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# MULTIPLE NON-RADIAL SOLUTIONS TO A MIXED DISPERSION NONLINEAR SCHRÖDINGER EQUATION

HE Ju-hua

(*School of Mathematics, Yunnan Normal University, Kunming 650500, China*)

**Abstract:** In this thesis, we consider the existence of solutions for the following mixed dispersion nonlinear Schrödinger equation

$$\Delta^2 u - \beta\Delta u - \frac{\lambda}{2}\Delta(u^2)u = g(u) \quad \text{in } \mathbb{R}^N, \tag{1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, $\lambda \geq 0$, $\beta \geq 0$. We shall prove that (1) has multiple non-radial solutions by variational method. This paper provides a method to prove compactness for the study of the mixed dispersion nonlinear Schrödinger equation with quasilinear terms.

**Keywords:** multiplicity; non-radial solutions; Quasilinar problem; fourth-order operator

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## 1 Introduction

In this paper we consider the existence of multiple non-radial solutions of the following nonlinear Schrödinger equation with a fourth-order dispersion term

$$\Delta^2 u - \beta\Delta u - \frac{\lambda}{2}\Delta(u^2)u = g(u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, $\lambda \geq 0$ and $\beta \geq 0$.

Nonlinear Schrödinger equation is the basic model of quantum mechanics. It is widely applied in fields such as physics, chemistry, biology, optics, fluid and so on. Schrödinger equations, especially the fourth-order case, have drawn much attention of many researchers all over the world. The existence of the fourth-order dispersion term has an effect on the saturation of the nonlinear term. The fourth-order dispersion equation can simulate the static deflection of an elastic plate in a fluid [1]. It can also be used to simulate the propagation of intense laser beams in bulk media with second-order dispersion terms.

The wide application of this type of equation motivates the relating research in mathematics. V.I. Karpman studied the fourth order nonlinear Schrödinger type equations in one,

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**Biography:** He Juhua (1998–), female, born in Kunming Yunnan, postgraduate, major in nonlinear functional analysis. E-mail: 1745190963@qq.com

two and three dimensions with power-law nonlinearities in [2, 3] and obtained the existence and stability of such solutions under certain conditions.

If $\lambda = 0$, (1.1) reduces to the following mixed dispersion nonlinear Schrödinger equation

$$\Delta^2 u - \beta\Delta u = g(u) \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

By the variational method, d'Avenia et al.[4] obtained infinitely many radial and non-radial solutions of (1.2) in the case of positive and zero mass regimes. The existence of multiple nonradially symmetric solutions of the conformally invariant problem was established by blow-up analysis and Pohozaev's identity in [5]. For other interesting results of the mixed dispersion nonlinear Schrödinger equation, we refer the readers to [6–10] and the references therein.

When the fourth-order dispersion term in (1.1) vanishes, it becomes the following equation with quasilinear operator $\Delta(u^2)u$

$$-\beta\Delta u - \frac{\lambda}{2}\Delta(u^2)u = g(u) \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

Equation (1.3) is related to the standing wave of the following quasilinear equation

$$\beta\Delta\psi + \frac{\lambda}{2}\Delta(\rho(|\psi|^2))\rho'(|\psi|^2)\psi + g(\psi) = 0 \quad \text{in } [0,\infty)\times\mathbb{R}^N, \tag{1.4}$$

where $g$ and $\rho$ are real functions, $\rho(s) = s$, $\beta \geq 0$ and $\lambda \geq 0$. Setting $\psi(t,u) = \exp(-i\alpha t)u(x)$, we obtain equation (1.3).

There are many results about equation (1.4) by variational method, in which different approaches were developed to prove the compactness and smoothness of the corresponding energy functional. In [11], by considering a perturbed functional with parameter $\mu$, and establishing an appropriate estimate as $\mu \to 0$, the authors obtained positive and multiple solutions of the following equation

$$\Delta u + \frac{1}{2}\Delta(u^2)u + f(x,u) = 0 \quad \text{in } \Omega \quad u = 0, \text{ on } \partial\Omega, \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain. The existence of positive and sign-changing solutions was established by Nehari method in [12]. In [13], the authors studied the following equation

$$\begin{cases} -\Delta u - \Delta(u^2)u = g(x,u) & \text{in } \Omega \\ u = 0, \text{ on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain. By using the critical point theorem, they proved the existence of positive solutions, negative solutions and sign-changing solutions. In [14], under certain suitable conditions, multiple nontrivial solutions of equation (1.6) were obtained by using Morse theory. The existence of solutions for quasilinear equations can also be obtained by minimization process [15, 16] and change of variables [17, 18].

Motivated by [4], we will prove the existence of non-radial solutions of (1.1). Assume that the nonlinearity $g$ satisfies the following conditions:

(g1) $g$ is continuous and odd;

(g2) $-\infty < \liminf\limits_{s\to 0} \frac{g(s)}{s} \leq \limsup\limits_{s\to 0} \frac{g(s)}{s} := -m < 0$;

(g3) $\lim\limits_{s\to+\infty} \frac{g(s)}{\exp^{\alpha s^2}} = 0$ for every $\alpha > 0$;

(g4) there exists $s_0 \neq 0$ such that $G(s_0) > 0$, where $G(s) := \int_0^s g(t)dt$; and $\beta \geq 0$.

This kind of conditions has been introduced in [19–21] to study the following equation

$$\begin{cases} -\Delta u = g(u) \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \quad u \not\equiv 0. \end{cases} \tag{1.7}$$

Compared to Equation (1.7), (1.1) contains the fourth-order dispersion and quasilinear terms, which make our study more difficult and interesting. We consider the case $N = 4$. From the viewpoint of Sobolev embedding, when $N = 4$, Sobolev embedding is different from the cases $N \in \{2,3\}$ and $N \geq 5$. When $N \in \{2,3\}$, $H^2(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ holds, and it is easy to deal with inequality scaling. When $N \geq 5$, $H^2(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$, $q \in [2, \frac{2N}{N-4}]$. In comparison, when $N = 4$, $H^2(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$, $q \in [2, +\infty)$. Moreover, with the quasilinear term, it is difficult to prove the smoothness of the energy functional corresponding to equation (1.1) in $H^2(\mathbb{R}^N)$. In the case $N > 6$, $\displaystyle\int_{\mathbb{R}^N} u^{22^*}dx$ is not well defined for any $u \in H^2(\mathbb{R}^N)$, then $\displaystyle\int_{\mathbb{R}^N} u^2|\nabla u|^2 dx$ is not bounded from below, which means that the smoothness of the functional in $H^2(\mathbb{R}^N)$ may not hold. However, when $N = 4$, we find that for every $u \in H^2(\mathbb{R}^N)$, $\displaystyle\int_{\mathbb{R}^N} u^{22^*}dx$ is finite and $\|\nabla u\|_4$ is smaller than or equal to $\|\Delta u\|_2$. So, $\int_{\mathbb{R}^N} u^2|\nabla u|^2 dx$ is well defined in $H^2(\mathbb{R}^N)$.

