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LIOUVILLE THEOREM FOR 3D STATIONARY *Q*-TENSOR SYSTEM OF LIQUID CRYSTAL

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Abstract: In this paper, we study Liouville theorem for the 3D stationary Q-tensor system of liquid crystal in Lorentz and Morrey spaces. Under some additional hypotheses, stated in terms of Lorentz and Morrey spaces, using energy estimation, we obtain that the trivial solution $u = Q \equiv 0$ is the unique solution. Our theorems correspond to improvements of some recent results and contain some known results as particular cases.

Keywords: Lorentz spaces; Morrey spaces; Q-tensor; liquid crystal; Liouville theorem

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1 Introduction

In this paper, we study the following 3D stationary Q-tensor system of liquid crystal:

$$\begin{cases} -\mu\Delta u + (u\cdot\nabla)u + \nabla P = -\nabla\cdot(\nabla Q\odot\nabla Q) - \lambda\nabla\cdot(|Q|H) + \nabla\cdot(Q\Delta Q - \Delta Q Q), \\ (u\cdot\nabla)Q + Q\Omega - \Omega Q - \lambda|Q|D = \Gamma H, \\ \nabla\cdot u = 0, \end{cases} \tag{1.1}$$

with

$$H = \Delta Q - aQ + b\left(Q^2 - \frac{\mathrm{tr}(Q^2)}{3}I_{3\times 3}\right) - cQ\mathrm{tr}(Q^2).$$

Here $u \in \mathbb{R}^3$, $P \in \mathbb{R}$ and

$$Q \in S_0^3 \triangleq \{A = (a_{ij})_{3\times 3} | a_{ij} = a_{ji},\ \mathrm{tr}(A) = 0\}$$

stand for the flow velocity, the scalar pressure and the nematic tensor order parameter, respectively. The parameters $\mu > 0$, $\Gamma^{-1} > 0$ and $\lambda \in \mathbb{R}$ represent the viscosity coefficient, the rotational viscosity and the nematic alignment, respectively. The coefficients $a, b, c \in \mathbb{R}$ with $c > 0$ are constants. $(\nabla Q \odot \nabla Q)_{i,j} = \partial_{x_i} Q : \partial_{x_j} Q$ is the symmetric additional stress

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tensor. $D \triangleq \frac{1}{2}(\nabla u + \nabla u^{\mathsf{T}})$ and $\Omega \triangleq \frac{1}{2}(\nabla u - \nabla u^{\mathsf{T}})$ are the symmetric and skew symmetric, respectively, where the notation T represents the transposition of a matrix.

When $Q = 0$, system (1.1) reduces to the 3D stationary Navier-Stokes system:

$$\begin{cases} -\Delta u + (u \cdot \nabla) u + \nabla P = 0, \\ \nabla \cdot u = 0. \end{cases} \tag{1.2}$$

In the past decades, there have been many results in the study of Liouville theorem for system (1.2). Specifically, Galdi [1] proved that when $u \in L^{\frac{9}{2}}(\mathbb{R}^3)$, then $u = 0$. Chae-Wölf [2] gave a logarithmic improvement, namely when the solution u satisfies

$$\int_{\mathbb{R}^3} |u|^{\frac{9}{2}} \left\{ \log \left(2 + \frac{1}{|u|} \right) \right\}^{-1} dx < \infty,$$

then $u = 0$. Kozono-Terasawa-Wakasugi [3] investigated Liouville theorem in Lorentz spaces $L^{\frac{9}{2},\infty}(\mathbb{R}^3)$. Li-Niu [4] further obtained sufficient conditions in Lorentz spaces, that is, $u \in L^{p,q}(\mathbb{R}^3)$ for $3 < p < \frac{9}{2}$, $3 \leq q \leq \infty$ or $p = q = 3$. Jarrin [5] proved in Morrey spaces that when $u \in L^2_{loc}(\mathbb{R}^3) \cap \dot{M}^{p,r}(\mathbb{R}^3)$ with $3 \leq p < r < \frac{9}{2}$, then $u = 0$. Later, Chamorro-Jarrin-Lemari-Rieusset [6] extended the result in [5] to $u \in \dot{M}^{2,3}(\mathbb{R}^3) \cap \dot{M}^{2,p}(\mathbb{R}^3)$ with $3 < p < \infty$. For more works on Liouville theorem for the 3D stationary Navier-Stokes system, one could refer to [7–12] and references therein.

