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INEQUALITIES FOR EIGENVALUES OF POLYNOMIAL OPERATOR OF THE DRIFTING LAPLACIAN ON THE CIGAR SOLITON

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Abstract: In this paper, we investigate the weighted Dirichlet eigenvalue problem of polynomial operator of the drifting Laplacian on the cigar soliton $(\mathbb{R}^2, g, \phi)$ as follows

$$\begin{cases} L_\phi^2 u - aL_\phi u + bu = \lambda\rho u, & u \in \Omega, \\ u = \dfrac{\partial u}{\partial v} = 0, & u \in \partial\Omega, \end{cases}$$

where ρ is a positive continuous function on Ω, v denotes the outward unit normal to the boundary $\partial\Omega$, and a, b are two nonnegative constants. We establish some universal inequalities for eigenvalues of this problem.

Keywords: drifting Laplacian; Cigar soliton; eigenvalue

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1 Introduction

Let M be an n-dimensional complete Riemannian manifold with a smooth metric g. The triple $(M, g, e^{-\phi}dv)$ is called a smooth metric measure space, where ϕ is a smooth function on M. A smooth metric measure space can also arise as the smooth collapse limit of a sequence of manifolds with lower bounds on Ricci curvature, under convergence in the Gromov-Hausdorff sense. As an important topic, smooth metric measure space has received lots of attention (cf. [1–3]).

The drifting Laplacian associated with $(M, g, e^{-\phi}dv)$ is defined by

$$L_\phi u = \triangle u - \langle \nabla\phi, \nabla u \rangle = e^{\phi}\mathrm{div}(e^{-\phi}\nabla u), \tag{1.1}$$

where $\triangle$ denotes the Laplacian on M. If ϕ is a constant, it is easy to see that the drifting Laplacian is exactly the Laplacian. In particular, when M is a self-shrinker and $\phi = \frac{1}{2}|x|^2$,

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the drifting Laplacian becomes $\mathfrak{L}$ operator introduced by Colding and Minicozzi [4]. Besides, it is a self-adjoint operator with respect to weighted volume density $d\mu = e^{-\phi}dv$. Namely, it holds

$$\int_{\Omega} u(L_{\phi}h)d\mu = -\int_{\Omega}\langle\nabla u, \nabla h\rangle d\mu = \int_{\Omega} h(L_{\phi}u)d\mu, \tag{1.2}$$

where Ω is a bounded domain of M. In recent years, the estimation of eigenvalues of the drifting Laplacian has received widespread attention (see [5, 6]).

In this paper, we investigate the weighted eigenvalue problem of polynomial operator of the drifting Laplacian as follows

$$\begin{cases} L_{\phi}^2 u - aL_{\phi}u + bu = \lambda\rho u, & u \in \Omega, \\ u = \dfrac{\partial u}{\partial v} = 0, & u \in \partial\Omega, \end{cases} \tag{1.3}$$

where ρ is a positive continuous function on Ω, v denotes the outward unit normal to the boundary $\partial\Omega$, and a, b are two nonnegative constants. It has the following real and discrete spectrum

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty, \tag{1.4}$$

where each eigenvalue is repeated according to its multiplicity.

Problem (1.3) has some interesting connection with some classical problems. On the one hand, when ϕ is a constant, it becomes the following weighted Dirichelet problem of quadratic polynomial operator of the Laplacian

$$\begin{cases} \triangle^2 u - a\triangle u + bu = \lambda\rho u, & u \in \Omega, \\ u = \dfrac{\partial u}{\partial v} = 0, & u \in \partial\Omega. \end{cases} \tag{1.5}$$

On the other hand, when $a = b = 0$ and $\rho \equiv 1$, problem (1.3) becomes the Dirichelet eigenvalue problem of the bi-drifting Laplacian

$$\begin{cases} L_{\phi}^2 u = \lambda u, & u \in \Omega, \\ u = \dfrac{\partial u}{\partial v} = 0, & u \in \partial\Omega. \end{cases} \tag{1.6}$$

Furthermore, if ϕ is a constant, problem (1.6) becomes the clamped plate problem

$$\begin{cases} \triangle^2 u = \lambda u, & u \in \Omega, \\ u = \dfrac{\partial u}{\partial v} = 0, & u \in \partial\Omega. \end{cases} \tag{1.7}$$

There have been some intereting results for problems (1.5-1.7). In 2007, Wang and Xia [7] established the following inequality for eigenvalues of problem (1.7) on a unit sphere

$$\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \leqslant \frac{8(n+2)}{n^2}\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)(\lambda_i^{\frac{1}{2}} + \frac{n^2}{2n+4})(\lambda_i^{\frac{1}{2}} + \frac{n^2}{4}). \tag{1.8}$$

In 2010, Cheng, Ichikawa and Mametsuka [8] established a Yang's inequality for problem (1.7) in an n-dimensional complete Riemannian manifold

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leqslant \frac{1}{n^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)[n^2H_0^2+(2n+4)\lambda_i^{\frac{1}{2}}](n^2H_0^2+4\lambda_i^{\frac{1}{2}}). \tag{1.9}$$

In 2019, for problem (1.6) on a bounded domain of the cigar soliton $(\mathbb{R}^2, g, \phi)$, Li and Xiong [15] obtained

$$\begin{aligned}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq & 2\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left[\frac{2(1+C_0)}{1+C_1}\lambda_i^{\frac{1}{2}}+\frac{1}{1+C_1}-3\right]\right\}^{\frac{1}{2}} \\ & \times\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left[\frac{1+C_0}{1+C_1}\lambda_i^{\frac{1}{2}}+\frac{1}{1+C_1}-3\right]\right\}^{\frac{1}{2}}\end{aligned} \tag{1.10}$$

and

$$\sum_{p=1}^{2}(\lambda_{p+1}-\lambda_1)^{\frac{1}{2}} \leq 4\left[\frac{2(1+C_0)}{1+C_1}\lambda_1^{\frac{1}{2}}+\frac{1}{1+C_1}-3\right]^{\frac{1}{2}}\left[\frac{1+C_0}{1+C_1}\lambda_1^{\frac{1}{2}}+\frac{1}{1+C_1}-3\right]^{\frac{1}{2}}, \tag{1.11}$$

where $C_0=\max\limits_{x\in\Omega}|x|^2$ and $C_1=\min\limits_{x\in\Omega}|x|^2$. For more reference on problems (1.5-1.7), we refer to [9–11] and the references therein.