As to the multiplicity of solutions, we need to prove a sequence of mini-max levels that diverges positively. To achieve this aim, we adopt the method in [22] and introduce a comparison function, follow the argument of [23, proof of Theorem 9.12] and prove that the sequence of mini-max levels is positively divergent. Finally, in a similar way to [24, 25], a bivariate function is introduced to prove compactness. Thus we obtain the multiplicity of solutions.

Recall the definition, (see [26, Definition 1.22]). A subgroup $O \subset O(N)$ is called compatible with $\mathbb{R}^N$ if and only if there exists an $r > 0$ such that

$$\lim_{|y|\to+\infty} m(y,r) = +\infty,$$

where $O(N)$ is an orthogonal group of order $N$ over $\mathbb{R}$ and

$$m(y,r) := \sup\{n \geq 1 : \exists\ \{g_i\}_{i=1}^n \subset O \text{ such that } i \neq j \Rightarrow B(g_i y, r) \cap B(g_j y, r) = \emptyset\}.$$

If $O$ is a subgroup of $O(N)$ compatible with $\mathbb{R}^N$, we define $H^2_O(\mathbb{R}^N)$ as the $O$-invariant functions subspace of $H^2(\mathbb{R}^N)$

$$H^2_O(\mathbb{R}^N) := \{u \in H^2(\mathbb{R}^N) : gu = u, \forall g \in O\}.$$

In order to obtain the non-radial solution for (1.1), define $H_X^2(\mathbb{R}^4) := H_O^2(\mathbb{R}^4) \cap X$, where

$$X := \{u \in D^2(\mathbb{R}^4) : u(x_1, x_2, x_3, x_4) = -u(x_3, x_4, x_1, x_2)\},$$

$$O = O(2) \times O(2).$$

$D^2(\mathbb{R}^4)$ is the completion of $C_0^\infty(\mathbb{R}^4)$ with respect to the norm

$$\|u\|_{D^2} = (\|\Delta u\|_2^2 + \|\nabla u\|_2^2)^{\frac{1}{2}},$$

and

$$D^2(\mathbb{R}^4) = \{u \in D^{1,2}(\mathbb{R}^4) | \Delta u \in L^2(\mathbb{R}^4)\}.$$

Our main result is the following theorem.

**Theorem 1.1** Assume that $N = 4$ and (g1)-(g4) hold. Then (1.1) has a sequence of non-radial solutions $\{u_n\} \subset H_X^2(\mathbb{R}^4)$ such that $I(u_n) \to +\infty$ as $n \to +\infty$.

**Remark 1** In the case $N = 6$, we have $22^* = 2^{**}$, where $2^{**} = \frac{2N}{N-4}$ is the critical Sobolev exponent. It means that both biharmonic operators and quasilinear operators involve critical growth, and our method in this paper can not be applied directly.

**Remark 2** Due to the existence of the biharmonic operators, it is difficult to deal with quasilinear terms by change of variables to get the solution of the equation (1.1).

This paper is organized as follows: In section 2, we introduce some inequalities and compactness results. In section 3, we introduce two functions to prove our main result.

In this paper, we use the following notations:
For $p \in (1, +\infty]$, we denote the usual $L^p(\mathbb{R}^4)$ norm by $\|\cdot\|_p$.
For $y \in \mathbb{R}^4$ and $r > 0$, we denote $B(y, r) := \{x \in \mathbb{R}^4 : |x - y| < r\}$, $B_r := B(0, r)$.
For every integer $k \geq 1$, $\mathbb{B}^k \subset \mathbb{R}^k$ is the closed unit ball centred at the origin, $\mathbb{S}^{k-1} := \partial\mathbb{B}^k$.
$C$ and $C_i$ denote any positive constants, whose values are not relevant.

## 2 Preliminaries

Define the Hilbert space

$$H^2(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : D^\alpha u \in L^2(\mathbb{R}^N), \forall \alpha \in \mathbb{Z}_+^N, |\alpha| \leq 2\},$$

with the inner product

$$\langle u, v\rangle_{H^2} = \int_{\mathbb{R}^N} [\Delta u \Delta v + \nabla u \cdot \nabla v + uv] dx$$

and the norm

$$\|u\|_{H^2} = \langle u, u\rangle_{H^2}^{\frac{1}{2}}.$$

If $\beta \geq 0$, then for any fixed $m' \in (0, m)$ such that $\beta > -2\sqrt{m'}$, the norm

$$\|u\| := (\|\Delta u\|_2^2 + \beta\|\nabla u\|_2^2 + m'\|u\|_2^2)^{\frac{1}{2}}$$

is equivalent to the standard norm $\|u\|_{H^2}$ (see [7]).

Let

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^4}[(\Delta u)^2+\beta|\nabla u|^2+\lambda u^2|\nabla u|^2]dx-\int_{\mathbb{R}^4}G(u)dx.$$

Note that $N=4$, we have $2^*=4$. Hence, in view of the proof of Proposition 2.1 in [4], for any $u\in C_0^\infty(\mathbb{R}^4)$, we conclude from Sobolev inequality that there exists a constant $C>0$ such that

$$\|u\|_4\le C\|\nabla u\|_2,\qquad \text{and}\qquad \|\partial_i u\|_4\le C\|\nabla\partial_i u\|_2,\quad i=1,2,3, or\, 4.$$

Notice that

$$\sum_{i,j}\int_{\mathbb{R}^4}|\partial_{ij}u|^2dx=\int_{\mathbb{R}^4}|\Delta u|^2dx,$$

we have

$$\|\nabla u\|_4\le C\|\Delta u\|_2. \tag{2.1}$$

Hence

$$\int_{\mathbb{R}^4}u^2|\nabla u|^2dx\le\frac{1}{2}\int_{\mathbb{R}^4}(|u|^4+|\nabla u|^4)dx\le C_1(\|u\|_4^4+\|\Delta u\|_2^4)\le C_2\|u\|^4<\infty. \tag{2.2}$$

In addition,

$$\int_{\mathbb{R}^4}u^2|\nabla u|^2dx=\frac{1}{4}\int_{\mathbb{R}^4}|\nabla(u^2)|^2dx\ge C(\int_{\mathbb{R}^4}(u^2)^{2^*}dx)^{\frac{2}{2^*}}.$$

Therefore, the functional $I$ is well defined in $H^2(\mathbb{R}^4)$.

