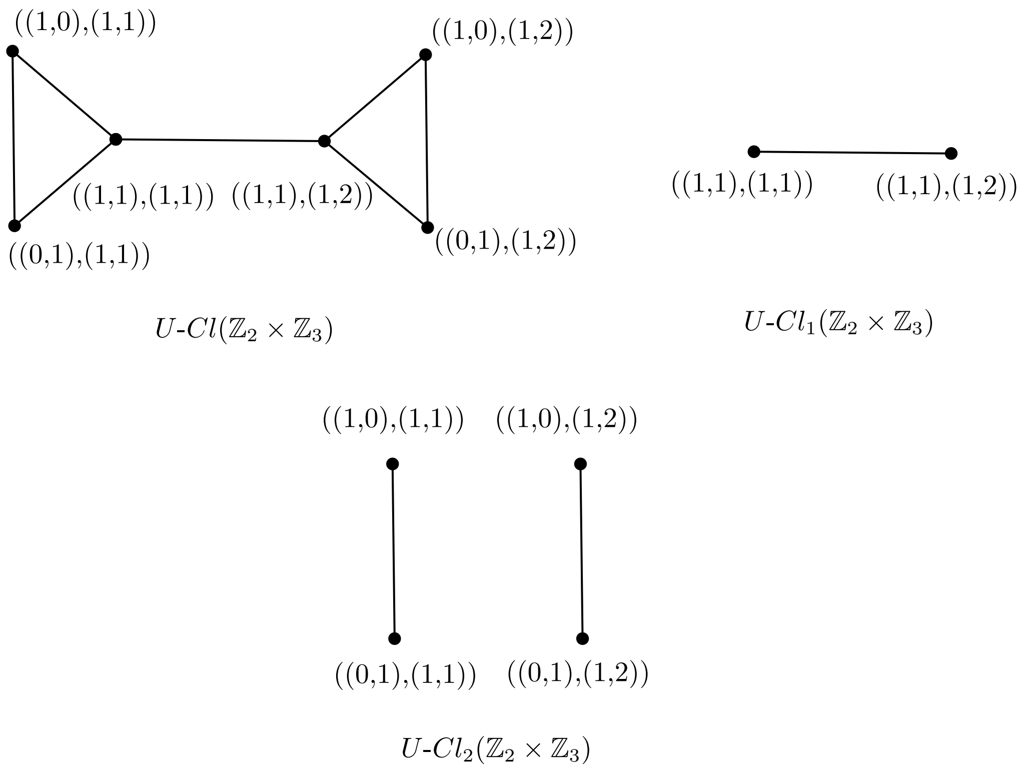
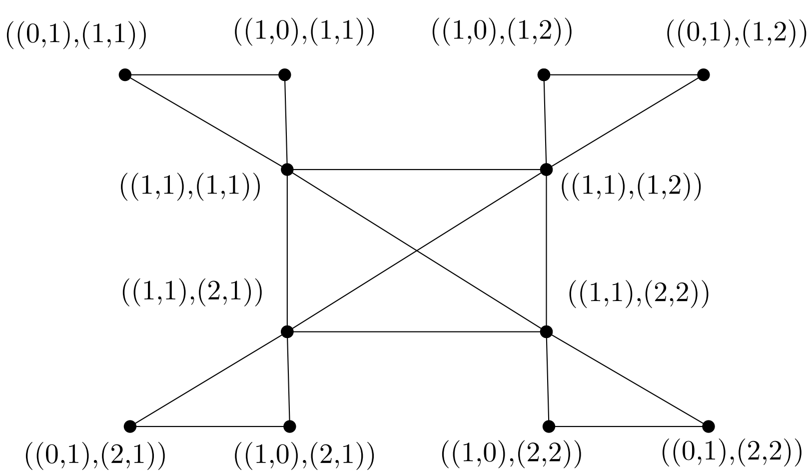


Figure 2: The U-clean graph of $\mathbb{Z}_2 \times \mathbb{Z}_3$

(3) Note that $V(U\text{-}Cl(\mathbb{Z}_3 \times \mathbb{Z}_3))$=$\{((0,1),(1,1)),((0,1),(1,2)),((0,1),(2,2)),$ $((0,1),(2,1)),((1,0),(1,1)),((1,0),(1,2)),((1,0),(2,1)),((1,0),(2,2)),((1,1),(1,1)),$ $((1,1),(1,2)),((1,1),(2,1)),((1,1),(2,2))\}$.