In recent years, the Q-tensor system of liquid crystal (1.1) has been received much attention, however there are few results on its Liouville theorem (see [13, 14]). Gong-Liu-Zhang [13] proved that when

$$u \in L^{\frac{9}{2}}(\mathbb{R}^3) \cap \dot{H}_1(\mathbb{R}^3),\ Q \in H^2(\mathbb{R}^3) \text{ and } b^2 - 24ac \leq 0, \tag{1.3}$$

then $u = Q \equiv 0$. Later, Lai and Wu [14] generalized the condition (1.3) to

$$u \in L^{\frac{9}{2},\infty}(\mathbb{R}^3) \cap \dot{H}_1(\mathbb{R}^3),\ Q \in H^2(\mathbb{R}^3) \text{ and } b^2 - 24ac \leq 0. \tag{1.4}$$

Concerning Liouville theorem for other systems, one can refer to [15–21] etc.

Motivated by the works [4–6], in this paper we will consider Liouville theorem for system (1.1) in Morrey and Lorentz spaces, respectively. First of all, we recall the definition of Lorentz spaces.

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a measurable function. The distribution function $d_f(\alpha)$ is defined as

$$d_f(\alpha) = dx(\{x \in \mathbb{R}^3 : |f(x)| > \alpha\}),$$

where dx denotes the Lebesgue measure. For $1 \leq p < +\infty$ and $1 \leq q \leq +\infty$, the Lortenz spaces $L^{p,q}(\mathbb{R}^3)$ are the spaces of measurable function $f : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\|f\|_{L^{p,q}(\mathbb{R}^3)} < +\infty,$$

where

$$\|f\|_{L^{p,q}(\mathbb{R}^3)} = \begin{cases} r^{\frac{1}{q}} \left(\int_0^{+\infty} \left(\alpha d_f^{\frac{1}{p}}(\alpha) \right)^q \frac{d\alpha}{\alpha} \right)^{\frac{1}{q}}, & \text{if } q < +\infty, \\ \sup\limits_{\alpha > 0} \left\{ \alpha d_f^{\frac{1}{p}}(\alpha) \right\}, & \text{if } q = +\infty. \end{cases}$$

This space is a homogeneous space of degree $-\frac{3}{r}$ and we have the following continuous embedding $L^p(\mathbb{R}^3) = L^{p,p}(\mathbb{R}^3) \subset L^{p,q}(\mathbb{R}^3)$ for $1 \le p < q \le +\infty$.

In the framework of Lorentz spaces, our result is stated as follows.

Theorem 1.1 Let $(u, Q) \in L^2_{loc}(\mathbb{R}^3)$ be a smooth solution to system (1.1). If one of the following holds,
(1) $u \in L^{p,q}(\mathbb{R}^3),\ Q \in H^2(\mathbb{R}^3),\ b^2 - 24ac \le 0$, with $3 < p \le \frac{9}{2},\ 3 \le q < \infty$, or $p = q = 3$,
(2) $u \in L^{p,q}(\mathbb{R}^3) \cap \dot{H}_1(\mathbb{R}^3),\ Q \in H^2(\mathbb{R}^3),\ b^2 - 24ac \le 0$, with $3 < p \le \frac{9}{2},\ 3 \le q \le \infty$, or $p = q = 3$, then $u = Q \equiv 0$.

Remark 1 The difference between (1) and (2) in Theorem 1.1 is that when $q = \infty$, the conditions for the velocity field need $u \in \dot{H}_1(\mathbb{R}^3)$.

Remark 2 When $(p, q) = (\frac{9}{2}, \frac{9}{2})$, we get the condition (1.3) in [13]; while $(p, q) = (\frac{9}{2}, \infty)$, that is the condition (1.4) in [13]. Therefore, Theorem 1.1 improves the results in [13] and [14].

We also establish the results in the framework of Morrey spaces. To this end, we first recall the definition of the homogeneous Morrey spaces $\dot{M}^{p,q}(\mathbb{R}^3)$.

For $1 < p < q < +\infty$, the homogeneous Morrey spaces $\dot{M}^{p,q}(\mathbb{R}^3)$ are the set of all $f \in L^p_{loc}(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{M}^{p,q}(\mathbb{R}^3)} = \sup_{R>0, x_0 \in \mathbb{R}^3} R^{\frac{3}{q}} \left(R^{-3} \int_{B(R(x_0))} |f(x)|^p dx \right)^{\frac{1}{p}} < +\infty,$$

where $B(R(x_0))$ is a ball of radius R centered at x_0. This space satisfies the following embedding relation

$$L^q(\mathbb{R}^3) \subset L^{q,r}(\mathbb{R}^3) \subset L^{q,\infty}(\mathbb{R}^3) \subset \dot{M}^{p,q}(\mathbb{R}^3),\ 1 < p < q \le r \le \infty.$$

In the framework of Morrey spaces, our two results are stated as follows.

Theorem 1.2 Let $(u, Q) \in L^2_{loc}(\mathbb{R}^3)$ be a smooth solution to system (1.1). Assume $u \in \dot{M}^{2,3}(\mathbb{R}^3) \cap \dot{M}^{2,q}(\mathbb{R}^3),\ Q \in H^2(\mathbb{R}^3),\ b^2 - 24ac \le 0$, with $3 < q < \infty$, then $u = Q \equiv 0$.