Ricci solitons are an important kind of complete metric measure spaces. They are corresponding to self-similar solutions of Hamilton's Ricci flow [12, 13]. (M, g, ϕ) is called a gradient Ricci soliton if there is a constant K, such that

$$\mathrm{Ric}+\mathrm{Hess}\phi = Kg. \tag{1.12}$$

The function ϕ is called a potential function of the gradient Ricci soliton. For $K>0$, $K=0$ and $K<0$, the Ricci soliton is called shrinking, steady or expanding respectively. When the dimension is two, Hamilton discovered the first complete non-compact example of a steady Ricci soliton on $\mathbb{R}^2$, called the cigar soliton. The metric and potential function of the cigar soliton $(\mathbb{R}^2, g, \phi)$ are given by

$$g=\frac{d(x^1)^2+d(x^2)^2}{1+|x|^2},$$

where $|x|^2=(x^1)^2+(x^2)^2$ and $\phi=-\log(1+|x|^2)$. In physics, the cigar soliton $(\mathbb{R}^2, g, \phi)$ is regarded as the Euclidean-Witten black hole under first-order Ricci flow of the world-sheet sigma model. Moreover, the cigar soliton was also studied by Witten as a target space in string theory [14]. Thus, it is of great importance both in geometry and physics.

In this paper, we obtain the following results for problem (1.3) on a bounded domain Ω of the cigar soliton $(\mathbb{R}^2, g, \phi)$.

Theorem 1.1 Let Ω be a bounded domain of the cigar soliton $(\mathbb{R}^2, g, \phi)$. Set $\rho_1=\min\limits_{x\in\Omega}\rho(x)$, $\rho_2=\max\limits_{x\in\Omega}\rho(x)$ and $\varpi_i=\frac{1}{2\rho_1}[-a+\sqrt{a^2+4\rho_1(\lambda_i-\frac{b}{\rho_2})}]$. Denote by λ_i the i-th eigenvalue of problem (1.3). Then we have

$$\begin{aligned}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq & \frac{2\rho_2}{\rho_1{}^{\frac{1}{2}}}\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left[\frac{2(1+C_0)}{1+C_1}\varpi_i+\frac{1}{\rho_2(1+C_1)}-\frac{3}{\rho_2}+\frac{a}{2\rho_1}\frac{1+C_0}{1+C_1}\right]\right\}^{\frac{1}{2}} \\ & \times\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left[\frac{1+C_0}{1+C_1}\varpi_i+\frac{1}{\rho_2(1+C_1)}-\frac{3}{\rho_2}\right]\right\}^{\frac{1}{2}},\end{aligned}\tag{1.13}$$

where $C_1=\min\limits_{x\in\Omega}|x|^2$ and $C_0=\max\limits_{x\in\Omega}|x|^2$.

Theorem 1.2 Under the same assumptions as Theorem 1.1, we have

$$\begin{aligned}\sum_{p=1}^{2}(\lambda_{p+1}-\lambda_1)^{\frac{1}{2}} \leq & 4\frac{\rho_2}{\rho_1^{\frac{1}{2}}}\left[\frac{2(1+C_0)}{1+C_1}\varpi_1+\frac{1}{\rho_2(1+C_1)}-\frac{3}{\rho_2}+\frac{a}{2\rho_1}\frac{1+C_0}{1+C_1}\right]^{\frac{1}{2}} \\ & \times\left[\frac{1+C_0}{1+C_1}\varpi_1+\frac{1}{\rho_2(1+C_1)}-\frac{3}{\rho_2}\right]^{\frac{1}{2}}.\end{aligned}\tag{1.14}$$

Remark 1.1 It is easy to find that when $a=b=0$ and $\rho=1$, (1.13) and (1.14) respectively become (1.10) and (1.11) for problem (1.6) in [15]. Therefore, our results generalize the results in [15].

2 Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. For this goal, we first establish a necessary lemma which plays a key role in the proof of Theorem 1.1.

Lemma 2.1 Let u_i be the orthonormal eigenfunction corresponding to the i-th eigenvalue λ_i of problem (1.3). Then, for any function $h\in C^4(M)\cap C^3(\partial M)$ and any positive integer k, we have

$$\begin{aligned}&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega u_i{}^2|\nabla h|^2d\mu \\ \leq&\delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega\left[(u_iL_\phi h+2\langle\nabla h,\nabla u_i\rangle)^2-2u_iL_\phi u_i|\nabla h|^2+au_i^2|\nabla h|^2\right]d\mu \\ &+\frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_\Omega\frac{1}{\rho}(\langle\nabla h,\nabla u_i\rangle+\frac{u_iL_\phi h}{2})^2d\mu,\end{aligned}\tag{2.1}$$

where δ is any positive constant.