$U\text{-}Cl(\mathbb{Z}_3 \times \mathbb{Z}_3)$

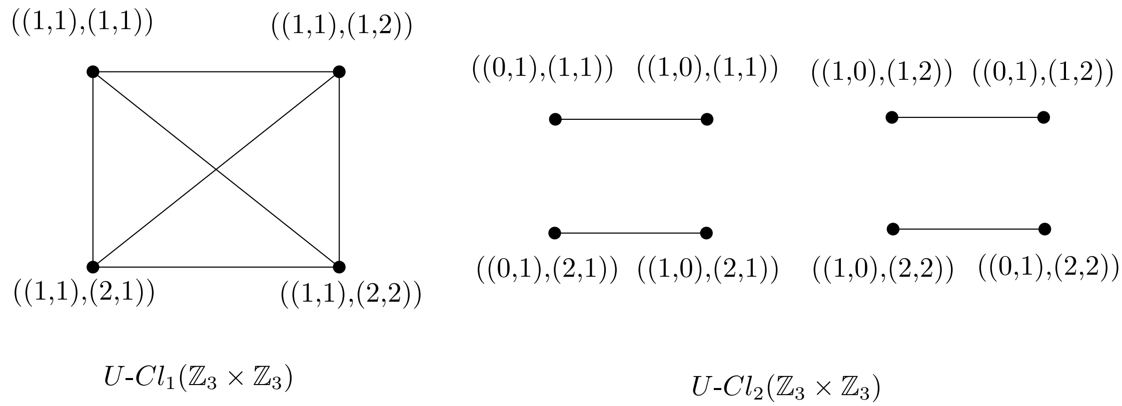


Figure 3: The U-clean graph of $\mathbb{Z}_3 \times \mathbb{Z}_3$

Theorem 2.5 Let R be a finite commutative ring. Then U-$Cl(R)$ is connected and $diam(U\text{-}Cl(R)) \leq 3$.

Proof Let $x = (e, u)$ and $y = (f, v)$ be two vertices of U-$Cl(R)$. It is clear that $(e, u) \sim (1, u^{-1}) \sim (1, v^{-1}) \sim (f, v)$, thus U-$Cl(R)$ is connected.

We discuss the following two cases.

Case 1. $e = f$. If $e = f = 1$, then $(1, u) \sim (1, v)$ is a path; if $e, f \neq 1$, then $(e, u) \sim (1, u^{-1}) \sim (1, v^{-1}) \sim (f, v)$, thus $d_G(x, y) = 3$.

Case 2. $e \neq f$. If e or f is equal to 1, we assume that $e = 1$ without loss of generality. It follows that $(1, u) \sim (1, v^{-1}) \sim (f, v)$ is a path of length 2; if $e, f \neq 1$, then $(e, u) \sim (1, u^{-1}) \sim (1, v^{-1}) \sim (f, v)$, thus $d_G(x, y) = 3$.

Hence, $diam(U\text{-}Cl(R)) \leq 3$.

This completes the proof of Theorem 2.5.

Lemma 2.6 Let R be a commutative ring. If R has two different nontrivial idempotents or three different units, then U-$Cl(R)$ has a triangle.

Proof If $e, f \in Id^*(R)\backslash\{1\}$ and $e \neq f$, then there is a cycle $(e, 1) \sim (1, 1) \sim (f, 1) \sim (e, 1)$. If $u, v, w \in U(R)$ and they are different, then there is a triangle $(1, u) \sim (1, v) \sim (1, w) \sim (1, u)$.

Thus the result follows.

Lemma 2.7 Let R be a commutative ring with nontrivial idempotents, and $x = (e, u)$ be a vertex of U-$Cl(R)$.

(1) If $e = 1$, then

$$d(x) = |Id^*(R)| + |U(R)| - 2. \tag{2.1}$$

(2) If $e \neq 1$, then

$$d(x) = \begin{cases} |Id^*(R)| - 1 & \text{if } u \in U_1(R), \\ |Id^*(R)| & \text{if } u \in U_2(R). \end{cases} \tag{2.2}$$

Proof (1) If $u \in U_1(R)$, then $(1, u) \sim (e_1, u)$, $e_1 \in Id^*(R)\backslash\{1\}$. $(1, u) \sim (1, u_1)$, $u_1 \in U(R)\backslash\{u\}$. Thus

$$d(x) = |Id^*(R)| + |U(R)| - 2.$$

If $u \in U_2(R)$, then $(1, u) \sim (e_1, u^{-1})$, $e_1 \in Id^*(R)\backslash\{1\}$. $(1, u) \sim (1, u_1)$, $u_1 \in U(R)\backslash\{u\}$. Thus

$$d(x) = |Id^*(R)| + |U(R)| - 2.$$

(2) If $u \in U_1(R)$, then $(e, u) \sim (e_1, u)$, $e_1 \in Id^*(R)\backslash\{e\}$. Thus

$$d(x) = |Id^*(R)| - 1.$$

If $u \in U_2(R)$, then $(e, u) \sim (e_1, u^{-1})$, $e_1 \in Id^*(R)$. Thus

$$d(x) = |Id^*(R)|.$$

Thus we are done.