Theorem 1.3 Let $(u, Q) \in L^2_{loc}(\mathbb{R}^3)$ be a smooth solution to system (1.1). Suppose $u \in \dot{M}^{p,q}(\mathbb{R}^3),\ Q \in H^2(\mathbb{R}^3),\ b^2 - 24ac \le 0$, with $3 \le p < q \le \frac{9}{2}$, then $u = Q \equiv 0$.

Remark 3 When $q = \frac{9}{2}$, compared to the conditions (1.3) and (1.4), due to the embedding relationship $L^{\frac{9}{2}}(\mathbb{R}^3) \subset L^{\frac{9}{2},\infty}(\mathbb{R}^3) \subset \dot{M}^{p,\frac{9}{2}}(\mathbb{R}^3)$, we know that Theorem 1.3 extends the results in [13, 14].

The rest of this paper is organized as follows. In Section 2 we will prove Theorem 1.1 in the Lorentz spaces, while Section 3 is devoted to the proofs of Theorem 1.2 and Theorem 1.3 in the Morrey spaces.

2 Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. Firstly, we provide two known results as follows.

Lemma 2.1 (see [22]) Let $\beta(Q) = 1 - 6\frac{(\mathrm{tr}Q^3)^2}{|Q|^6}$. Assume $Q \in S_0^3$, then $0 \le \beta(Q) \le 1$.

Lemma 2.2 (see [6]) Let $U \in L^\infty(\mathbb{R}^3) \cap \dot{M}^{2,3}(\mathbb{R}^3)$. Then $\|U\|_{\dot{M}^{3,\frac{9}{2}}(\mathbb{R}^3)} \le C\|U\|^{\frac{1}{3}}_{L^\infty(\mathbb{R}^3)}\|U\|^{\frac{2}{3}}_{\dot{M}^{2,3}(\mathbb{R}^3)}$.

We consider a smooth function $\phi \in C_c^\infty(\mathbb{R}^3)$ as

$$\phi(x) = \begin{cases} 1, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 2 \end{cases}$$

and define $\phi_R(x) = \phi(\frac{x}{R})$, $x \in \mathbb{R}^3$, so we have supp $(\nabla\phi_R) \subset \{x \in \mathbb{R}^3; R < |x| < 2R\} = B(2R/R)$.

Proof Multiplying the first equation and the second equation in (1.1) by $u\phi_R$ and $-H\phi_R$, respectively, integrating them over $\mathbb{R}^3$, and integration by parts, we have

$$\begin{aligned}
&\mu\int_{\mathbb{R}^3}|\nabla u|^2\phi_R(x)\mathrm{d}x + \Gamma\int_{\mathbb{R}^3}|H|^2\phi_R(x)\mathrm{d}x \\
={}&\left(\int_{\mathbb{R}^3}(u\cdot\nabla Q):\Delta Q\phi_R(x)\mathrm{d}x - \int_{\mathbb{R}^3}(\Omega Q - Q\Omega):\Delta Q\phi_R(x)\mathrm{d}x\right) \\
&-\int_{\mathbb{R}^3}(u\cdot\nabla Q):\left[aQ + b\left(Q^2 - \frac{\mathrm{tr}(Q^2)}{3}I_{3\times3}\right) + cQ|Q|^2\right]\phi_R(x)\mathrm{d}x \\
&+\int_{\mathbb{R}^3}(\Omega Q - Q\Omega):\left[aQ + b\left(Q^2 - \frac{\mathrm{tr}(Q^2)}{3}I_{3\times3}\right) + cQ|Q|^2\right]\phi_R(x)\mathrm{d}x \\
&-\left(\lambda\int_{\mathbb{R}^3}|Q|D:H\phi_R(x)\mathrm{d}x + \int_{\mathbb{R}^3}\nabla\cdot(\nabla Q\odot\nabla Q)\cdot u\phi_R(x)\mathrm{d}x\right) \\
&+\left(\lambda\int_{\mathbb{R}^3}|Q|H:\nabla(u\phi_R(x))\mathrm{d}x - \int_{\mathbb{R}^3}(Q\Delta Q - \Delta QQ):\nabla(u\phi_R(x))\mathrm{d}x\right) \\
&+\int_{\mathbb{R}^3}\frac{1}{2}|u|^2u\cdot\nabla\phi_R(x)\mathrm{d}x + \left(\int_{\mathbb{R}^3}Pu\cdot\nabla\phi_R(x) + \frac{\mu}{2}|u|^2\Delta\phi_R(x)\mathrm{d}x\right) \\
\triangleq{}&\sum_{i=1}^{7}I_i.
\end{aligned} \tag{2.1}$$

Firstly, we deduce that $I_4 = 0$ due to Q is symmetric and Ω is skew symmetric. In what follows, we estimate the remaining terms on the right-hand side of (2.1) one by one.