Proof Set $\varphi_i=hu_i-\sum_{j=1}^{k}\alpha_{ij}u_j$, where $\alpha_{ij}=\int_M\rho hu_iu_jd\mu$. Then we have

$$\int_\Omega\rho\varphi_iu_jd\mu=0,\qquad\forall i,j=1,2,\cdots,k.\tag{2.2}$$

Using the Rayleigh-Ritz inequality, we get

$$\lambda_{k+1}\int_\Omega\rho\varphi_i^2d\mu\leqslant\int_\Omega\varphi_i(L_\phi^2-aL_\phi+b)\varphi_id\mu.\tag{2.3}$$

According to the definition of φ_i, we have

$$L_\phi(hu_i) = hL_\phi u_i + u_i L_\phi h + 2\langle\nabla h, \nabla u_i\rangle \tag{2.4}$$

and

$$L^2_\phi(hu_i) = hL^2_\phi u_i + 2\langle\nabla h, \nabla L_\phi u_i)\rangle + L_\phi h L_\phi u_i + 2L_\phi\langle\nabla h, \nabla u_i\rangle + L_\phi(u_i L_\phi h). \tag{2.5}$$

Therefore, using (2.4) and (2.5), we get

$$\begin{aligned}(L^2_\phi - aL_\phi + b)(hu_i) =& h(L^2_\phi u_i - aL_\phi u_i + bu_i) + \Psi_i\\ =& h\lambda_i\rho u_i + \Psi_i,\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}\Psi_i =& 2\langle\nabla h, \nabla(L_\phi u_i))\rangle + 2L_\phi h L_\phi u_i + u_i L^2_\phi h + 2\langle\nabla u_i, \nabla(L_\phi h))\rangle + 2L_\phi\langle\nabla h, \nabla u_i\rangle\\ &- au_i L_\phi h - 2a\langle\nabla h, \nabla u_i\rangle.\end{aligned}$$

Then it follows from (2.6) that

$$\begin{aligned}\int_\Omega \varphi_i(L^2_\phi - aL_\phi + b)\varphi_i d\mu &= \int_M \varphi_i(L^2_\phi - aL_\phi + b)(hu_i)d\mu\\ &= \lambda_i\int_\Omega \varphi_i\rho hu_i d\mu + \int_\Omega \varphi_i\Psi_i d\mu\\ &= \lambda_i\int_\Omega \rho{\varphi_i}^2 d\mu + \int_\Omega h\Psi_i u_i d\mu - \sum_{j=1}^k \alpha_{ij}\beta_{ij},\end{aligned} \tag{2.7}$$

where $\beta_{ij} = \int_\Omega \Psi_i u_j$. Hence, substituting (2.6) into (2.3), we derive

$$(\lambda_{k+1} - \lambda_i)\int_\Omega \rho{\varphi_i}^2 d\mu \le \int_\Omega h\Psi_i u_i d\mu - \sum_{j=1}^k \alpha_{ij}\beta_{ij}. \tag{2.8}$$

Using the divergence theorem, we deduce

$$\begin{aligned}\int_\Omega u_j\langle\nabla h, \nabla(L_\phi u_i))\rangle d\mu &= -\int_\Omega L_\phi u_i \mathrm{div}_\phi(u_j\nabla h)d\mu\\ &= -\int_\Omega u_j L_\phi u_i L_\phi h d\mu - \int_\Omega L_\phi u_i\langle\nabla u_j, \nabla h\rangle d\mu\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}&\int_\Omega (2u_j\langle\nabla h, \nabla u_i\rangle + u_i u_j L_\phi h)\, d\mu\\ =& -\int_\Omega [2h\mathrm{div}_\phi(u_j\nabla u_i) + hL_\phi(u_i u_j)]\, d\mu\\ =& -\int_\Omega [2h(u_j L_\phi u_i + \langle\nabla u_j, \nabla u_i\rangle) + h(u_i L_\phi u_j + u_j L_\phi u_i + 2\langle\nabla u_i, \nabla u_j\rangle)]\, d\mu\\ =& \int_\Omega (hu_i L_\phi u_j - hu_j L_\phi u_i)\, d\mu.\end{aligned} \tag{2.10}$$

Moreover, since

$$\begin{aligned}\int_\Omega L_\phi u_j\langle\nabla h,\nabla u_i\rangle d\mu &= -\int_\Omega h\mathrm{div}_\phi(L_\phi u_j\nabla u_i)d\mu\\ &= -\int_\Omega h\left(L_\phi u_j L_\phi u_i+\langle\nabla(L_\phi u_j),\nabla u_i\rangle\right)d\mu,\end{aligned}\tag{2.11}$$

we have

$$\begin{aligned}&\int_\Omega\left(L_\phi u_j\langle\nabla h,\nabla u_i\rangle-L_\phi u_i\langle\nabla h,\nabla u_j\rangle\right)d\mu\\ =&\int_\Omega(h\langle\nabla(L_\phi u_i,\nabla u_j\rangle-h\langle\nabla(L_\phi(u_j),\nabla u_i\rangle)d\mu\\ =&-\int_\Omega u_j\mathrm{div}_\phi(h\nabla(L_\phi u_i))d\mu+\int_\Omega u_i\mathrm{div}_\phi(h\nabla(L_\phi u_j))d\mu\\ =&\int_\Omega\left[hu_i{L_\phi}^2u_j-hu_j{L_\phi}^2u_i+u_i\langle\nabla h,\nabla(L_\phi u_j)\rangle-u_j\langle\nabla h,\nabla(L_\phi u_i)\rangle\right]d\mu\\ =&\int_\Omega\left(hu_i{L_\phi}^2u_j-hu_j{L_\phi}^2u_i+u_jL_\phi u_iL_\phi h-u_iL_\phi u_jL_\phi h\right)d\mu\\ &-\int_\Omega\left(L_\phi u_j\langle\nabla h,\nabla u_i\rangle-L_\phi u_i\langle\nabla h,\nabla u_j\rangle\right)d\mu.\end{aligned}\tag{2.12}$$

It implies

$$\begin{aligned}&2\int_\Omega(L_\phi u_j\langle\nabla h,\nabla u_i\rangle-L_\phi u_i\langle\nabla h,\nabla u_j\rangle)d\mu\\ =&\int_\Omega(hu_iL_\phi^2u_j-hu_jL_\phi^2u_i+u_jL_\phi u_iL_\phi h-u_iL_\phi u_jL_\phi h)d\mu.\end{aligned}\tag{2.13}$$

Moreover, we have

$$\int_\Omega 2u_jL_\phi\langle\nabla h,\nabla u_i\rangle d\mu=\int_\Omega 2L_\phi u_j\langle\nabla h,\nabla u_i\rangle d\mu.\tag{2.14}$$