Theorem 2.8 Let R be a commutative ring. Then $U\text{-}Cl(R)$ is regular if and only if $|U(R)| = 1$ or R has no nontrivial idempotent.

proof If $|U(R)| = 1$, then

$$V(U\text{-}Cl(R))=\{(e, 1) : e \in Id^*(R)\},\ U\text{-}Cl(R) \cong K_{|Id^*(R)|}.$$

If R has no nontrivial idempotent, then

$$V(U\text{-}Cl(R))=\{(1, u) : u \in U(R)\},\ U\text{-}Cl(R) \cong K_{|U(R)|}.$$

Conversely, let $U\text{-}Cl(R)$ be regular. Assume that $|U(R)| \geq 2$ and e is a nontrivial idempotent. Let $x = (e, 1)$, $y = (1, 1)$. Then by (2.1), (2.2), we have $d(x) = |Id^*(R)| - 1$ and $d(y) = |Id^*(R)| + |U(R)| - 2$, thus $d(x) < d(y)$. Hence $U\text{-}Cl(R)$ is not regular, a contradiction.

This completes the proof of Theorem 2.8.

Proposition 2.9 Let R be a finite commutative ring. Then the following statements hold:

(1)

$$\Delta(U\text{-}Cl(R)) = |U(R)| + |Id^*(R)| - 2. \tag{2.3}$$

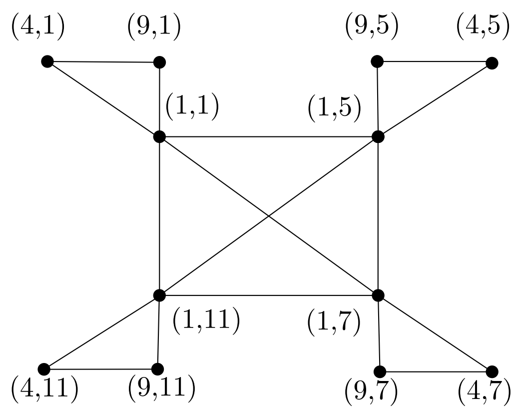
(2)

$$\delta(U\text{-}Cl(R)) = \begin{cases} |Id^*(R)| - 1 & \text{if } R \text{ has nontrivial idempotents,} \\ |U(R)| - 1 & \text{if } R \text{ has no nontrivial idempotent.} \end{cases} \quad (2.4)$$

Example 2.10 (1) Let $\mathbb{Z}_{12}$ be the ring of integers modulo 12. Note that $Id^*(R) = \{1, 4, 9\}$, $U(R) = \{1, 5, 7, 11\}$. By (2.3), (2.4), we have

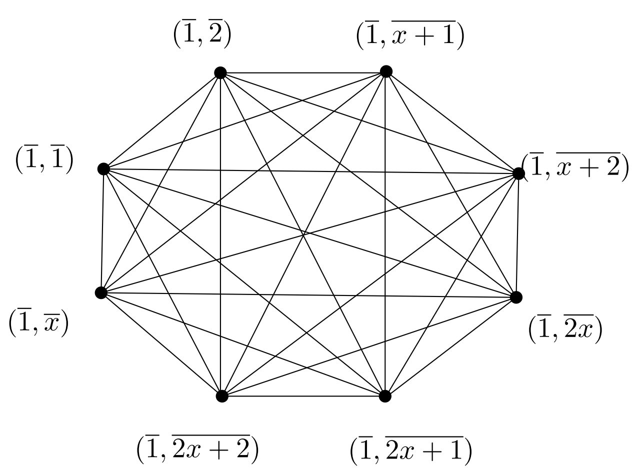
$$\Delta(U\text{-}Cl(\mathbb{Z}_{12})) = 5,\ \delta(U\text{-}Cl(\mathbb{Z}_{12})) = 2,$$

and the U-$Cl(\mathbb{Z}_{12})$.



U-$Cl(\mathbb{Z}_{12})$

(2) Let R be $\mathbb{Z}_3[x]/(x^2+1)$. Then U-$Cl(R)$ is regular, because the degree of any vertex in U-$Cl(R)$ is 7. We give U-$Cl(R)$ as follows.



U-$Cl(\mathbb{Z}_3[x]/(x^2+1))$

3 The structure of U-clean graph of ring $\mathbb{Z}_p \times \mathbb{Z}_q$

Let p and q be primes such that $p \leq q$. In this part, we show the degree of each vertex of $U\text{-}Cl(\mathbb{Z}_p \times \mathbb{Z}_q)$. It is obvious that $Id^*(\mathbb{Z}_p \times \mathbb{Z}_q) = \{(0,1),(1,0),(1,1)\}$ and $|U(\mathbb{Z}_p \times \mathbb{Z}_q)| = (p-1)(q-1)$. From Theorem 2.5, $U\text{-}Cl(\mathbb{Z}_p \times \mathbb{Z}_q)$ is connected.