For I_1, applying Hölder inequality in Lorentz spaces, we see that

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^3}(u\cdot\nabla Q):\Delta Q\phi_R dx - \int_{\mathbb{R}^3}\nabla\cdot(\nabla Q\odot\nabla Q)\cdot u\phi_R\mathrm{d}x \\
&= \int_{\mathbb{R}^3}(u\cdot\nabla Q):\Delta Q\phi_R dx - \int_{\mathbb{R}^3}(u\cdot\nabla Q):\Delta Q\phi_R dx - \frac{1}{2}\int_{\mathbb{R}^3}u\cdot\nabla|\nabla Q|^2\phi_R\mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}^3}u|\nabla Q|^2\nabla\phi_R\mathrm{d}x \\
&\le \frac{C}{R}\||\nabla Q|^2\|_{L^2(B(2R\backslash R))}\|u\|_{L^{p,q}(B(2R\backslash R))}\|1\|_{L^{\frac{2p}{p-2},\frac{2q}{q-2}}(B(2R\backslash R))} \\
&\le CR^{\frac{1}{2}-\frac{3}{p}}\||\nabla Q|^2\|_{L^2(B(2R\backslash R))}\|u\|_{L^{p,q}(B(2R\backslash R))} \to 0 \quad (as \quad R\to\infty).
\end{aligned} \tag{2.2}$$

By Sobolev imbedding theorem, we have

$$\begin{aligned}
I_2 &= -\int_{\mathbb{R}^3}(u\cdot\nabla Q):\left[aQ-b\left(Q^2-\frac{\mathrm{tr}(Q^2)}{3}I_{3\times 3}\right)+cQ|Q|^2\right]\phi_R\mathrm{d}x\\
&=\int_{\mathbb{R}^3}\left(\frac{a}{2}|Q|^2-\frac{b}{3}\mathrm{tr}(Q^3)+\frac{c}{4}|Q|^4\right)u\cdot\nabla\phi_R\mathrm{d}x\\
&\le\frac{C}{R}\left(1+\|Q\|_{L^\infty(B(2R\backslash R))}+\|Q\|^2_{L^\infty(B(2R\backslash R))}\right)\||Q|^2\|_{L^2(B(2R\backslash R))}\|u\|_{L^{p,q}(B(2R\backslash R))}\|1\|_{L^{\frac{2p}{p-2},\frac{2q}{q-2}}(B(2R\backslash R))}\\
&\le CR^{\frac{1}{2}-\frac{3}{p}}\||Q|^2\|_{L^2(B(2R\backslash R))}\|u\|_{L^{p,q}(B(2R\backslash R))}\to 0\quad(as\quad R\to\infty).
\end{aligned}\tag{2.3}$$

For I_3, we obtain

$$\begin{aligned}
I_3 &= -\int_{\mathbb{R}^3}(\Omega Q-Q\Omega):\Delta Q\phi_R-\int_{\mathbb{R}^3}(Q\Delta Q-\Delta QQ):\nabla(u\phi_R)\mathrm{d}x\\
&=-\int_{\mathbb{R}^3}(Q\Delta Q-\Delta QQ):u\otimes\nabla\phi_R\mathrm{d}x\\
&\le\frac{C}{R}\|\nabla^2Q\|_{L^2(B(2R\backslash R))}\|Q\|_{L^\infty(B(2R\backslash R))}\|u\|_{L^{p,q}(B(2R\backslash R))}\|1\|_{L^{\frac{2p}{p-2},\frac{2q}{q-2}}(B(2R\backslash R))}\\
&\le CR^{\frac{1}{2}-\frac{3}{p}}\|\nabla^2Q\|_{L^2(B(2R\backslash R))}\|u\|_{L^{p,q}(B(2R\backslash R))}\to 0\quad(as\quad R\to\infty).
\end{aligned}\tag{2.4}$$

With regard to I_5, we observe

$$\begin{aligned}
I_5 &= -\lambda\int_{\mathbb{R}^3}|Q|D:H\phi_R(x)\mathrm{d}x+\lambda\int_{\mathbb{R}^3}|Q|H:\nabla(u\phi_R(x))\mathrm{d}x\\
&=\lambda\int_{\mathbb{R}^3}|Q|H:(u\otimes\nabla\phi_R(x))\mathrm{d}x\\
&\le\frac{C}{R}\|H\|_{L^2(B(2R\backslash R))}\|Q\|_{L^\infty(B(2R\backslash R))}\|u\|_{L^{p,q}(B(2R\backslash R))}\|1\|_{L^{\frac{2p}{p-2},\frac{2q}{q-2}}(B(2R\backslash R))}\\
&\le CR^{\frac{1}{2}-\frac{3}{p}}\|H\|_{L^2(B(2R\backslash R))}\|u\|_{L^{p,q}(B(2R\backslash R))}\to 0\quad(as\quad R\to\infty).
\end{aligned}\tag{2.5}$$

