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一类多维随机微分方程的对数截断 EM 方法

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摘 要: 本文研究了一类多维随机微分方程 (SDEs) 的保正性数值求解问题. 利用修正原有理论中的假设条件并结合对数变换的性质, 构造新型截断函数的方法. 获得了该数值方法的强收敛性结果, 证明了在一定条件下数值解的 L^p 收敛性, 且强收敛速率可达 1/2 阶. 推广了原有方法的结果, 成功将对数截断 EM 方法的应用范围从标量 SDEs 拓展至多维 SDEs.

关 键 词: 多维随机微分方程 ; 保正性 ; 对数截断 EM 方法 ; 数值模拟
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1 引言

随着微分方程在生物学、金融学、工程学和环境科学等领域的广泛应用, 如何有效刻画具有正性约束的随机系统已成为当前研究的重点问题. 这类系统在描述种群演化、资产定价、化学反应等实际过程时, 其数学模型的解必须严格保持正值才具有物理意义.

近年来, 众多学者（见文献 [1–8]）致力于发展具有保正性、精度与稳定性的数值方法. Neuenkirch 与 Szpruch 在文献 [9] 中针对取值于特定区域的 SDE, 提出了漂移隐式 Euler-Maruyama(EM) 方法. 然而值得注意的是, 尽管该方法在理论上具有重要意义, 但其隐式特性导致计算成本较高. 此后, Mao[10] 开创性地提出截断 EM 方法, 这是一种用于研究高度非线性随机微分方程数值解的显式方法. Mao 和 Hu[11, 12] 通过改进步长约束条件, 进一步提升了该方法的适用性. 然而, 上述方法在处理解必须保持正性的 SDEs 时存在固有缺陷. 难以直接应用于正约束随机模型. 针对这一挑战, 众多学者相继提出了显式保正 EM 格式的创新解法. 其中, 对数截断 EM 方法（见文献 [13, 14]）因其计算效率与保正特性的良好平衡而备受关注. 最近, 文献 [15] 通过弱化约束条件, 建立了标量 SDEs 对数截断 EM 方法的理论框架. 但需要指出的是, 其框架无法直接推广至多维情形. 因此, 需对假设条件进行适应性修正以拓展至多维 SDEs. 文献 [16–18] 研究了具体的一类多维的随机微分方程的保正数值方法, 但该方法限制于标量的布朗运动.

本文核心目标是进一步研究对数截断 EM 方法, 将对数截断 EM 方法拓展到多维情况, 并将布朗运动拓广到 m 维. 对原 SDE 施加假设, 结合对数变换相关的性质, 进而使得原随机微分方程数值解保持正性. 在此框架下, 我们通过构造新型截断函数, 证明了数值解的 L^p 收敛性, 其收敛速率在适当条件下可达经典的 1/2 阶. 同时, 该方法可有效应用于含负幂项系数的一类多维 SDEs, 使得该方法可处理更广泛的非线性项.

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全文结构如下: 第 2 节介绍符号体系与基本假设, 建立关键引理; 第 3 节构建多维 SDEs 的对数截断 EM 方法; 第 4 节分析其收敛速率; 第 5 节通过一个典型算例验证该方法在一般参数下对多维 SDEs 的有效性; 第 6 节基于算例进行数值模拟以佐证理论结论.

2 符号和假设

在全文论述中, 若无特殊说明, 我们设定以下基础框架: 设 $(\Omega, \mathcal{F}, \mathcal{P})$ 为完备概率空间, 其配备满足常规条件的滤子 $\{\mathcal{F}_t\}_{t\geqslant 0}$ (即右连续递增且 $\mathcal{F}_0$ 包含所有 $\mathcal{P}$- 零测集). 记 $\mathbb{E}$ 为关于测度 $\mathbb{P}$ 的数学期望算子. 在此空间上定义 m 维 $\mathcal{F}_t$- 适应布朗运动 $B(t) = (B_1(t), B_2(t), \ldots, B_m(t))^T$. 其中各分量 $B_i(t)$ $(i = 1, \ldots, m)$ 是相互独立、标准的一维布朗运动. 由布朗运动的性质可知 $B_i(t) - B_i(s)$ 服从以 0 为均值、以 $t - s$ 为方差的正态分布.