Thus, using (2.13) and (2.14), we derive

$$\begin{aligned}\beta_{ij}=&\int_\Omega hu_i(L_\phi^2u_j-aL_\phi u_j+bu_j)d\mu-\int_\Omega hu_j(L_\phi^2u_i-aL_\phi u_i+bu_i)d\mu\\ =&\int_\Omega\lambda_j\rho hu_iu_jd\mu-\int_\Omega\lambda_i\rho hu_iu_jd\mu\\ =&(\lambda_j-\lambda_i)\alpha_{ij}.\end{aligned}\tag{2.15}$$

Therefore, we obtain

$$(\lambda_{k+1}-\lambda_i)\int_\Omega\rho\varphi_i^2d\mu\le\int_\Omega h\Psi_iu_id\mu-\sum_{j=1}^k(\lambda_j-\lambda_i)\alpha_{ij}^2.\tag{2.16}$$

Set $\xi_{ij}=\int_\Omega u_j(\langle\nabla h,\nabla u_i\rangle+\frac{1}{2}u_iL_\phi h)d\mu$. Then it holds $\xi_{ij}=\xi_{ji}$. Moreover, since

$$\int_\Omega hu_i\langle\nabla h,\nabla u_i\rangle d\mu=-\int_\Omega(u_i^2|\nabla h|^2-hu_i^2L_\phi h-hu_i\langle\nabla u_i,\nabla h\rangle)d\mu,\tag{2.17}$$

we obtain

$$2\int_\Omega hu_i\langle\nabla h,\nabla u_i\rangle d\mu=-\int_\Omega(u_i^2|\nabla h|^2-hu_i^2L_\phi h)d\mu. \tag{2.18}$$

Hence we derive

$$-2\int_\Omega\varphi_i(\langle\nabla h,\nabla u_i\rangle+\frac{1}{2}u_iL_\phi h)d\mu=\int_\Omega u_i^2|\nabla h|^2d\mu+2\sum_{j=1}^k a_{ij}\xi_{ij}. \tag{2.19}$$

Multiplying both sides of (2.17) by $(\lambda_{k+1}-\lambda_i)^2$, and using the Schwarz inequality, we get

$$\begin{aligned}&(\lambda_{k+1}-\lambda_i)^2(\int_\Omega u_i^2|\nabla h|^2d\mu+2\sum_{j=1}^k\alpha_{ij}\xi_{ij})\\ =&(\lambda_{k+1}-\lambda_i)^2\int_\Omega-2\sqrt{\rho}\varphi_i\left[\frac{1}{\sqrt{\rho}}(\langle\nabla h,\nabla u_i\rangle+\frac{u_iL_\phi h}{2})-\sum_{j=1}^k\sqrt{\rho}\xi_{ij}u_j\right]d\mu\\ \leq&\delta(\lambda_{k+1}-\lambda_i)^3\int_\Omega\rho\varphi_i^2d\mu+\frac{\lambda_{k+1}-\lambda_i}{\delta}\int_\Omega[\frac{1}{\sqrt{\rho}}(\langle\nabla h,\nabla u_i\rangle+\frac{u_iL_\phi h}{2})-\sum_{j=1}^k\sqrt{\rho}\xi_{ij}u_j]^2d\mu,\end{aligned} \tag{2.20}$$

where δ is any positive constant. Summing over i from 1 to k, we have

$$\begin{aligned}&\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int_\Omega u_i^2|\nabla h|^2d\mu+2\sum_{i,j=1}^k(\lambda_{k+1}-\lambda_i)^2\alpha_{ij}\xi_{ij}\\ \leq&\delta\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int_\Omega h\Psi_iu_id\mu-\delta\sum_{i,j=1}^k(\lambda_{k+1}-\lambda_i)^2(\lambda_j-\lambda_i)\alpha_{ij}^2\\ &+\frac{1}{\delta}\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\int_\Omega\frac{1}{\rho}(\langle\nabla h,\nabla u_i\rangle+\frac{u_iL_\phi h}{2})^2d\mu-\frac{1}{\delta}\sum_{i,j=1}^k(\lambda_{k+1}-\lambda_i)\xi_{ij}^2.\end{aligned} \tag{2.21}$$

Since α_{ij} is symmetric and ξ_{ij} is anti-symmetric, we deduce

$$\sum_{i,j=1}^k(\lambda_{k+1}-\lambda_i)^2\alpha_{ij}\xi_{ij}=-\sum_{i,j=1}^k(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)\alpha_{ij}\xi_{ij} \tag{2.22}$$

and

$$\sum_{i,j=1}^k(\lambda_{k+1}-\lambda_i)^2(\lambda_i-\lambda_j)\alpha_{ij}^2=-\sum_{i,j=1}^k(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)^2\alpha_{ij}^2. \tag{2.23}$$

Therefore, combining (2.23) and (2.23) with (2.21), we obtain

$$\begin{aligned}&\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int_\Omega u_i^2|\nabla h|^2d\mu\\ \leq&\delta\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int_\Omega h\Psi_iu_id\mu+\frac{1}{\delta}\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\int_\Omega\frac{1}{\rho}(\langle\nabla h,\nabla u_i\rangle+\frac{u_iL_\phi h}{2})^2d\mu.\end{aligned} \tag{2.24}$$