By Hölder inequality in Lorentz spaces, we see

$$\begin{aligned}
I_6 &= \frac{1}{2}\int_{\mathbb{R}^3}|u|^2u\cdot\nabla\phi_R\mathrm{d}x\\
&\le\frac{C}{R}\|u\|^3_{L^{p,q}(B(2R\backslash R))}\|1\|_{L^{\frac{p}{p-3},\frac{q}{q-3}}(B(2R\backslash R))}\\
&\le CR^{2-\frac{9}{p}}\|u\|^3_{L^{p,q}(B(2R\backslash R))}\to 0\quad(as\quad R\to\infty).
\end{aligned}\tag{2.6}$$

To deal with I_7, we first need to estimate the pressure. For that, taking the divergence to the first equation of (1.1) yields

$$\Delta P=-\mathrm{divdiv}\left(u\otimes u+\nabla Q\odot\nabla Q+\lambda|Q|H+Q\Delta Q-\Delta QQ\right).\tag{2.7}$$

Let $P=P_1+P_2$ such that $\Delta P_1=-\mathrm{divdiv}(f_1)$ and $\Delta P_2=-\mathrm{divdiv}(f_2)$, where

$$f_1=u\otimes u,\qquad f_2=\nabla Q\odot\nabla Q+\lambda|Q|H+Q\Delta Q-\Delta QQ.\tag{2.8}$$

In view of the conditions $u\in L^{p,q}(\mathbb{R}^3)$ and $Q\in H^2(\mathbb{R}^3)$, it is not difficult to obtain that

$$f_1\in L^{\frac{p}{2},\frac{q}{2}}(\mathbb{R}^3),\quad f_2\in L^2(\mathbb{R}^3).$$

By Calderón-Zygmund theorem, we have $P_1 \in L^{\frac{p}{2},\frac{q}{2}}(\mathbb{R}^3),\ P_2 \in L^2(\mathbb{R}^3)$, and then

$$\begin{aligned}
I_7 &= \int_{\mathbb{R}^3} \Big(Pu\cdot\nabla\phi_R + \frac{\mu}{2}|u|^2\Delta\phi_R\Big)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} P_1u\cdot\nabla\phi_R\mathrm{d}x + \int_{\mathbb{R}^3} P_2u\cdot\nabla\phi_R\mathrm{d}x + \frac{\mu}{2}\int_{\mathbb{R}^3}|u|^2\Delta\phi_R\mathrm{d}x \\
&\le \frac{C}{R}\|P_1\|_{L^{\frac{p}{2},\frac{q}{2}}(B(2R\backslash R))}\|u\|_{L^{p,q}(B(2R\backslash R))}\|1\|_{L^{\frac{p}{p-3},\frac{q}{q-3}}(B(2R\backslash R))} \\
&\quad + \frac{C}{R}\|P_2\|_{L^2(B(2R\backslash R))}\|u\|_{L^{p,q}(B(2R\backslash R))}\|1\|_{L^{\frac{2p}{p-2},\frac{2q}{q-2}}(B(2R\backslash R))} \\
&\quad + \frac{C}{R^2}\|u\|^2_{L^{p,q}(B(2R\backslash R))}\|1\|_{L^{\frac{p}{p-2},\frac{q}{q-2}}(B(2R\backslash R))} \\
&\le CR^{2-\frac{9}{p}}\|P_1\|_{L^{\frac{p}{2},\frac{q}{2}}(B(2R\backslash R))}\|u\|_{L^{p,q}(B(2R\backslash R))} \\
&\quad + CR^{\frac{1}{2}-\frac{3}{p}}\|P_2\|_{L^2(B(2R\backslash R))}\|u\|_{L^{p,q}(B(2R\backslash R))} \\
&\quad + CR^{1-\frac{6}{p}}\|u\|^2_{L^{p,q}(B(2R\backslash R))} \to 0 \quad (as \quad R\to\infty).
\end{aligned} \tag{2.9}$$

Based on the above estimates $I_1 - I_7$, returning to (2.1), we find that

$$\lim_{R\to+\infty}\Big(\mu\int_{B(R)}|\nabla u|^2\mathrm{d}x + \Gamma\int_{B(R)}|H|^2\mathrm{d}x\Big) = 0. \tag{2.10}$$

Thanks to the Sobolev embedding, we have

$$\|u\|_{L^6(\mathbb{R}^3)} \le C\|\nabla u\|_{L^2(\mathbb{R}^3)}, \tag{2.11}$$

which implies $u = H \equiv 0$.