We show an explicit structure of $U\text{-}Cl(\mathbb{Z}_p \times \mathbb{Z}_q)$ by considering $U_2(\mathbb{Z}_p \times \mathbb{Z}_q)$. If $U_2(\mathbb{Z}_p \times \mathbb{Z}_q) = \varnothing$, we have shown as Example 2.4. If $U_2(\mathbb{Z}_p \times \mathbb{Z}_q) \neq \varnothing$, we consider $U\text{-}Cl(\mathbb{Z}_p \times \mathbb{Z}_q)$ by partitioning the vertex set of $U\text{-}Cl(\mathbb{Z}_p \times \mathbb{Z}_q)$:

$$\begin{cases} A_1 = \{((0,1),(x,y)) | (x,y)^{-1} = (x,y)\}, \\ A_2 = \{((1,0),(x,y)) | (x,y)^{-1} = (x,y)\}, \\ A_3 = \{((1,1),(x,y)) | (x,y)^{-1} = (x,y)\}, \\ A_4 = \{((0,1),(x,y)) | (x,y)^{-1} \neq (x,y)\}, \\ A_5 = \{((1,0),(x,y)) | (x,y)^{-1} \neq (x,y)\}, \\ A_6 = \{((1,1),(x,y)) | (x,y)^{-1} \neq (x,y)\}. \end{cases} \tag{3.1}$$

Proposition 3.1 Let R be $\mathbb{Z}_p \times \mathbb{Z}_q$. Then

$$|U_1(R)| = \begin{cases} 1 & \text{if } p = q = 2, \\ 2 & \text{if } p = 2 \text{ and } q \geq 3, \\ 4 & \text{if } p \geq 3 \text{ and } q \geq 3, \end{cases}$$

and

$$|U_2(R)| = \begin{cases} (p-1)(q-1) - 1 & \text{if } p = q = 2, \\ (p-1)(q-1) - 2 & \text{if } p = 2 \text{ and } q \geq 3, \\ (p-1)(q-1) - 4 & \text{if } p \geq 3 \text{ and } q \geq 3. \end{cases}$$

Proposition 3.2 Let R be $\mathbb{Z}_p \times \mathbb{Z}_q$, and A_i be in (3.1). Then the following statements hold:

(1) $\bigcup_{i=1}^{6} A_i = V(U\text{-}Cl(R))$ and $A_i \cap A_j = \varnothing$, $1 \leq i < j \leq 6$.

(2) For $i \in \{1,2,3\}$,

$$|A_i| = \begin{cases} 1 & \text{if } p = q = 2, \\ 2 & \text{if } p = 2 \text{ and } q \geq 3, \\ 4 & \text{if } p \geq 3 \text{ and } q \geq 3. \end{cases} \tag{3.2}$$

For $j \in \{4,5,6\}$,

$$|A_j| = \begin{cases} 0 & \text{if } p = q = 2, \\ q-3 & \text{if } p = 2 \text{ and } q \geq 3, \\ (p-1)(q-1) - 4 & \text{if } p \geq 3 \text{ and } q \geq 3, \end{cases} \tag{3.3}$$

Proposition 3.3 Let p, q be primes with $p+q>6$. For A_i in (3.1), the following statements hold:

(1) No two distinct vertices in A_i are adjacent, $i\in\{1,2\}$.

(2) For each vertex in A_i, there exists a unique distinct vertex in A_i such that they are adjacent, $i\in\{4,5\}$.

(3) The subgraph induced by A_i is complete, $i\in\{3,6\}$.

(4) The subgraph induced by $A_3\cup A_6$ is complete.

(5) The subgraph induced by $A_i\cup A_j$ is bipartite graph, $(i,j)\in\{(1,2),(4,5)\}$.

(6) Each vertex in A_i is not adjacent to any vertex in A_j, $(i,j)\in\{(1,4),(1,5),(1,6),$ $(2,4),(2,5),(2,6),(3,4),(3,5)\}$.

Proof (1) It suffices to prove the case $i=1$. Let $((0,1),(x,y))$ and $((0,1),(a,b))$ be two distinct vertices in A_1. Since $(0,1)\cdot(0,1)\neq(1,1)$ and $(x,y)\cdot(a,b)\neq(1,1)$, we get that $((0,1),(x,y))$ and $((0,1),(a,b))$ are not adjacent.