Following [4], when $N=4$, we recall the following results.

**Corollary 2.2** Let $\sigma\ge 2$, $M>0$ and $\alpha>0$ such that $\alpha M^2<32\pi^2$. Then there exists a $C>0$ such that for every $\tau\in(1,32\pi^2/(\alpha M^2)]$ and $u\in H^2(\mathbb{R}^4)$ with $\|u\|\le M$,

$$\int_{\mathbb{R}^4}|u|^\sigma(\exp^{\alpha u^2}-1)dx\le C\|u\|^\sigma_{\frac{\sigma\tau}{\tau-1}}.$$

**Corollary 2.3** Let $O\subset O(4)$ a subgroup compatible with $\mathbb{R}^4$. Then the following embedding is compact:

$$H^2_O(\mathbb{R}^4)\hookrightarrow L^p(\mathbb{R}^4),\qquad \text{for all } p\in(2,+\infty).$$

**Proposition 2.1** Let $F\in C^1(\mathbb{R}^4)$ be a function such that $F(0)=0$ and

$$\lim_{s\to 0}\frac{F^{'}(s)}{|s|}=\lim_{|s|\to+\infty}\frac{F^{'}(s)}{\exp^{\alpha s^2}}=0,\qquad \text{for all } \alpha>0,$$

and let $\{u_n\}$ be a bounded sequence of O-invariant functions in $H^2(\mathbb{R}^4)$, for a suitable subgroup $O\subset O(4)$ compatible with $\mathbb{R}^4$, such that $u_n\to u_0$ a.e. in $\mathbb{R}^4$ for some $u_0\in H^2(\mathbb{R}^4)$. Then

$$\lim_n\int_{\mathbb{R}^4}F^{'}(u_n)u_ndx=\int_{\mathbb{R}^4}F^{'}(u_0)u_0dx.$$

# 3 Proof of the Main Result

Following [4] and [22], we introduce the functions $h(s) \in C(\mathbb{R},\mathbb{R})$ and $\overline{h}(s) \in C(\mathbb{R},\mathbb{R})$ by

$$h(s) := \begin{cases} (m's + g(s))^+ & \text{for} \quad s \geq 0, \\ -h(-s) & \text{for} \quad s < 0, \end{cases}$$

$$\overline{h}(s) := \begin{cases} s^{q-1} \sup\limits_{0<t\leq s} \dfrac{h(t)}{t^{q-1}} & \text{for} \quad s > 0, \\ 0 & \text{for} \quad s = 0, \\ -\overline{h}(-s) & \text{for} \quad s < 0, \end{cases}$$

where $q \in (4, +\infty)$.

We define

$$H(s) := \int_0^s h(t)dt \quad \text{and} \quad \overline{H}(s) := \int_0^s \overline{h}(t)dt.$$

Similar to [4, Lemma 2.13], according to the definition of $h$, $\overline{h}$, $H$ and $\overline{H}$, we have the following lemma.

**Lemma 3.4** The following properties hold.

(a) There exists a $\delta_0 > 0$ such that $h(s) = \overline{h}(s) = H(s) = \overline{H}(s) = 0$ for every $s \in [-\delta_0, \delta_0]$.

(b) The functions $h$ and $\overline{h}$ satisfy (g3). Moreover, for every $\alpha > 0$

$$\lim_{s\to+\infty} \frac{H(s)}{\exp^{\alpha s^2}} = \lim_{s\to+\infty} \frac{\overline{H}(s)}{\exp^{\alpha s^2}} = 0.$$

(c) For every $s \geq 0$, we have that $\overline{h}(s) \geq h(s) \geq g(s) + m's$ and $\overline{H}(s) \geq H(s) \geq G(s) + m's^2/2$.

(d) The function $s \longmapsto \overline{h}(s)/s^{q-1}$ is non-decreasing on $(0, +\infty)$ and $\overline{h}(s)s \geq q\overline{H}(s) \geq 0$ for all $s \in \mathbb{R}$.

By the definition of $\overline{h}$ and $\overline{H}$, we conclude that

$$\forall\ \alpha > 0, \sigma \geq 2, \exists\ C > 0 \text{ such that } \overline{h}(s) \leq C(\exp^{\alpha s^2} - 1)s^{\sigma-1} \text{ for } s \geq 0, \tag{3.1}$$

and

$$\forall\ \alpha > 0, \sigma \geq 2, \exists\ C > 0 \text{ such that } \overline{H}(s) \leq C(\exp^{\alpha s^2} - 1)|s|^{\sigma} \text{ for } s \in \mathbb{R}. \tag{3.2}$$

The same estimates hold for the functions $h$ and $H$.

Note that, since the function $g : \mathbb{R} \to \mathbb{R}$ satisfies conditions (g1)-(g3), it holds that

$$\forall \alpha > 0, \sigma \geq 2, \exists\ C > 0 \text{ such that } g(s) \leq C(\exp^{\alpha s^2} - 1)s^{\sigma-1} \text{ for } s \geq 0. \tag{3.3}$$

We define a comparison $C^1$ functional $\overline{I}: H^2(\mathbb{R}^4) \to \mathbb{R}$ by

$$\overline{I}(u) = \frac{1}{2}\int_{\mathbb{R}^4} [(\Delta u)^2 + \beta|\nabla u|^2 + m'u^2 + \lambda u^2|\nabla u|^2]dx - \int_{\mathbb{R}^4} \overline{H}(u)dx.$$

Then, we can prove the following proposition([4, Proposition 2.14]).

**Proposition 3.2** The following properties hold.

(a) $\overline{I}(u) \le I(u)$ for any $u \in H^2_X(\mathbb{R}^4)$;

(b) There exist $\mu, \rho > 0$ such that $I(u) \ge \overline{I}(u) \ge 0$ for any $\|u\| \le \rho$ and $I(u) \ge \overline{I}(u) \ge \mu$ for any $\|u\| = \rho$;

(c) For every integer $k \ge 1$, there exists an odd mapping $\gamma_k \in C(\mathbb{S}^{k-1}, H^2_X(\mathbb{R}^4))$ such that $\overline{I} \circ \gamma_k \le I \circ \gamma_k < 0$;

(d) $\overline{I}(u)$ satisfies the Palais-Smale condition for any $u \in H^2_X(\mathbb{R}^4)$.