Using the divergence theorem, we deduce

$$\int_\Omega hu_i\langle\nabla h,\nabla(L_\phi u_i)\rangle d\mu=\int_\Omega(-hu_iL_\phi u_iL_\phi h-hL_\phi u_i\langle\nabla u_i,\nabla h\rangle-u_iL_\phi u_i|\nabla h|^2)d\mu, \tag{2.25}$$

$$\int_\Omega hu_i\langle\nabla u_i,\nabla(L_\phi h)\rangle d\mu=\int_\Omega(-hu_iL_\phi hL_\phi u_i-u_iL_\phi h\langle\nabla h,\nabla u_i\rangle-hL_\phi h|\nabla u_i|^2)d\mu, \tag{2.26}$$

$$2\int_\Omega hu_i\langle\nabla h,\nabla u_i\rangle d\mu=-\int_\Omega hu_i^2L_\phi h-u_i^2|\nabla h|^2d\mu \tag{2.27}$$

and

$$\int_\Omega hu_iL_\phi\langle\nabla h,\nabla u_i\rangle d\mu=\int_\Omega(hL_\phi u_i\langle\nabla h,\nabla u_i\rangle+u_iL_\phi h\langle\nabla h,\nabla u_i\rangle+2\langle\nabla h,\nabla u_i\rangle^2)d\mu. \tag{2.28}$$

Using (2.25-2.28), we have

$$\int_\Omega h\Psi_iu_id\mu=\int_\Omega\left[(2\langle\nabla h,\nabla u_i\rangle+u_iL_\phi h)^2-2u_iL_\phi u_i|\nabla h|^2\right]d\mu+a\int_\Omega u_i^2|\nabla h|^2d\mu. \tag{2.29}$$

Substituting (2.29) into (2.24), we obtain (2.1). This completes the proof of Lemma 2.1.

Now we give the proof of Theorem 1.1 by using Lemma 1.1.

Proof of Theorem 1.1 Suppose that x^p is the p-th local coordinate of $x_0\in\Omega\subset\mathbb{R}^2$, where $p=1,2$. Taking $h=x^p$ in Lemma 2.1, we have

$$\begin{aligned}&\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int_\Omega {u_i}^2|\nabla x^p|^2d\mu\\ \le&\delta\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int_\Omega\left[(u_iL_\phi x^p+2\langle\nabla x^p,\nabla u_i\rangle)^2-2u_iL_\phi u_i|\nabla x^p|^2+au_i^2|\nabla x^p|^2\right]d\mu\\ &+\frac{1}{\delta}\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\int_\Omega\frac{1}{\rho}(\langle\nabla x^p,\nabla u_i\rangle+\frac{u_iL_\phi x^p}{2})^2d\mu.\end{aligned} \tag{2.30}$$

It is not difficult to obtain

$$|\nabla x^p|^2=1+|x|^2,\quad\forall p=1,2, \tag{2.31}$$

$$\langle\nabla x^1,\nabla x^2\rangle=0 \tag{2.32}$$

and

$$\Delta x^1=\Delta x^2=0. \tag{2.33}$$

Using (2.31) and (2.32), we get

$$\langle\nabla\phi,\nabla x^p\rangle=-\langle\nabla(-\log(1+|x|^2)),\nabla x^p\rangle=2x^p \tag{2.34}$$

and

$$\sum_{p=1}^2\langle\nabla x^p,\nabla u_i\rangle^2=(1+|x|^2)|\nabla u_i|^2. \tag{2.35}$$

Taking the sum over p from 1 to 2 on (2.30), and using (2.31), (2.34) and (2.35), we have

$$\begin{aligned}
&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega\sum_{p=1}^{2}u_i^2|\nabla x^p|^2d\mu\\
\leq&\sum_{i=1}^{k}\delta(\lambda_{k+1}-\lambda_i)^2\int_\Omega\bigg[4u_i^2|x|^2-4u_iL_\phi u_i(1+|x|^2)+4(1+|x|^2)|\nabla u_i|^2\\
&+\sum_{p=1}^{2}8u_ix^p\langle\nabla x^p,\nabla u_i\rangle+2au_i^2(1+|x|^2)\bigg]d\mu\\
&+\frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_\Omega\frac{1}{\rho}\left[(1+|x|^2)|\nabla u_i|^2+\sum_{p=1}^{2}2u_ix^p\langle\nabla x^p,\nabla u_i\rangle+u_i^2|x|^2\right]d\mu.
\end{aligned}\tag{2.36}$$

Since

$$\begin{aligned}
\sum_{p=1}^{2}\int_\Omega u_ix^p\langle\nabla x^p,\nabla u_i\rangle d\mu&=-\sum_{p=1}^{2}\int_\Omega u_i\mathrm{div}_\phi(u_ix^p\nabla x^p)d\mu\\
&=-\sum_{p=1}^{2}\int_\Omega u_i\left(u_ix^pL_\phi x^p+u_i|\nabla x^p|^2+x^p\langle\nabla u_i,\nabla x^p\rangle\right)d\mu\\
&=-4\int_\Omega u_i^2|x|^2d\mu-2\int_\Omega u_i^2d\mu-\sum_{p=1}^{2}\int_\Omega u_ix^p\langle\nabla u_i,\nabla x^p\rangle d\mu,
\end{aligned}$$

we obtain

$$\sum_{p=1}^{2}\int_\Omega u_ix^p\langle\nabla x^p,\nabla u_i\rangle d\mu=-2\int_\Omega u_i^2|x|^2d\mu-\int_\Omega u_i^2d\mu.\tag{2.37}$$

Moreover, using

$$\lambda_i=\int_\Omega(L_\phi u_i)^2d\mu-a\int_\Omega u_iL_\phi u_id\mu+b\int_\Omega u_i^2d\mu\tag{2.38}$$

and

$$\int_\Omega|\nabla u_i|^2d\mu=-\int_\Omega u_iL_\phi u_id\mu\leq(\int_\Omega u_i^2d\mu)^{\frac{1}{2}}(\int_\Omega(L_\phi u_i)^2d\mu)^{\frac{1}{2}}.\tag{2.39}$$

It yields

$$\rho_1\rho_2(\int_\Omega|\nabla u_i|^2)^2d\mu+a\rho_2\int_\Omega|\nabla u_i|^2d\mu+b-\lambda_i\rho_2\leq 0.\tag{2.40}$$