关于符号体系作如下约定:

- 向量或矩阵 A 的转置记为 A^T;
- 对 $x = (x_1, \ldots, x_d)^T \in \mathbb{R}_+^d \triangleq (0, \infty)^d$, 其欧几里得范数为 $\|x\|$;
- 若 A 表示矩阵, 用 $\|A\| = \sqrt{\mathrm{trace}(A^TA)}$ 表示其迹范数;
- 向量 x 的逐元倒数记为 (x^{-1}) 或 $\left(\frac{1}{x}\right)$, 即 $(x_1^{-1}, \ldots, x_d^{-1})^T$;
- 若 A 表示一个矩阵或向量, 则用 A_i 表示 A 的第 i 行;
- 对任意实数 a, b, 定义 $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$;
- 集合 G 的示性函数记为 I_G, 即 $I_G(x) = 1$ 当且仅当 $x \in G$, 否则 $I_G(x) = 0$;
- 规定空集 $\emptyset$ 的下确界为 ∞;
- 定义非负闭区域 $\bar{\mathbb{R}}_+^d \triangleq [0, \infty)^d$.

考虑如下定义在时间区间 $[0, T]$ 上的 d 维非线性随机微分方程:

$$dx(t) = f(x(t))dt + g(x(t))dB(t), \tag{2.1}$$

其中初始值 $x(0) = x_0 \in \mathbb{R}_+^d$, T 为固定正数. 漂移项 $f : \mathbb{R}^d \to \mathbb{R}^d$ 与扩散项 $g : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ 均为 Borel 可测函数. 本文设定以下三个基本假设条件:

(A_1) 假设漂移系数满足局部 Lipschitz: 存在实数 $K_1 > 0$, 以及 $\alpha > 0, \beta > 0$ 使得对任意 $x, y \in \mathbb{R}_+^d$, 有

$$\|f(x) - f(y)\| \leqslant K_1 \left(1 + \|x\|^\alpha + \|y\|^\alpha + \|x^{-1}\|^\beta + \|y^{-1}\|^\beta\right) \|x - y\|.$$

(A_2) 存在正实数 $x^* > 0, p > 1, q > 0$ 以及 $K_2 > 0$ 使得对任意 $x \in \mathbb{R}_+^d$, 有

$$x^T f(x) + \frac{p-1}{2}\|g(x)\|^2 \leqslant K_2(1 + \|x\|^2),$$

并且对 $i = 1, \ldots, d$ 和任意 $x_i \in (0, x^*), x_j \in \mathbb{R}_+$, 其中 $j \neq i$, 有

$$x_i f_i(x) - \frac{q+1}{2}\|g_i(x)\|^2 \geqslant 0.$$

(A_3) 存在一对正实数 $r^*>2$ 和 $H>0$ 使得对任意 $x,y\in\mathbb{R}_+^d$, 有

$$(x-y)^T(f(x)-f(y))+\frac{r^*-1}{2}\|g(x)-g(y)\|^2\leqslant H\|x-y\|^2.$$

注释 2.1 由 (A_1) 可得

$$\|f(x)\|\leqslant\|f(x)-f(\mathbf{1})\|+\|f(\mathbf{1})\|\leqslant 3(\sqrt{d})^{\alpha\vee\beta}K_1\left(1+\|x\|^\alpha+\left\|x^{-1}\right\|^\beta\right)\|x-\mathbf{1}\|+\|f(\mathbf{1})\|,$$

其中 $\mathbf{1}=(1,\ldots,1)^T$, 同时由 Young 不等式有

$$\begin{aligned}\|f(x)\|&\leqslant 3(\sqrt{d})^{\alpha\vee\beta}K_1\left(1+\|x\|^\alpha+\left\|x^{-1}\right\|^\beta\right)(\|x\|+\|\mathbf{1}\|)+\|f(\mathbf{1})\|\\&\leqslant 3(\sqrt{d})^{(\alpha\vee\beta)+1}K_1\left(1+\|x\|+\|x\|^\alpha+\left\|x^{-1}\right\|^\beta+\|x\|^{\alpha+1}+\|x\|\cdot\left\|x^{-1}\right\|^\beta\right)+\|f(\mathbf{1})\|\\&\leqslant 9(\sqrt{d})^{(\alpha\vee\beta)+1}K_1\left(1+\|x\|^{\alpha+1}+\|x\|^{\beta+1}+\left\|x^{-1}\right\|^{\beta+1}\right)+\|f(\mathbf{1})\|.\end{aligned}$$