**Proof** (a) By Lemma 3.4(c) and functions $\overline{H}, G$ being even, we conclude that $\overline{I}(u) \le I(u)$ for any $u \in H^2_X(\mathbb{R}^4)$.

(b) Note that (a), it suffices to prove the statement holds for $\overline{I}$.

Let $\alpha \in (0, 32\pi^2)$ and $\sigma > 2$, it follows from (3.2) and Corollary 2.2 that there exists a $C > 0$ such that for any $u \in H^2(\mathbb{R}^4)$ with $\|u\| \le 1$

$$\begin{aligned}\overline{I}(u) &\ge \frac{1}{2}\|\Delta u\|_2^2 + \frac{\beta}{2}\|\nabla u\|_2^2 + \frac{m'}{2}\|u\|_2^2 + \frac{\lambda}{2}\int_{\mathbb{R}^4} u^2|\nabla u|^2 dx - C\int_{\mathbb{R}^4} (\exp^{\alpha u^2} - 1)|u|^\sigma dx \\ &\ge \frac{1}{2}\|\Delta u\|_2^2 + \frac{\beta}{2}\|\nabla u\|_2^2 + \frac{m'}{2}\|u\|_2^2 + \frac{\lambda}{2}\int_{\mathbb{R}^4} u^2|\nabla u|^2 dx - C\|u\|^\sigma_{\frac{\sigma\tau}{\tau-1}}\end{aligned}$$

for some fixed $\tau \in (1, 32\pi^2/\alpha]$. By the Sobolev inequality, it is easy to get the conclusion.

(c) By (a), it suffices to prove the statement holds for $I$. It follows from Lemma 3.4 in [27] that for every integer $k \ge 1$ there exists an odd continuous mapping $\pi_k : \mathbb{S}^{k-1} \to H^2_X(\mathbb{R}^4)$ such that

$$\int_{\mathbb{R}^4} G(\pi_k(\zeta))dx \ge 1, \qquad \text{for all } \zeta \in \mathbb{S}^{k-1}.$$

Set $\alpha > 0$, define $\gamma_k(\zeta) := \pi_k(\zeta)(\cdot/\alpha)$. Then

$$\begin{aligned}I(\gamma_k(\zeta)) &= \frac{1}{2}\|\Delta\pi_k(\zeta)\|_2^2 + \frac{\beta\alpha^2}{2}\|\nabla\pi_k(\zeta)\|_2^2 + \frac{\lambda\alpha^2}{2}\int_{\mathbb{R}^4} \pi_k^2(\zeta)|\nabla\pi_k(\zeta)|^2 dx - \alpha^4\int_{\mathbb{R}^4} G(\pi_k(\zeta))dx \\ &\le \frac{1}{2}\|\Delta\pi_k(\zeta)\|_2^2 + \frac{\beta\alpha^2}{2}\|\nabla\pi_k(\zeta)\|_2^2 + \frac{\lambda\alpha^2}{2}\int_{\mathbb{R}^4} \pi_k^2(\zeta)|\nabla\pi_k(\zeta)|^2 dx - \alpha^4.\end{aligned}$$

Therefore, the statement holds for sufficiently large $\alpha$.

(d) Let $\{u_n\} \subset H_X^2(\mathbb{R}^4)$ be a Palais-Smale sequence of the functional $\overline{I}$. By Lemma 3.4(d), we have

$$
\begin{aligned}
&\overline{I}(u_n) - \frac{1}{q}\langle \overline{I}'(u_n), u_n\rangle \\
=&\frac{1}{2}\|u_n\|^2 + \frac{\lambda}{2}\int_{\mathbb{R}^4} u_n^2|\nabla u_n|^2 dx - \int_{\mathbb{R}^4} \overline{H}(u_n)dx - \frac{1}{q}\|u_n\|^2 - \frac{2\lambda}{q}\int_{\mathbb{R}^4} u_n^2|\nabla u_n|^2 dx \\
&+ \frac{1}{q}\int_{\mathbb{R}^4} \overline{h}(u_n)u_n dx \\
=&(\frac{1}{2} - \frac{1}{q})\|u_n\|^2 + (\frac{\lambda}{2} - \frac{2}{q}\lambda)\int_{\mathbb{R}^4} u_n^2|\nabla u_n|^2 dx + \int_{\mathbb{R}^4} (\frac{1}{q}\overline{h}(u_n)u_n - \overline{H}(u_n))dx \\
\geq&(\frac{1}{2} - \frac{1}{q})\|u_n\|^2 + (\frac{\lambda}{2} - \frac{2}{q}\lambda)\int_{\mathbb{R}^4} u_n^2|\nabla u_n|^2 dx.
\end{aligned}
$$

Thus, $\|u_n\|$ is bounded.

Up to a subsequence, we can assume that $u_n \rightharpoonup u$ in $H_X^2(\mathbb{R}^4)$. By Corollary 2.3, $u_n \to u$ in $L^p(\mathbb{R}^4)$ for $p \in (2, +\infty)$, then we have

$$
\begin{aligned}
o(1) &= \langle \overline{I}'(u_n) - \overline{I}'(u), u_n - u\rangle \\
&= \|u_n - u\|^2 + \lambda\int_{\mathbb{R}^4} (u_n^2\nabla u_n\nabla(u_n - u) - u^2\nabla u\nabla(u_n - u))dx \\
&\quad + \lambda\int_{\mathbb{R}^4} (u_n(u_n - u)|\nabla u_n|^2 - u(u_n - u)|\nabla u|^2)dx - \int_{\mathbb{R}^4} (\overline{h}(u_n) - \overline{h}(u))(u_n - u)dx \\
&= \|u_n - u\|^2 + \lambda\int_{\mathbb{R}^4} u_n^2|\nabla u_n - \nabla u|^2 dx + \lambda\int_{\mathbb{R}^4} (u_n^2 - u^2)\nabla u(\nabla u_n - \nabla u)dx \\
&\quad + \lambda\int_{\mathbb{R}^4} (u_n(u_n - u)|\nabla u_n|^2 - u(u_n - u)|\nabla u|^2)dx - \int_{\mathbb{R}^4} (\overline{h}(u_n) - \overline{h}(u))(u_n - u)dx.
\end{aligned}
$$