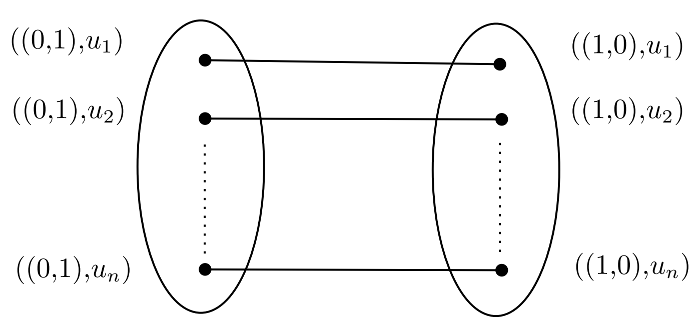
(2) Suppose that $((0,1),(x,y))\in A_4$. $(x,y)^{-1}\neq(x,y)$, $((0,1),(x,y)^{-1})\in A_4$. We know that $((0,1),(x,y))$ and $((0,1),(x,y)^{-1})$ are adjacent, because $(x,y)\cdot(x,y)^{-1}=(1,1)$. Next, let $((0,1),(a,b))\in A_4$ such that $(a,b)\neq(x,y)^{-1}$. Since $(0,1)\cdot(0,1)\neq(1,1)$ and $(x,y)\cdot(a,b)\neq(1,1)$, we have $((0,1),(x,y))$ is not adjacent to $((0,1),(a,b))$. The case $i=5$ can be proved similarly.

(3) Since $(1,1)\cdot(1,1)=(1,1)$, there is an edge between $((1,1),(x,y))$ and $((1,1),(a,b))$.

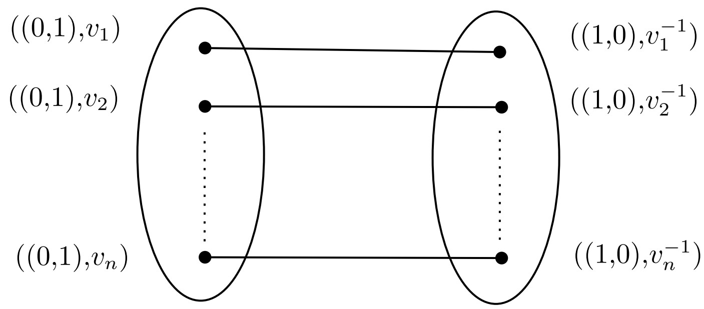
(4) The proof is similar to (3).

(5) The proof splits into the following cases:

Case 1. $(i,j)=(1,2)$. The subgraph induced by $A_1\cup A_2$ is shown as follows:



Case 2. $(i,j)=(4,5)$. The subgraph induced by $A_4\cup A_5$ is shown as follows:

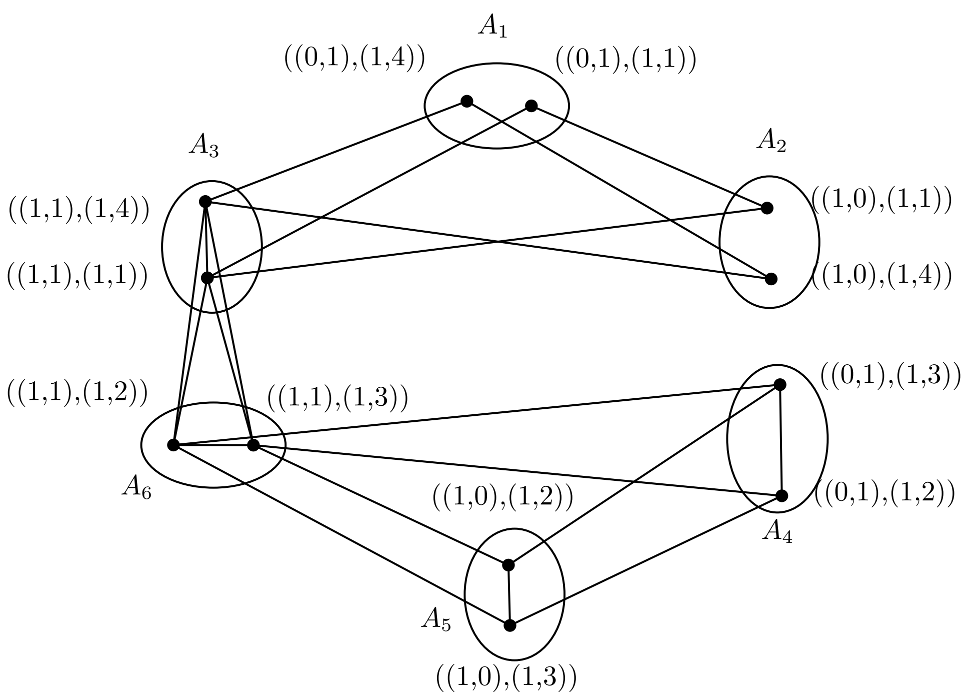


Note that $\{u_1, u_2, \cdots, u_n\} \in U_1(R)$ and $\{v_1, v_2, \cdots, v_n, v_1^{-1}, v_2^{-1}, \cdots, v_n^{-1}\} \in U_2(R)$.