This is a quadratic inequality of $\int_\Omega|\nabla u_i|^2d\mu$. Hence we get

$$\int_\Omega|\nabla u_i|^2d\mu\leq\varpi_i.\tag{2.41}$$

Substituting (2.37) and (2.41) into (3.25), we infer

$$\begin{aligned}\frac{2}{\rho_2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq&\delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left[8\frac{1+C_0}{1+C_1}\varpi_i+\frac{4}{\rho_2(1+C_1)}-\frac{12}{\rho_2}+\frac{2a}{\rho_1}\frac{1+C_0}{1+C_1}\right]\\&+\frac{1}{\rho_1\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left[\frac{1+C_0}{1+C_1}\varpi_i+\frac{1}{\rho_2(1+C_1)}-\frac{3}{\rho_2}\right].\end{aligned}\tag{2.42}$$

Taking

$$\delta=\left\{\frac{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left[\frac{1+C_0}{1+C_1}\varpi_i+\frac{1}{\rho_2(1+C_1)}-\frac{3}{\rho_2}\right]}{\rho_1\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left[8\frac{1+C_0}{1+C_1}\varpi_i+\frac{4}{\rho_2(1+C_1)}-\frac{12}{\rho_2}+\frac{2a}{\rho_1}\frac{1+C_0}{1+C_1}\right]}\right\}^{\frac{1}{2}}$$

in (2.42), we obtain (1.13). The proof of Theorem 1.1 is finished.

3 Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. For this goal, we first prove the following lemma.

Lemma 3.1 Under the same assumptions as Lemma 2.1, for any function $\zeta^p \in C^4(M)\cap C^3(\partial M)$ $(p=1,2)$ and any positive integer k, we have

$$\begin{aligned}&(\lambda_{p+1}-\lambda_1)^{\frac{1}{2}}\int_\Omega u_1^2|\nabla\zeta^p|^2d\mu\\ \leq&\frac{\delta}{2}\left(\|u_1L_\phi\zeta^p+2\langle\nabla\zeta^p,\nabla u_1\rangle\|^2-2\int_\Omega|\nabla\zeta^p|^2u_1L_\phi u_1d\mu+a\int_\Omega u_1^2|\nabla\zeta^p|^2d\mu\right)\\&+\frac{1}{2\delta}\int_\Omega\frac{1}{\rho}\left(u_1L_\phi\zeta^p+2\langle\nabla u_1,\nabla\zeta^p\rangle\right)^2d\mu,\end{aligned}\tag{3.1}$$

where δ is any positive constant.

Proof Set $\psi^p=(\zeta^p-\gamma^p)u_1$, where $\gamma^p=\int_\Omega\rho\zeta^pu_1^2d\mu$. It implies $\int_\Omega\rho\psi^pu_1d\mu=0$. Noticing that $\int_\Omega\rho\zeta^pu_1u_{q+1}d\mu=0$, for $1\leq q<p$, we have

$$\int_\Omega\rho\psi^pu_{q+1}d\mu=0,\quad 0\leq q<p.\tag{3.2}$$

From the Rayleigh-Ritz inequality, we have

$$\lambda_{p+1}\int_\Omega\rho(\psi^p)^2d\mu\leq\int_\Omega\psi^p(L_\phi^2-aL_\phi+b)\psi^pd\mu.\tag{3.3}$$

According to the definition of ψ^p, we have

$$L_\phi\psi^p=u_1L_\phi\zeta^p+2\langle\nabla\zeta^p,\nabla u_1\rangle+\zeta^pL_\phi u_1-\gamma^pL_\phi u_1\tag{3.4}$$

and

$$\begin{aligned}L_\phi^2\psi^p=&u_1L_\phi^2\zeta^p+2\langle\nabla u_1,\nabla(L_\phi\zeta^p)\rangle+2L_\phi\langle\nabla\zeta^p,\nabla u_1\rangle+2L_\phi\zeta^pL_\phi u_1\\&+\zeta^pL_\phi^2u_1+2\langle\nabla\zeta^p,\nabla(L_\phi u_1)\rangle-\gamma^pL_\phi^2u_1.\end{aligned}\tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned}(L_\phi^2 - aL_\phi + b)\psi^p =& u_1 L_\phi^2\zeta^p + 2\langle\nabla u_1, \nabla(L_\phi\zeta^p)\rangle + 2L_\phi\zeta^p L_\phi u_1 + 2L_\phi\langle\nabla\zeta^p, \nabla u_1\rangle\\ &+ \zeta^p L_\phi^2 u_1 + 2\langle\nabla\zeta^p, \nabla(L_\phi u_1)\rangle - \gamma^p L_\phi^2 u_1 - a u_1 L_\phi\zeta^p\\ &- 2a\langle\nabla\zeta^p, \nabla u_1\rangle - a\zeta^p L_\phi u_1 + a\gamma^p L_\phi u_1\\ =&\lambda_1\rho\psi^p + \Theta^p,\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}\Theta^p =& u_1 L_\phi^2\zeta^p + 2\langle\nabla u_1, \nabla(L_\phi\zeta^p)\rangle + 2L_\phi\zeta^p L_\phi u_1 + 2L_\phi\langle\nabla\zeta^p, \nabla u_1\rangle\\ &+ 2\langle\nabla\zeta^p, \nabla(L_\phi u_1\rangle - a u_1 L_\phi\zeta^p - 2a\langle\nabla\zeta^p, \nabla u_1\rangle.\end{aligned}$$

Using the divergence theorem, we deduce

$$\int_\Omega u_1\langle\nabla u_1, \nabla(L_\phi\zeta^p)\rangle d\mu = -\int_\Omega u_1 L_\phi\zeta^p L_\phi u_1 d\mu - \int_\Omega |\nabla u_1|^2 L_\phi\zeta^p d\mu, \tag{3.7}$$

$$\int_\Omega u_1\langle\nabla\zeta^p, \nabla(L_\phi u_1)\rangle d\mu = -\int_\Omega u_1 L_\phi\zeta^p L_\phi u_1 d\mu - \int_\Omega L_\phi u_1\langle\nabla u_1, \nabla\zeta^p\rangle d\mu \tag{3.8}$$

and

$$\int_\Omega u_1^2 L_\phi^2\zeta^p d\mu = 2\int_\Omega (u_1 L_\phi u_1 L_\phi\zeta^p + |\nabla u_1|^2 L_\phi\zeta^p) d\mu. \tag{3.9}$$