According to Hölder inequality and (2.1), we have

$$
\begin{aligned}
|\lambda\int_{\mathbb{R}^4} (u_n^2 - u^2)\nabla u(\nabla u_n - \nabla u)dx| &\leq\lambda\int_{\mathbb{R}^4} |u_n - u||u_n + u|\nabla u(\nabla u_n - \nabla u)dx \\
&\leq\lambda\|u_n - u\|_4\|u_n + u\|_4\|\nabla u\|_4\|\nabla u_n - \nabla u\|_4 \\
&\leq\lambda\|u_n - u\|_4(\|u_n\|_4 + \|u\|_4)\|\Delta u\|_2\|\Delta u_n - \Delta u\|_2 \\
&\leq\lambda\|u_n - u\|_4(\|u_n\|_4 + \|u\|_4)\|u\|(\|u_n\| + \|u\|) \\
&=o(1),
\end{aligned}
$$

and

$$
\begin{aligned}
|\lambda\int_{\mathbb{R}^4} (u_n(u_n - u)|\nabla u_n|^2 - u(u_n - u)|\nabla u|^2)| &\leq\lambda\int_{\mathbb{R}^4} (|u_n|\nabla u_n|^2 - u|\nabla u|^2|)|u_n - u|dx \\
&\leq\lambda\|u_n\|_4\|\nabla u_n\|_4^2\|u_n - u\|_4 + \lambda\|u\|_4\|\nabla u\|_4^2\|u_n - u\|_4 \\
&\leq\lambda(\|u_n\|_4\|u_n\| + \|u\|_4\|u\|)\|u_n - u\|_4 \\
&=o(1).
\end{aligned}
$$

Arguing in a similar way to [4, proof of Proposition 2.10]. Let $M > 0$ be such that $\|u_n\| \le M$, choose $\alpha > 0$ and $p_1, p_2, p_3 > 1$ such that $1/p_1 + 1/p_2 + 1/p_3 = 1$, $\alpha p_1 M^2 \le 32\pi^2$, $p_2 \ge 2/(\sigma - 1)$ and $p_3 > 2$. Combining with (3.1), we have

$$
\begin{aligned}
&|\int_{\mathbb{R}^4} (\overline{h}(u_n) - \overline{h}(u))(u_n - u)dx| \\
\le& C\int_{\mathbb{R}^4} (\exp^{\alpha u_n^2} - 1)|u_n|^{\sigma-1}|u_n - u|dx + C\int_{\mathbb{R}^4} (\exp^{\alpha u^2} - 1)|u|^{\sigma-1}|u_n - u|dx \\
\le& C(\int_{\mathbb{R}^4} (\exp^{\alpha u_n^2} - 1)^{p_1} dx)^{\frac{1}{p_1}} \|u_n\|^{\sigma-1}_{(\sigma-1)p_2}\|u_n - u\|_{p_3} + C(\int_{\mathbb{R}^4} (\exp^{\alpha u^2} - 1)^{p_1} dx)^{\frac{1}{p_1}} \|u\|^{\sigma-1}_{(\sigma-1)p_2}\|u_n - u\|_{p_3} \\
\le& C(\int_{\mathbb{R}^4} (\exp^{\alpha u_n^2 p_1} - 1)dx)^{\frac{1}{p_1}} \|u_n\|^{\sigma-1}_{(\sigma-1)p_2}\|u_n - u\|_{p_3} + C(\int_{\mathbb{R}^4} (\exp^{\alpha u^2 p_1} - 1)dx)^{\frac{1}{p_1}} \|u\|^{\sigma-1}_{(\sigma-1)p_2}\|u_n - u\|_{p_3} \\
\le& C\|u_n - u\|_{p_3} \\
=& o(1).
\end{aligned}
$$

Hence

$$
o(1) = \|u_n - u\|^2 + \lambda \int_{\mathbb{R}^4} u_n^2 |\nabla u_n - \nabla u|^2 dx + o(1),
$$

which implies that $\|u_n - u\| \to 0$. The proof is completed.

Following [4], we define

$$
\Gamma_k := \{\gamma \in C(\mathbb{B}^k, H^2_X(\mathbb{R}^4)), \gamma \text{ is odd and } \gamma|_{\partial \mathbb{B}^k} = \gamma_k\},
$$

where $\gamma_k : \mathbb{S}^{k-1} \to H^2_X(\mathbb{R}^4)$ is given in Proposition 3.2(c). We remark that

$$
\widetilde{\gamma}_k(\zeta) := \begin{cases} |\zeta|\gamma_k(\dfrac{\zeta}{|\zeta|}) & \text{for} \quad \zeta \ne 0 \\ 0 & \text{for} \quad \zeta = 0 \end{cases}
$$

belongs to $\Gamma_k$, hence $\Gamma_k \ne \varnothing$ for any positive integer $k$.

Define

$$
d_k := \inf_{\gamma \in \Gamma_k} \sup_{\zeta \in \mathbb{B}^k} I(\gamma(\zeta)), \qquad c_k := \inf_{\gamma \in \Gamma_k} \sup_{\zeta \in \mathbb{B}^k} \overline{I}(\gamma(\zeta)).
$$

We will prove that each $d_k$ is a critical value of $I$.

It follows from Proposition 3.2(b) that $d_k \ge c_k \ge \mu > 0$. In view of [4, 23], we have the following result.

**Lemma 3.5** $\lim_{k \to +\infty} c_k = +\infty$.

**Proof** For any integer $k \ge 1$, we apply an argument in [23, Chapter 9]. Let

$$
\Sigma_k := \{\gamma(\overline{\mathbb{B}^n \backslash Y}) : \gamma \in \Gamma_n, n \ge k, \mathbb{R}^n \backslash \{0\} \supset Y = \overline{Y} = -Y, \text{genus}(Y) \le n - k\},
$$

where genus(Y) is the Krasnoselski's genus of Y. Next we define the sequence of values

$$
b_k := \inf_{A \in \Sigma_k} \sup_{u \in A} \overline{I}(u).
$$

Then we have $c_k \ge b_k$ for any integer $k \ge 1$ and $\{b_k\}$ is nondecreasing.