(6) Let $((0,1),(x,y)) \in A_1$ and $((0,1),(a,b)) \in A_4$. Then $(x,y)^{-1} = (x,y)$ and $(a,b)^{-1} \neq (a,b)$. It follows that $(x,y) \neq (a,b)$. So because of $(0,1)\cdot(0,1) \neq (1,1)$ and $(x,y)\cdot(a,b) \neq (1,1)$, we see that$((0,1),(x,y))$ is not adjacent to $((0,1),(a,b))$.

Thus we are done.

Example 3.4 Let $p = 2$, $q = 5$. Then $|U_1(\mathbb{Z}_2 \times \mathbb{Z}_5)| = 2$, $|U_2(\mathbb{Z}_2 \times \mathbb{Z}_5)| = 2$, $|A_1| = |A_2| = |A_3| = 2$ and $|A_4| = |A_5| = |A_6| = 2$, $U\text{-}Cl(\mathbb{Z}_2 \times \mathbb{Z}_5)$ is shown as follows:



Lemma 3.5 [10, Theorem 1.8.1] A connected graph is Eulerian if and only if every vertex has even degree.

Lemma 3.6 Let (x,y) be a vertex in $U\text{-}Cl(\mathbb{Z}_p\times\mathbb{Z}_q)$. If $p+q>6$, then

$$d(x,y)=\begin{cases}2 & \text{if } (x,y)\in A_1,\\ 2 & \text{if } (x,y)\in A_2,\\ 1+(p-1)(q-1) & \text{if } (x,y)\in A_3,\\ 3 & \text{if } (x,y)\in A_4,\\ 3 & \text{if } (x,y)\in A_5,\\ 1+(p-1)(q-1) & \text{if } (x,y)\in A_6.\end{cases} \tag{3.4}$$

Proof Case 1. Let (x,y) be the vertex in A_1. By Proposition 3.3, each vertex in A_1 is adjacent to a unique vertex in A_2 and A_3, respectively. Then $d(x,y)=2$.

Case 2. It is similar to Case 1.

Case 3. Let (x,y) be a vertex in A_3. By Proposition 3.3, the subgraph induced by $A_3\cup A_6$ is a complete graph. Each vertex in A_3 is adjacent to a unique vertex in A_1 and A_2, respectively. Then

$$d(x,y)=|A_3|+|A_6|+1.$$

Let $p=q=2$. Then by (3.2), (3.3), we obtain

$$d(x,y)=1+(p-1)(q-1).$$

Let $p=2$ and $q\geq 3$. Then by (3.2), (3.3), we have

$$d(x,y)=1+(p-1)(q-1).$$

Let $p\geq 3$ and $q\geq 3$. Then by (3.2), (3.3), we get

$$d(x,y)=1+(p-1)(q-1).$$

Case 4. Let (x,y) be the vertex in A_4. By Proposition 3.3, each vertex in A_4 is adjacent to a unique vertex in A_5 and A_6, respectively. For each vertex in A_4, there exists a unique vertex in A_4, which is adjacent. Then $d(x,y)=3$.

Case 5. It is similar to Case 4.

Case 6. It is similar to Case 3.

This completes the proof of Lemma 3.6.

Theorem 3.7 Let p, q be primes. Then $U\text{-}Cl(\mathbb{Z}_p\times\mathbb{Z}_q)$ is Eulerian if and only if $p=2$, $q=2$.

Proof On the one hand, we consider the case $U_2(\mathbb{Z}_p\times\mathbb{Z}_q)=\varnothing$.

Case 1. $p=q=2$. By Figure 1, $U\text{-}Cl(\mathbb{Z}_2\times\mathbb{Z}_2)$ is an Eulerian.

Case 2. $p=2$, $q=3$. By Figure 2, $U\text{-}Cl(\mathbb{Z}_2\times\mathbb{Z}_3)$ has a vertex of odd degree.

Case 3. $p=3$, $q=3$. By Figure 3, $U\text{-}Cl(\mathbb{Z}_3\times\mathbb{Z}_3)$ has a vertex of odd degree.

On the other hand, we consider the case $p+q>6$. If $v_1 \in A_4$, then $d(v_1)=3$ by (3.4). Thus $U\text{-}Cl(\mathbb{Z}_p\times\mathbb{Z}_q)$ has a vertex of odd degree.

This completes the proof of Theorem 3.7.

In next section, we will give some parameters of U-clean graph of every finite commutative ring R. Moreover, the exact value of these parameters is given.