Therefore, using (3.7-3.9), we have

$$\begin{aligned}\int_\Omega \Theta^p u_1 d\mu =& \int_\Omega u_1\bigg(u_1 L_\phi^2\zeta^p + 2\langle\nabla u_1, \nabla(L_\phi\zeta^p)\rangle + 2L_\phi\zeta^p L_\phi u_1 + 2L_\phi\langle\nabla\zeta^p, \nabla u_1\rangle\\ &+ 2\langle\nabla\zeta^p, \nabla(L_\phi u_1\rangle - a u_1 L_\phi\zeta^p - 2a\langle\nabla\zeta^p, \nabla u_1\rangle\bigg) d\mu\\ =& \int_\Omega (u_1^2 L_\phi^2\zeta^p - 2|\nabla u_1|^2 L_\phi\zeta^p - 2u_1 L_\phi\zeta^p L_\phi u_1) d\mu\\ =&0.\end{aligned} \tag{3.10}$$

Using (3.3), (3.6) and (3.10), we derive

$$(\lambda_{p+1} - \lambda_1)\int_\Omega \rho(\psi^p)^2 d\mu \leq \int_\Omega \zeta^p\Theta^p u_1 d\mu. \tag{3.11}$$

Furthermore, using the similiar computation, we have

$$2\int_\Omega \zeta^p u_1\langle\nabla\zeta^p, \nabla u_1\rangle d\mu = -\int_\Omega u_1^2(\zeta^p L_\phi\zeta^p + |\nabla\zeta^p|^2) d\mu, \tag{3.12}$$

$$2\int_\Omega \zeta^p u_1\langle\nabla u_1, \nabla(L_\phi\zeta^p)\rangle d\mu = \int_\Omega \left[2u_1 L_\phi\zeta^p\langle\nabla u_1, \nabla\zeta^p\rangle + u_1^2(L_\phi\zeta^p)^2 - \zeta^p u_1^2 L_\phi^2\zeta^p\right] d\mu, \tag{3.13}$$

$$\begin{aligned}\int_\Omega \zeta^p u_1 L_\phi\langle\nabla\zeta^p, \nabla u_1\rangle d\mu =& \int_\Omega \bigg(u_1 L_\phi\zeta^p\langle\nabla\zeta^p, \nabla u_1\rangle + 2\langle\nabla\zeta^p, \nabla u_1\rangle^2\\ &+ \zeta^p L_\phi u_1\langle\nabla\zeta^p, \nabla u_1\rangle\bigg) d\mu\end{aligned} \tag{3.14}$$

and

$$2\int_\Omega \zeta^p u_1\langle\nabla\zeta^p,\nabla(L_\phi u_1)\rangle d\mu=-\int_\Omega\left(2|\nabla\zeta^p|^2u_1L_\phi u_1-2\zeta^pL_\phi u_1\langle\nabla\zeta^p,\nabla u_1\rangle-2\zeta^pu_1L_\phi\zeta^pL_\phi u_1\right)d\mu. \tag{3.15}$$

Therefore, it follows from (3.11-3.15) that

$$(\lambda_{p+1}-\lambda_1)\int_\Omega\rho(\psi^p)^2d\mu\le\|u_1(L_\phi\zeta^p)+2\langle\nabla\zeta^p,\nabla u_1\rangle\|^2-2\int_\Omega|\nabla\zeta^p|^2u_1L_\phi u_1d\mu+a\int_\Omega u_1^2|\nabla\zeta^p|^2d\mu. \tag{3.16}$$

Since $\int_\Omega\zeta^pu_1^2L_\phi\zeta^pd\mu=-\int_\Omega u_1^2|\nabla\zeta^p|^2d\mu-2\int_\Omega\zeta^pu_1\langle\nabla u_1,\nabla\zeta^p\rangle d\mu$, we have

$$\begin{aligned}&\int_\Omega\psi^p(u_1L_\phi\zeta^p+2\langle\nabla u_1,\nabla\zeta^p\rangle)d\mu\\=&\int_\Omega(\zeta^pu_1^2L_\phi\zeta^p+2\zeta^pu_1\langle\nabla u_1,\nabla\zeta^p\rangle)d\mu-\gamma^p\int_\Omega(u_1^2L_\phi\zeta^p+2u_1\langle\nabla u_1,\nabla\zeta^p\rangle)d\mu\\=&-\int_\Omega u_1^2|\nabla\zeta^p|^2d\mu.\end{aligned} \tag{3.17}$$

Multiplying both sides of (3.17) by $(\lambda_{p+1}-\lambda_1)^{\frac12}$, and using (3.16), we derive

$$\begin{aligned}&(\lambda_{p+1}-\lambda_1)^{\frac12}\int_\Omega u_1^2|\nabla\zeta^p|^2d\mu\\=&-(\lambda_{p+1}-\lambda_1)\int_\Omega\sqrt{\rho}\psi^p\frac{1}{\sqrt{\rho}}(u_1L_\phi\zeta^p+2\langle\nabla u_1,\nabla\zeta^p\rangle)d\mu\\\le&\frac{\delta}{2}(\lambda_{p+1}-\lambda_1)\int_\Omega\rho(\psi^p)^2d\mu+\frac{1}{2\delta}\int_\Omega\frac{1}{\rho}(u_1L_\phi\zeta^p+2\langle\nabla u_1,\nabla\zeta^p\rangle)^2d\mu\\\le&\frac{\delta}{2}\left(\|u_1L_\phi\zeta^p+2\langle\nabla\zeta^p,\nabla u_1\rangle\|^2-2\int_\Omega|\nabla\zeta^p|^2u_1L_\phi u_1d\mu+a\int_\Omega u_1^2|\nabla\zeta^p|^2d\mu\right)\\&+\frac{1}{2\delta}\int_\Omega\frac{1}{\rho}(u_1L_\phi\zeta^p+2\langle\nabla u_1,\nabla\zeta^p\rangle)^2d\mu.\end{aligned} \tag{3.18}$$

The proof of Lemma 3.1 is ended.