Since $\overline{I}(u)$ satisfies the Palais-Smale condition, arguing in a similar way to [23, proof of Theorem 9.12]. Assume that $\{b_k\}$ is bounded, then $b_k \to \overline{b} < \infty$ as $k \to \infty$. If $b_k = \overline{b}$ for sufficiently large $k$, from [23, Proposition 9.30] we have genus$(K_{\overline{b}}) = \infty$, where $K_{\overline{b}} = \{u \in H^2(\mathbb{R}^4) | \overline{I}(u) = \overline{b}, \overline{I}'(u) = 0\}$. But by Palais-Smale condition, $K_{\overline{b}}$ is compact, then genus$(K_{\overline{b}}) < \infty$ by [23, Proposition 7.5 of 5°]. Thus $\overline{b} > b_k$ for every $k \in \mathbb{N}$. Let

$$W \equiv \{u \in H^2(\mathbb{R}^4) | b_{j+1} \leq \overline{I}(u) \leq \overline{b} \text{ and } \overline{I}'(u) = 0\},$$

then $W$ is compact. Again by [23, Proposition 7.5 of 5°], we see that genus$(W) < \infty$ and there exists a $\delta > 0$ such that genus$(N_\delta(W))$ = genus$(W)$, where $N_\delta(W) = \{u \in H^2(\mathbb{R}^4) : \|u - W\| \leq \delta\}$. Let $s = \max\{\text{genus}(W), j+1\}$, we apply deformation theorem [23, Theorem A.4] with $c = \overline{b}$, $\overline{\varepsilon} = \overline{b} - b_s$ and $O = N_\delta(W)$ yields an $\varepsilon$ and $\eta$ such that

$$\eta(1, A_{\overline{b}+\varepsilon} \setminus O) \subset A_{\overline{b}-\varepsilon}, \tag{3.4}$$

where $A_b = \{u \in H^2(\mathbb{R}^4) | \overline{I}(u) \leq d\}$.

Choose $k \in \mathbb{N}$ such that $b_k > \overline{b} - \varepsilon$ and $A \in \Sigma_{k+s}$ such that

$$\sup_A \overline{I} \leq \overline{b} + \varepsilon. \tag{3.5}$$

From [23, Proposition 9.18 of 4°] we have $\overline{A \setminus O} \in \Sigma_k$. Furthermore, for each finite dimensional subspace $E \subset H^2(\mathbb{R}^4)$ by the equivalency of all norms in the finite dimensional space, there exists a constant $C_1 > 0$ such that

$$\|u\|_{\frac{\sigma\tau}{\tau-1}} \geq C_1 \|u\|, \quad \forall u \in E, \tag{3.6}$$

where $\sigma > 2$. Then, it follows from (3.6), (2.2), (3.2) and Corollary 2.2 that there exists a large $r > 0$ such that $\overline{I} < 0$ on $E \setminus B_r$ and $\overline{I} \leq 0$ on $\partial B_r$, then $\eta(1, \cdot) = id$ on $\partial B_r$ by [23, Theorem A.4 of 2°]. Consequently, [23, Proposition 9.18 of 3°] implies that $\eta(1, \overline{A \setminus O}) \in \Sigma_k$. According to (3.4), (3.5) and the definition of $b_k$, we have

$$b_k \leq \sup_{\eta(1, \overline{A \setminus O})} \overline{I} \leq \overline{b} - \varepsilon < b_k.$$

This is a contradiction. We have $\lim\limits_{k \to +\infty} b_k = +\infty$. Thus $\lim\limits_{k \to +\infty} c_k = +\infty$.

Motivated by [22], we introduce an auxiliary functional $J(s, u) \in C^1(\mathbb{R} \times H^2(\mathbb{R}^4), \mathbb{R})$ by

$$J(s, u) = \frac{1}{2}\|\Delta u\|_2^2 + \beta \frac{\exp^{2s}}{2} \|\nabla u\|_2^2 + \lambda \frac{\exp^{2s}}{2} \int_{\mathbb{R}^4} u^2 |\nabla u|^2 dx - \exp^{4s} \int_{\mathbb{R}^4} G(u) dx.$$

For every $(s, u) \in \mathbb{R} \times H^2(\mathbb{R}^4)$, $J(0, u) = I(u), J(s, u) = I(u(\exp^{-s} \cdot))$. We equip $\mathbb{R} \times H^2(\mathbb{R}^4)$ with a standard product norm $\|(s, u)\|_{\mathbb{R} \times H^2(\mathbb{R}^4)} = (|s|^2 + \|u\|^2)^{1/2}$.

Let

$$\widetilde{d}_k := \inf_{\widetilde{\gamma} \in \widetilde{\Gamma}_k} \sup_{\zeta \in \mathbb{B}^k} J(\widetilde{\gamma}(\zeta)),$$

where $\widetilde{\Gamma_k} := \{\widetilde{\gamma} = (\widetilde{\gamma}_1, \widetilde{\gamma}_2) \in C(\mathbb{B}^k, \mathbb{R} \times H_X^2(\mathbb{R}^4)) : \widetilde{\gamma}_1 \text{ is even}, \ \widetilde{\gamma}_2 \text{ is odd}, \text{ and } \widetilde{\gamma}|_{\partial \mathbb{B}^k} = (0, \gamma_k)\}$ and $\gamma_k$ is given in Proposition 3.2(c). Then applying [22, Section 4], for any integer $k \geq 1$, we see that $d_k = \widetilde{d}_k$ and the following properties.

**Proposition 3.3** For every integer $k \geq 1$ there exists a sequence $\{(s_n, u_n)\} \subset \mathbb{R} \times H^2_X(\mathbb{R}^4)$ such that

(a) $\lim\limits_{n\to\infty} s_n = 0$;

(b) $\lim\limits_{n\to\infty} J(s_n, u_n) = d_k$;

(c) $\lim\limits_{n\to\infty} \partial_s J(s_n, u_n) = 0$;

(d) $\lim\limits_{n\to\infty} \partial_u J(s_n, u_n) = 0$ in $(H^2_X(\mathbb{R}^4))^*$.

The proof of this proposition can be found in [22, Proposition 4.2]. We omit it here.

We have some further results about the sequence found in Proposition 3.3.

**Lemma 3.6** Assume that $\{(s_n, u_n)\} \subset \mathbb{R} \times H^2_X(\mathbb{R}^4)$ satisfies (a)-(d) of Proposition 3.3. Then $\{u_n\}$ is bounded.