In what follows, we give the estimate about $|\nabla Q|$. By the definition of H, we have

$$-\Delta Q = -aQ + b\left(Q^2 - \frac{tr(Q^2)}{3}I_{3\times3}\right) - cQ\mathrm{tr}(Q^2). \tag{2.12}$$

Multiplying (2.12) by $Q\phi_R$ and integrating over $\mathbb{R}^3$, one has from Lemma 2.1 that

$$\begin{aligned}
\int_{\mathbb{R}^3}|\nabla Q|^2\phi_R\mathrm{d}x &= \int_{\mathbb{R}^3}|Q|^2\Delta\phi_R\mathrm{d}x - \int_{\mathbb{R}^3}\Big(a\mathrm{tr}(Q^2) - b\mathrm{tr}(Q^3) + c[\mathrm{tr}(Q^2)]^2\phi_R\Big)\mathrm{d}x \\
&\le \frac{C}{R^2}\int_{B(2R\backslash R)}|Q|^2\mathrm{d}x + \int_{\mathbb{R}^3}\left(\frac{b^2-24ac}{24c}\right)|Q|^2\mathrm{d}x \\
&\le \frac{C}{R^2}\int_{B(2R\backslash R)}|Q|^2\mathrm{d}x \to 0 \quad (as \quad R\to\infty).
\end{aligned} \tag{2.13}$$

In view of $Q \in H^2(\mathbb{R}^3)$, we obtain $Q = 0$. The proof of Theorem 1.1 is finished.

3 Proofs of Theorem 1.2 and Theorem 1.3

In this section, we are going to prove Theorem 1.2 and Theorem 1.3. Theorem 1.2 is demonstrated through the essential application of Lemma 2.2. To this end, we first claim that $u \in L^\infty(\mathbb{R}^3)$. In fact, according to the conditions of Theorem 1.2, we know that $u \in \dot{M}^{2,q}(\mathbb{R}^3),\ Q \in H^2(\mathbb{R}^3)$, with $3 < q < \infty$. By [23], and using the same method as in [6], we could obtain $u \in L^\infty(\mathbb{R}^3)$.

Proof of Theorem 1.2 We reestimate the terms $I_1 - I_7$ on the right-hand side of (2.1) in the framework of the Morrey spaces. For I_1, making use of (2.2), and the Hölder inequality, we observe

$$\begin{aligned} I_1 &= \frac{1}{2}\int_{\mathbb{R}^3} u|\nabla Q|^2 \nabla \phi_R \mathrm{d}x \\ &\le \frac{C}{R}\left(\int_{B(2R\backslash R)} |u|^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{B(2R\backslash R)} |\nabla Q|^4 \mathrm{d}x\right)^{\frac{1}{2}} \\ &\le CR^{-\frac{1}{2}}\|u\|_{\dot{M}^{2,3}(B(2R\backslash R))}\||\nabla Q|^2\|_{L^2(B(2R\backslash R))} \to 0 \quad (as \quad R \to \infty). \end{aligned} \tag{3.1}$$

With respect to I_2, it follows from (2.3) and the Sobolev imbedding that

$$\begin{aligned} I_2 &= \int_{\mathbb{R}^3} \left(\frac{a}{2}|Q|^2 - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}|Q|^4\right) u \cdot \nabla \phi_R \mathrm{d}x \\ &\le \frac{C}{R}\left(1 + \|Q\|_{L^\infty(B(2R\backslash R))} + \|Q\|^2_{L^\infty(B(2R\backslash R))}\right)\|Q\|^2_{L^4(B(2R\backslash R))}\|u\|_{L^2(B(2R\backslash R))} \\ &\le CR^{-\frac{1}{2}}\|u\|_{\dot{M}^{2,3}(B(2R\backslash R))}\|Q\|^2_{L^4(B(2R\backslash R))} \to 0 \quad (as \quad R \to \infty). \end{aligned} \tag{3.2}$$

One deduces from (2.4) and Lemma 2.2 that

$$\begin{aligned} I_3 &= -\int_{\mathbb{R}^3} (Q\Delta Q - \Delta Q Q) : u \otimes \nabla \phi_R \mathrm{d}x \\ &\le \frac{C}{R}\|\nabla^2 Q\|_{L^2(B(2R\backslash R))}\|u\|_{L^3(B(2R\backslash R))}\|Q\|_{L^6(B(2R\backslash R))} \\ &\le CR^{-\frac{2}{3}}\|u\|_{\dot{M}^{3,\frac{9}{2}}(B(2R\backslash R))}\|\nabla^2 Q\|_{L^2(B(2R\backslash R))}\|Q\|_{L^6(B(2R\backslash R))} \to 0 \quad (as \quad R \to \infty). \end{aligned} \tag{3.3}$$