Now we give the proof of Theorem 1.2 by using Lemma 3.1.

Proof of Theorem 1.2 Define a (2×2)-matrix $B=(\epsilon_{pt})_{2\times2}$, where $\epsilon_{pt}=\int_\Omega\rho x^pu_1u_{t+1}d\mu$. Using the orthogonalization of Gram-Schmidt, we know that there exists an upper triangle matrix $U=(\vartheta_{pt})_{2\times2}$ and an orthogonal matrix $P=(\varsigma_{pt})_{2\times2}$ such that $U=PB$. That is to say, for $1\leqslant t<p\leqslant2$, we have

$$\vartheta_{pt}=\sum_{s=1}^2\varsigma_{ps}\epsilon_{st}=\int_\Omega\rho\sum_{s=1}^2\varsigma_{ps}x^su_1u_{t+1}d\mu=0. \tag{3.19}$$

Setting $y^p=\sum_{s=1}^2\varsigma_{ps}x^s$, we obtain

$$\int_\Omega\rho y_pu_1u_{t+1}d\mu=0,\quad 1\le t<p\le2. \tag{3.20}$$

Taking $\zeta^p = y^p$ in (3.1), and taking sum on p from 1 to 2, we get

$$\begin{aligned}
&\sum_{p=1}^{2}(\lambda_{p+1}-\lambda_1)^{\frac{1}{2}}\int_\Omega u_1^2|\nabla y^p|^2 d\mu \\
\leq&\frac{\delta}{2}\left(\|u_1 L_\phi y^p + 2\langle\nabla y^p,\nabla u_1\rangle\|^2 - 2\int_\Omega |\nabla y^p|^2 u_1 L_\phi u_1 d\mu + a\int_\Omega u_1^2|\nabla y^p|^2 d\mu\right) \\
&+\frac{1}{2\delta}\int_\Omega \frac{1}{\rho}(u_1 L_\phi y^p + 2\langle\nabla y^p,\nabla u_1\rangle)^2 d\mu.
\end{aligned} \tag{3.21}$$

Since $y^p = \sum_{s=1}^2 \varsigma_{ps}x^s$ and P is an orthogonal matrix, we know that y^1 and y^2 are the standard coordinate functions of $\mathbb{R}^2$. It is not difficult to check that

$$|y|^2 = |x|^2, \tag{3.22}$$

$$|\nabla y^p|^2 = 1 + |x|^2 \tag{3.23}$$

and

$$L_\phi y^p = 2y^p. \tag{3.24}$$

Substituting it into (3.21), we have

$$\begin{aligned}
&\sum_{p=1}^{2}(\lambda_{p+1}-\lambda_1)^2\int_\Omega u_1^2(1+|x|^2)d\mu \\
\leq&2\delta\int_\Omega\Bigg[u_1^2|x|^2 + (1+|x|^2)|\nabla u_1|^2 + 4\sum_{p=1}^2 u_1 x^p\langle\nabla x^p,\nabla u_1\rangle - (1+|x|^2)u_1 L_\phi u_1 \\
&+\frac{a}{2}(1+|x|^2)u_1^2\Bigg]d\mu + \frac{2}{\delta}\int_\Omega\frac{1}{\rho}\left[u_1^2|x|^2 + (1+|x|^2)|\nabla u_1|^2 + 4\sum_{p=1}^2 u_1 x^p\langle\nabla x^p,\nabla u_1\rangle\right]d\mu.
\end{aligned} \tag{3.25}$$

Similar to the computation as (2.42), we obtain

$$\begin{aligned}
\frac{1}{\rho_2}\sum_{p=1}^{2}(\lambda_{p+1}-\lambda_1)^{\frac{1}{2}} \leq&2\delta\left[2\frac{1+C_0}{1+C_1}\varpi_1 + \frac{1}{\rho_2(1+C_1)} - \frac{3}{\rho_2} + \frac{a}{2\rho_1}\frac{1+C_0}{1+C_1}\right] \\
&+\frac{2}{\delta}\frac{1}{\rho_1}\left[\frac{1+C_0}{1+C_1}\varpi_1 + \frac{1}{\rho_2(1+C_1)} - \frac{3}{\rho_2}\right].
\end{aligned} \tag{3.26}$$

Taking $\delta = \left\{\frac{\frac{1}{\rho_1}\left[\frac{1+C_0}{1+C_1}\varpi_1 + \frac{1}{\rho_2(1+C_1)} - \frac{3}{\rho_2}\right]}{\left[2\frac{1+C_0}{1+C_1}\varpi_1 + \frac{1}{\rho_2(1+C_1)} - \frac{3}{\rho_2} + \frac{a}{2\rho_1}\frac{1+C_0}{1+C_1}\right]}\right\}^{\frac{1}{2}}$ in (3.26), we obtain (1.14). The proof of Theorem 1.2 is completed.

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Cigar孤立子上漂移Laplace算子的多项式算子的特征值不等式

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摘要: 本文研究了cigar孤立子$(\mathbb{R}^2, g, f)$上漂移Laplace算子的多项式算子的加权Dirichlet特征值问题:

$$\begin{cases} L_\phi^2 u - aL_\phi u + bu = \lambda\rho u, & u \in \Omega, \\ u = \dfrac{\partial u}{\partial v} = 0, & u \in \partial\Omega, \end{cases}$$

其中ρ是Ω上的正连续函数,v是$\partial\Omega$的单位外法向量,a, b是两个非负常数. 我们建立了该问题的一些特征值不等式.

关键词: 漂移Laplace算子; Cigar孤立子; 特征值

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