4 U-clean graphs of finite commutative rings

For any finite commutative ring R with $Id^*(R)=\{1\}$. It follows that

$$V(U\text{-}Cl(R))=\{(1,u)|u\in U(R)\}, U\text{-}Cl(R)=K_{|U(R)|},$$

thus

$$\omega(U\text{-}Cl(R))=\chi(U\text{-}Cl(R))=|U(R)|.$$

Theorem 4.1 Let R be a finite commutative ring and $R_1,\cdots,R_n$ be rings with $Id^*(R_i)=\{1\}$. If $R\cong R_1\times\cdots\times R_n$, then

$$\omega(U\text{-}Cl(R))=\chi(U\text{-}Cl(R))=max\{|U(R)|,\ 2^n-1\}.$$

Proof Assume that 1_R is identity of R. We divide the set V as follows:

$$V_1=\{(e,u)|e\in Id^*(R),u\in U_1(R)\},$$

$$V_2=\{(e,u)|(1_R\neq e\in Id^*(R),u\in U_2(R)\},$$

$$V_3=\{(1_R,u)|u\in U_2(R)\}.$$

Clearly, the subgraphs induced by $\{(1_R,u)|u\in U_1(R)\}\in V_1$ and $\{(e,u_1)|e\in Id^*(R),u_1\in U_1(R)\}\in V_1$ are both complete. By $K_{max\{|U_1(R)|,\ 2^n-1\}}$, K_2, $K_{|U_2(R)|}$, we mean the largest complete subgraphs of V_1, V_2 and V_3, respectively. Each vertex in V_1 is not adjacent to any vertex in V_2, because $e\neq 1_R$ and $u_1u_2\neq 1_R$, where $u_1\in U_1(R)$, $u_2\in U_2(R)$. There is no vertex in V_3 that is adjacent to both vertices in $\{(e^{'},u),(e^{''},u^{-1})\}$, where $e^{'},e^{''}\neq 1_R$ and $u\in U_2(R)$. Then each vertex in V_3 is adjacent to all vertices in $\{(1_R,u)|u\in U_1(R)\}\subseteq V_1$. Thus

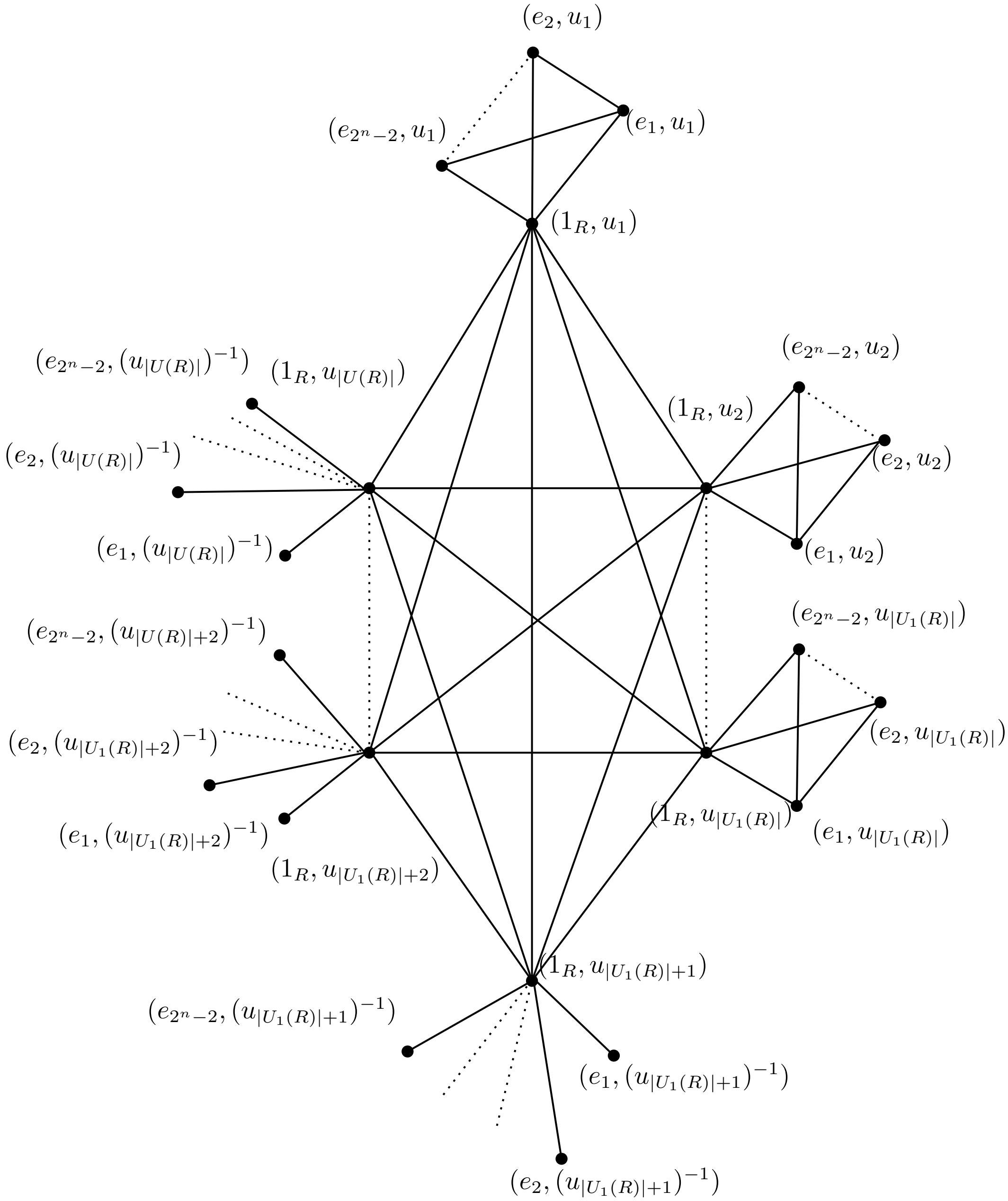
$$\omega(U\text{-}Cl(R))=max\{|U(R)|,\ 2^n-1\}.$$