We get from (2.5) that when $R \to \infty$,

$$\begin{aligned} I_5 &= \lambda \int_{\mathbb{R}^3} |Q|H : (u \otimes \nabla \phi_R(x))\mathrm{d}x \\ &\le \frac{C}{R}\|H\|_{L^2(B(2R\backslash R))}\|u\|_{L^3(B(2R\backslash R))}\|Q\|_{L^6(B(2R\backslash R))} \\ &\le CR^{-\frac{2}{3}}\|u\|_{\dot{M}^{3,\frac{9}{2}}(B(2R\backslash R))}\|H\|_{L^2(B(2R\backslash R))}\|Q\|_{L^6(B(2R\backslash R))} \to 0. \end{aligned} \tag{3.4}$$

For I_6, similar to I_1, we see by (2.6) that

$$\begin{aligned} I_6 &= \frac{1}{2}\int_{\mathbb{R}^3} |u|^2 u \cdot \nabla \phi_R \mathrm{d}x \\ &\le \frac{C}{R}\int_{B(2R\backslash R)} |u|^3 \mathrm{d}x \le \frac{C}{R}\left(R^{9(\frac{1}{3}-\frac{2}{9})}\|u\|^3_{\dot{M}^{3,\frac{9}{2}}(B(2R\backslash R))}\right) \\ &\le C\|u\|^3_{\dot{M}^{3,\frac{9}{2}}(B(2R\backslash R))} \to 0 \quad (as \quad R \to \infty). \end{aligned} \tag{3.5}$$

In view of (2.7)-(2.9), we can get

$$\begin{aligned}
I_7 &= \int_{\mathbb{R}^3} \Big(Pu \cdot \nabla\phi_R + \frac{\mu}{2}|u|^2 \Delta\phi_R \Big) \mathrm{d}x \\
&= \int_{\mathbb{R}^3} P_1 u \cdot \nabla\phi_R \mathrm{d}x + \int_{\mathbb{R}^3} P_2 u \cdot \nabla\phi_R \mathrm{d}x + \frac{\mu}{2}\int_{\mathbb{R}^3} |u|^2 \Delta\phi_R \mathrm{d}x \\
&\le \frac{C}{R}\int_{B(2R\backslash R)} |u|^3 \mathrm{d}x + \frac{C}{R}\|u\|_{L^2(B(2R\backslash R))}\|P_2\|_{L^2(B(2R\backslash R))} + \frac{C}{R^2}\int_{B(2R\backslash R)} |u|^2 \mathrm{d}x \\
&\le C\|u\|^3_{\dot{M}^{3,\frac{9}{2}}(B(2R\backslash R))} + CR^{-\frac{1}{2}}\|u\|_{\dot{M}^{2,3}(B(2R\backslash R))}\|P_2\|_{L^2(B(2R\backslash R))} \\
&\quad + CR^{-\frac{1}{3}}\|u\|^2_{\dot{M}^{3,\frac{9}{2}}(B(2R\backslash R))} \to 0 \quad (as \quad R \to \infty).
\end{aligned} \tag{3.6}$$

Setting $R \to +\infty$, by (2.10) and (2.11), yields $u = H \equiv 0$. Similarly, from (2.12) and (2.13), we also obtain $Q \equiv 0$. Therefore, we have completed the proof of Theorem 1.2.

Proof of Theorem 1.3 To prove Theorem 1.3, we use different Hölder indicators to reestimate the terms $I_1 - I_7$ in (2.1). For I_1, by Hölder inequality and the definition of Morrey spaces, we deduce from (3.1) that

$$\begin{aligned}
I_1 &= \frac{1}{2}\int_{\mathbb{R}^3} u|\nabla Q|^2 \nabla\phi_R \mathrm{d}x \\
&\le \frac{C}{R}\Big(\int_{B(2R\backslash R)} |u|^p \mathrm{d}x\Big)^{\frac{1}{p}} \Big(\int_{B(2R\backslash R)} |\nabla Q|^4 \mathrm{d}x\Big)^{\frac{1}{2}} \Big(\int_{B(2R\backslash R)} 1 \,\mathrm{d}x\Big)^{\frac{1}{2}-\frac{1}{p}} \\
&\le CR^{-1}R^{(\frac{3}{2}-\frac{3}{p})}R^{3(\frac{1}{p}-\frac{1}{q})}\|u\|_{\dot{M}^{p,q}(B(2R\backslash R))}\||\nabla Q|^2\|_{L^2(B(2R\backslash R))} \\
&\le CR^{\frac{1}{2}-\frac{3}{q}}\|u\|_{\dot{M}^{p,q}(B(2R\backslash R))}\||\nabla Q|^2\|_{L^2(B(2R\backslash R))}.
\end{aligned}$$