**Proof** By Proposition 3.3(b) and (c), we have

$$\frac{1}{2}\|\Delta u_n\|_2^2 + \beta\frac{\exp^{2s_n}}{2}\|\nabla u_n\|_2^2 + \lambda\frac{\exp^{2s_n}}{2}\int_{\mathbb{R}^4} u_n^2|\nabla u_n|^2dx - \exp^{4s_n}\int_{\mathbb{R}^4} G(u_n)dx \to d_k, \quad (3.7)$$

$$\beta\exp^{2s_n}\|\nabla u_n\|_2^2 + \lambda\exp^{2s_n}\int_{\mathbb{R}^4} u_n^2|\nabla u_n|^2dx - 4\exp^{4s_n}\int_{\mathbb{R}^4} G(u_n)dx \to 0,$$

which implies that

$$2\|\Delta u_n\|_2^2 + \beta\exp^{2s_n}\|\nabla u_n\|_2^2 + \lambda\exp^{2s_n}\int_{\mathbb{R}^4} u_n^2|\nabla u_n|^2dx \to 4d_k. \quad (3.8)$$

By (a) of Proposition 3.3, there exists a $N_1 > 0$ such that $\exp^{2s_n} \geq \frac{1}{2}$ as $n > N_1$, then

$$2\|\Delta u_n\|_2^2+\beta\exp^{2s_n}\|\nabla u_n\|_2^2+\lambda\exp^{2s_n}\int_{\mathbb{R}^4} u_n^2|\nabla u_n|^2dx \geq 2\|\Delta u_n\|_2^2+\frac{\beta}{2}\|\nabla u_n\|_2^2+\frac{\lambda}{2}\int_{\mathbb{R}^4} u_n^2|\nabla u_n|^2dx.$$

Since (3.8), there exists $N_2 > 0$ such that

$$2\|\Delta u_n\|_2^2 + \beta\exp^{2s_n}\|\nabla u_n\|_2^2 + \lambda\exp^{2s_n}\int_{\mathbb{R}^4} u_n^2|\nabla u_n|^2dx \leq 4d_k + 1$$

as $n > N_2$. Let $N = \max\{N_1, N_2\}$, then

$$2\|\Delta u_n\|_2^2 + \frac{\beta}{2}\|\nabla u_n\|_2^2 + \frac{\lambda}{2}\int_{\mathbb{R}^4} u_n^2|\nabla u_n|^2dx \leq 4d_k + 1$$

as $n > N$. Set $a_n = 2\|\Delta u_n\|_2^2 + \frac{\beta}{2}\|\nabla u_n\|_2^2 + \frac{\lambda}{2}\int_{\mathbb{R}^4} u_n^2|\nabla u_n|^2dx$ as $n = 1, 2, \cdots, N-1$, $M = \max\{4d_k + 1, a_1, \cdots, a_{N-1}\}$, then

$$2\|\Delta u_n\|_2^2 + \frac{\beta}{2}\|\nabla u_n\|_2^2 + \frac{\lambda}{2}\int_{\mathbb{R}^4} u_n^2|\nabla u_n|^2dx \leq M$$

for all $n$. Hence, $\{\|\Delta u_n\|_2\}$, $\{\|\nabla u_n\|_2\}$, $\{\int_{\mathbb{R}^4} u_n^2|\nabla u_n|^2dx\}$ are bounded. By (3.7), $\{\int_{\mathbb{R}^4} G(u_n)dx\}$ is bounded.

Up to a subsequence, assume that $t_n := \|u_n\|_2^{1/2} \to +\infty$ and define $v_n(x) = u_n(t_n x)$. Then

$$\|v_n\|_2^2 = 1, \quad \|\nabla v_n\|_2^2 = t_n^{-2}\|\nabla u_n\|_2^2,$$
$$\|\Delta v_n\|_2^2 = \|\Delta u_n\|_2^2, \quad \int_{\mathbb{R}^4} v_n^2|\nabla v_n|^2dx = t_n^{-2}\int_{\mathbb{R}^4} u_n^2|\nabla u_n|^2dx,$$

which implies that $\{v_n\}$ is bounded in $H^2(\mathbb{R}^4)$. Since $|\nabla v_n| \to 0$ in $L^2(\mathbb{R}^4)$, $v_n \rightharpoonup 0$ in $H^2(\mathbb{R}^4)$.

Moreover,

$$\begin{aligned}
&t_n^4|t_n^{-4}\|\Delta v_n\|_2^2 + \beta \exp^{2s_n} t_n^{-2}\|\nabla v_n\|_2^2 + 2\lambda \exp^{2s_n} t_n^{-2}\int_{\mathbb{R}^4} v_n^2|\nabla v_n|^2dx - \exp^{4s_n}\int_{\mathbb{R}^4} g(v_n)v_ndx| \\
=&|\|\Delta u_n\|_2^2 + \beta\exp^{2s_n}\|\nabla u_n\|_2^2 + 2\lambda\exp^{2s_n}\int_{\mathbb{R}^4} u_n^2|\nabla u_n|^2dx - \exp^{4s_n}\int_{\mathbb{R}^4} g(u_n)u_ndx| \\
=&|\partial_u J(s_n,u_n)[u_n]| \\
\le&\|\partial_u J(s_n,u_n)\|_{(H_X^2(\mathbb{R}^4))^*}\|u_n\| \\
=&\|\partial_u J(s_n,u_n)\|_{(H_X^2(\mathbb{R}^4))^*}\sqrt{\|\Delta v_n\|_2^2 + \beta t_n^2\|\nabla v_n\|_2^2 + m' t_n^4}.
\end{aligned}$$

We obtain

$$\delta_n := t_n^{-4}\|\Delta v_n\|_2^2 + \beta\exp^{2s_n}t_n^{-2}\|\nabla v_n\|_2^2 + 2\lambda\exp^{2s_n}t_n^{-2}\int_{\mathbb{R}^4} v_n^2|\nabla v_n|^2dx - \exp^{4s_n}\int_{\mathbb{R}^{4N}} g(v_n)v_ndx \to 0.$$

Hence, applying Proposition 2.1 to the function $F' = h$, by Lemma 3.4(c) and (g1), for large $n$ we have

$$\begin{aligned}
\frac{m'}{2} &\le t_n^{-4}\|\Delta v_n\|_2^2 + \beta\exp^{2s_n}t_n^{-2}\|\nabla v_n\|_2^2 + 2\lambda\exp^{2s_n}t_n^{-2}\int_{\mathbb{R}^4} v_n^2|\nabla v_n|^2dx + m'\exp^{4s_n} \\
&= \exp^{4s_n}\int_{\mathbb{R}^4} m' v_n^2dx + \exp^{4s_n}\int_{\mathbb{R}^4} g(v_n)v_ndx + \delta_n \\
&\le \exp^{4s_n}\int_{\mathbb{R}^4} h(v_n)v_ndx \to 0,
\end{aligned}$$

which is a contradiction.

**Lemma 3.7** Assume that $\{(s_n,u_n)\} \subset \mathbb{R}\times H_X^2(\mathbb{R}^4)$ satisfies (a)-(d) of Proposition 3.3. Then $\{u_n\}$ contains a convergent subsequence.

**Proof** Since $\{u_n\}$ is a bounded sequence in $H^2(\mathbb{R}^4)$ from Lemma 3.6, up to a subsequence, assume that $u_n \rightharpoonup u$ in $H^2(\mathbb{R}^4)$ and $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^4$. By Proposition 3.3(d), we obtain $|\partial_u J(s_n,u_n)[u_n-u]| \to 0$.