Note that

$$\omega(U\text{-}Cl(R))\leq \chi(U\text{-}Cl(R)).$$

Next, we consider the following two cases:

Case 1. $|U(R)|\geq 2^n-1$. Let $V_{11}=\{(1_R,u)|u\in U_1(R)\}$. We color the vertices of V_{11} by $1,\cdots,|U_1(R)|$ and V_3 by $|U_1(R)|+1,\cdots,|U(R)|$, respectively. We show the following figure to provide a more intuitive experience of the coloring situation:



Note that $u_1, \cdots, u_{|U_1(R)|} \in U_1(R), u_{|U_1(R)|+1}, \cdots, u_{|U(R)|} \in U_2(R)$ and $e_1, \cdots, e_{2^n-2} \in Id^*(R)$ are not equal to 1_R. Clearly,

$$\chi(U\text{-}Cl(R)) \leq \omega(U\text{-}Cl(R)),$$

and thus,

$$\omega(U\text{-}Cl(R)) = \chi(U\text{-}Cl(R)) = |U(R)|.$$

Case 2. $|U(R)| < 2^n - 1$. Let $V_{e1} = \{(e_i, u_1)|e_i \in Id^*(R), u_1 \in U_1(R)\}$. Then we color the vertices of V_{e1} by $1, \cdots, 2^n - 1$. Similar to Case 1, we have

$$\chi(U\text{-}Cl(R)) \leq \omega(U\text{-}Cl(R)),$$

and thus,

$$\omega(U\text{-}Cl(R)) = \chi(U\text{-}Cl(R)) = 2^n - 1.$$

Hence

$$\omega(U\text{-}Cl(R)) = \chi(U\text{-}Cl(R)) = max\{|U(R)|,\ 2^n - 1\}.$$

This completes the proof of Theorem 4.1.

In [9, Theorem 8.7], it was proved that every commutative Artin ring is uniquely a finite direct product of Artin local rings. Then we have the following result.

Corollary 4.2 Let R be a commutative Artin ring with n distinct maximal ideals. Then we have

$$\omega(U\text{-}Cl(R)) = \chi(U\text{-}Cl(R)) = \begin{cases} |U(R)| & \text{if } n = 1, \\ max\{|U(R)|,\ 2^n - 1\} & \text{if } n \geq 2. \end{cases}$$

Proof It is clear that $R \cong R_1 \times \cdots \times R_n$, where R_i is a local ring for each $i \in \{1, \cdots, n\}$. The proof is directly from Theorem 4.1.

Corollary 4.3 Let m, n be positive integers, and p, q be primes with $p \leq q$. Then

$$\omega(U\text{-}Cl(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})) = \chi(U\text{-}Cl(\mathbb{Z}_{p^m} \times \mathbb{Z}_{q^n})) = \begin{cases} 3 & \text{if } p^m + q^n \leq 5, \\ (p^m - p^{m-1})(q^n - q^{n-1}) & \text{if } p^m + q^n > 5. \end{cases}$$

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有限交换环的U-clean 图

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摘要: 设 R 为含幺的有限交换环, 本文引入了环 R 的 U-clean 图 $U\text{-}Cl(R)$ 的概念, 研究了 $U\text{-}Cl(R)$ 的基本性质. 设 p, q 为素数, 给出了 $U\text{-}Cl(\mathbb{Z}_p \times \mathbb{Z}_q)$ 的结构, 证明了 $U\text{-}Cl(\mathbb{Z}_p \times \mathbb{Z}_q)$ 是欧拉图当且仅当 $p = 2$, $q = 2$, 并探究了一些特殊环的 U-clean 图的团数以及点染色数.

关键词: 关U-clean; 欧拉图; 可逆元

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