Similarly, we could obtain the following new estimations of (3.2)-(3.6) as

$$\begin{aligned}
I_2 &\le CR^{\frac{1}{2}-\frac{3}{q}}\|u\|_{\dot{M}^{p,q}(B(2R\backslash R))}\|Q\|^2_{L^4(B(2R\backslash R))}, \\
I_3 &\le CR^{\frac{1}{2}-\frac{3}{q}}\|u\|_{\dot{M}^{p,q}(B(2R\backslash R))}\|\nabla^2 Q\|_{L^2(B(2R\backslash R))}, \\
I_5 &\le CR^{\frac{1}{2}-\frac{3}{q}}\|u\|_{\dot{M}^{p,q}(B(2R\backslash R))}\|H|_{L^2(B(2R\backslash R))}, \\
I_6 &\le CR^{2-\frac{9}{q}}\|u\|^3_{\dot{M}^{p,q}(B(2R\backslash R))}, \\
I_7 &\le CR^{2-\frac{9}{q}}\|u\|_{\dot{M}^{p,q}(B(2R\backslash R))}\|P_1\|_{\dot{M}^{\frac{p}{2},\frac{q}{2}}(B(2R\backslash R))} \\
&\quad + CR^{\frac{1}{2}-\frac{3}{q}}\|u\|_{\dot{M}^{p,q}(B(2R\backslash R))}\|P_2\|_{L^2(B(2R\backslash R))} + CR^{1-\frac{6}{q}}\|u\|^2_{\dot{M}^{p,q}(B(2R\backslash R))}.
\end{aligned}$$

Collecting all the above estimations implies

$$\begin{aligned}
&\mu\int_{\mathbb{R}^3} |\nabla u|^2 \phi_R(x) \mathrm{d}x + \Gamma\int_{\mathbb{R}^3} |H|^2 \phi_R(x) \mathrm{d}x \\
\le& CR^{\frac{1}{2}-\frac{3}{q}}\|u\|_{\dot{M}^{p,q}(B(2R\backslash R))}\||\nabla Q|^2\|_{L^2(B(2R\backslash R))} + CR^{\frac{1}{2}-\frac{3}{q}}\|u\|_{\dot{M}^{p,q}(B(2R\backslash R))}\|Q\|^2_{L^4(B(2R\backslash R))} \\
&+ CR^{\frac{1}{2}-\frac{3}{q}}\|u\|_{\dot{M}^{p,q}(B(2R\backslash R))}\|\nabla^2 Q\|_{L^2(B(2R\backslash R))} + CR^{\frac{1}{2}-\frac{3}{q}}\|u\|_{\dot{M}^{p,q}(B(2R\backslash R))}\|H|_{L^2(B(2R\backslash R))} \\
&+ CR^{2-\frac{9}{q}}\|u\|^3_{\dot{M}^{p,q}(B(2R\backslash R))} + CR^{2-\frac{9}{q}}\|u\|_{\dot{M}^{p,q}(B(2R\backslash R))}\|P_1\|_{\dot{M}^{\frac{p}{2},\frac{q}{2}}(B(2R\backslash R))} \\
&+ CR^{\frac{1}{2}-\frac{3}{q}}\|u\|_{\dot{M}^{p,q}(B(2R\backslash R))}\|P_2\|_{L^2(B(2R\backslash R))} + CR^{1-\frac{6}{q}}\|u\|^2_{\dot{M}^{p,q}(B(2R\backslash R))}.
\end{aligned} \tag{3.7}$$

By the condition $3 \le p < q \le \frac{9}{2}$, we have

$$\frac{1}{2} - \frac{3}{q} \le 0,\ 2 - \frac{9}{q} \le 0,\ 1 - \frac{6}{q} \le 0.$$

Thus, returning to (3.7), we get

$$\lim_{R \to +\infty} \left(\mu \int_{B(R)} |\nabla u|^2 \mathrm{d}x + \Gamma \int_{B(R)} |H|^2 \mathrm{d}x \right) = 0.$$

By virtue of the Sobolev embedding $\|u\|_{L^6(\mathbb{R}^3)} \le C \|\nabla u\|_{L^2(\mathbb{R}^3)}$, we have $u = H \equiv 0$. And using the same method dealing with (2.12) and (2.13) one concludes $Q \equiv 0$, which ends the proof of Theorem 1.3.

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三维稳态 Q-tensor 液晶流系统的 Liouville 定理

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摘要: 本文研究了三维稳态 Q-tensor 液晶系统在 Lorentz 和 Morrey 空间中的 Liouville 定理. 在 Lorentz 和 Morrey 空间的一些附加假设下, 利用能量估计, 获得了平凡解 $u = Q \equiv 0$ 是唯一解. 我们的定理是对最近一些结果的改进, 并包含一些已知的结果作为特例.

关键词: Lorentz 空间; Morrey 空间; Q-tensor; 液晶; Liouville 定理

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