Moreover

$$\begin{aligned}
|\partial_u J(s_n,u)[u_n-u]| =& \int_{\mathbb{R}^4}\Delta u\Delta(u_n-u)dx + \beta\exp^{2s_n}\int_{\mathbb{R}^4}\nabla u\nabla(u_n-u)dx \\
&+ \lambda\exp^{2s_n}\int_{\mathbb{R}^4} u^2\nabla u\nabla(u_n-u)dx + \lambda\exp^{2s_n}\int_{\mathbb{R}^4} u(u_n-u)|\nabla u|^2dx \\
&- \exp^{4s_n}\int_{\mathbb{R}^4} g(u)(u_n-u)dx.
\end{aligned}$$

We may estimate the terms involved as follows:

Define $f(\varphi) := \int_{\mathbb{R}^4} \Delta u \Delta \varphi dx$, it is clear that $f$ is a bounded linear functional in $H^2(\mathbb{R}^4)$. Then, by the definition of weak convergence, we conclude that $\int_{\mathbb{R}^4} \Delta u \Delta (u_n - u) dx \to 0$. Similarly, by Proposition 3.3(a), we can prove that $\beta \exp^{2s_n} \int_{\mathbb{R}^4} \nabla u \nabla (u_n - u) dx \to 0$ and $\lambda \exp^{2s_n} \int_{\mathbb{R}^4} u^2 \nabla u \nabla (u_n - u) dx \to 0$.

From the Hölder inequality, Corollary 2.3 and (2.1), we also have

$$\lambda \exp^{2s_n} \int_{\mathbb{R}^4} u(u_n - u)|\nabla u|^2 dx \to 0.$$

Following (3.3) and the same arguments as in the proof of Proposition 3.2(d), choose $p_1, p_2, p_3$ such that $1/p_1 + 1/p_2 + 1/p_3 = 1$, then

$$|\int_{\mathbb{R}^4} g(u)(u_n - u) dx| \le C \int_{\mathbb{R}^4} |u|^{\sigma-1}(\exp^{\alpha u^2} - 1)|u_n - u| \le C\|u_n - u\|_{p_3} \to 0.$$

Hence, according to Lemma 3.4(c) and again by the proof of Proposition 3.2(d), we have

$$\begin{aligned}
o(1) &= (\partial_u J(s_n, u_n) - \partial_u J(s_n, u))[u_n - u] \\
&= \int_{\mathbb{R}^4} (\Delta(u_n - u))^2 dx + \beta \exp^{2s_n} \int_{\mathbb{R}^4} |\nabla(u_n - u)|^2 dx \\
&\quad + \lambda \exp^{2s_n} \int_{\mathbb{R}^4} (u_n^2 \nabla u_n - u^2 \nabla u)\nabla(u_n - u) dx + \lambda \exp^{2s_n} \int_{\mathbb{R}^4} (u_n|\nabla u_n|^2 - u|\nabla u|^2)(u_n - u) dx \\
&\quad - \exp^{4s_n} \int_{\mathbb{R}^4} (g(u_n) - g(u))(u_n - u) dx \\
&\ge \frac{1}{2}\|\Delta(u_n - u)\|_2^2 + \frac{\beta}{2}\|\nabla(u_n - u)\|_2^2 + \frac{m'}{2}\|(u_n - u)\|_2^2 - \frac{m'}{2}\|(u_n - u)\|_2^2 \\
&\quad + \frac{\lambda}{2}\int_{\mathbb{R}^4} u_n^2 |\nabla u_n - \nabla u|^2 dx + \frac{\lambda}{2}\int_{\mathbb{R}^4} (u_n^2 - u^2)\nabla u(\nabla u_n - \nabla u) dx \\
&\quad + \frac{\lambda}{2}\int_{\mathbb{R}^4} (u_n|\nabla u_n|^2 - u|\nabla u|^2)(u_n - u) dx - \frac{1}{2}\int_{\mathbb{R}^4} (g(u_n) - g(u))(u_n - u) dx \\
&= \frac{1}{2}\|u_n - u\|^2 + \frac{\lambda}{2}\int_{\mathbb{R}^4} u_n^2 |\nabla u_n - \nabla u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^4} (m'(u_n - u)^2 + g(u_n - u)(u_n - u)) dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^4} g(u_n - u)(u_n - u) dx + o(1) \\
&\ge \frac{1}{2}\|u_n - u\|^2 + \frac{\lambda}{2}\int_{\mathbb{R}^4} u_n^2 |\nabla u_n - \nabla u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^4} h(u_n - u)(u_n - u) dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^4} g(u_n - u)(u_n - u) dx + o(1) \\
&= \frac{1}{2}\|u_n - u\|^2 + \frac{\lambda}{2}\int_{\mathbb{R}^4} u_n^2 |\nabla u_n - \nabla u|^2 dx + o(1).
\end{aligned}$$

It follows that $\|u_n - u\| \to 0$.

**Proof** [The proof of Theorem 1.1] Fix $k \geq 1$, we prove that $d_k = \widetilde{d}_k$ is a critical value of $I$. Let $\{(s_n, u_n)\} \subset \mathbb{R} \times H_X^2(\mathbb{R}^4)$ be a sequence obtained in Proposition 3.3, in view of Lemma 3.7, we may assume that there exists a $u_0 \in H_X^2(\mathbb{R}^4)$ such that $u_n \to u_0$ in $H_X^2(\mathbb{R}^4)$. Note that, $\lim\limits_{n\to\infty} s_n = 0$, we obtain

$$I(u_0) = J(0, u_0) = d_k \quad \text{and} \quad I'(u_0) = \partial_u J(0, u_0) = 0.$$

This shows that $u_0$ is non-radial solution of (1.1).

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# 混合色散非线性薛定谔方程的无穷多非径向解

何菊花

(云南师范大学数学学院, 云南 昆明 650500)

**摘 要:** 本文考虑了下列混合色散非线性薛定谔方程解的存在性

$$\Delta^2 u - \beta\Delta u - \frac{\lambda}{2}\Delta(u^2)u = g(u), \qquad x \in \mathbb{R}^N, \tag{1}$$

其中 $g : \mathbb{R} \to \mathbb{R}$ 是一个连续函数, $\lambda \geq 0$, $\beta \geq 0$. 利用变分法证明了方程 (1)有无穷多非径向解. 为带有拟线性项的混合色散非线性薛定谔方程的研究提供了一个证明紧性的方法.

**关键词:** 多重性; 非径向解; 拟线性问题; 四阶算子

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