进一步, 有

$$\|f(x)\|\leqslant\begin{cases}27(\sqrt{d})^{(\alpha\vee\beta)+1}K_1\left(1+\left\|x^{-1}\right\|^{\beta+1}\right)+\|f(\mathbf{1})\|, & \|x\|\leqslant 1\\ 18(\sqrt{d})^{(\alpha\vee\beta)+1}K_1\left(1+\|x\|^{(\alpha\vee\beta)+1}+\left\|x^{-1}\right\|^{\beta+1}\right)+\|f(\mathbf{1})\|, & \|x\|>1.\end{cases}$$

因此, 对任意 $x\in\mathbb{R}_+^d$ 有

$$\|f(x)\|\leqslant\left(27(\sqrt{d})^{(\alpha\vee\beta)+1}K_1\vee\|f(\mathbf{1})\|\right)\left(1+\|x\|^{(\alpha\vee\beta)+1}+\left\|x^{-1}\right\|^{\beta+1}\right).$$

同理, 由 (A_2) 可得, 对任意 $x\in\mathbb{R}_+^d$ 有

$$\begin{aligned}\|g(x)\|^2&\leqslant\frac{2}{p-1}\left(\|x\|\cdot\|f(x)\|+K_2(1+\|x\|^2)\right)\\&\leqslant\frac{2}{p-1}\left(M\left(\|x\|+\|x\|^{(\alpha\vee\beta)+2}+\|x\|\cdot\left\|x^{-1}\right\|^{\beta+1}\right)+K_2(1+\|x\|^2)\right),\end{aligned}$$

其中 $M=27(\sqrt{d})^{(\alpha\vee\beta)+1}K_1\vee\|f(\mathbf{1})\|$. 因此, 存在实数 C, 使得

$$\|f(x)\|\leqslant C\left(1+\|x\|^{(\alpha\vee\beta)+1}+\left\|x^{-1}\right\|^{\beta+1}\right),\quad \|g(x)\|^2\leqslant C\left(1+\|x\|^{(\alpha\vee\beta)+2}+\left\|x^{-1}\right\|^{\beta+2}\right).$$

注意到, 上式对 $\alpha=0$ 或 $\beta=0$ 也成立. 另外若 $x\in\mathbb{R}_+^d$, 则由 Hölder 不等式有

$$\left\|x^{-1}\right\|^q=\left(\sum_{i=1}^d\left(x_i^{-2}\right)\right)^{\frac{q}{2}}\leqslant\left(\left(\sum_{i=1}^d 1^{\frac{q}{q-2}}\right)^{1-\frac{2}{q}}\left(\sum_{i=1}^d x_i^{-q}\right)^{\frac{2}{q}}\right)^{\frac{q}{2}}\leqslant d^{\frac{q}{2}-1}\sum_{i=1}^d x_i^{-q},$$

其中, $q\geqslant 2$.

下述引理表明, 随机微分方程 (2.1) 在时间区间 $[0,T]$ 上存在唯一的强解, 且该解具有严格正性, 即满足

$$\mathbb{P}\left(x(t)\in\mathbb{R}_+^d,\forall t\in[0,T]\right)=1.$$

引理 2.1 在 (A_1)、(A_2)、(A_3) 及 $(\alpha+2)\vee(\beta+2)\leqslant p\wedge q$ 成立的条件下, 随机微分方程 (2.1) 在区间 $[0,T]$ 上存在唯一的强解. 进一步地, 存在常数 $C>0$, 使得以下估计成立:

$$\sup_{t\in[0,T]}\mathbb{E}\|x(t\wedge\theta)\|^p<C \quad 和 \quad \sup_{t\in[0,T]}\mathbb{E}\|x(t\wedge\theta)^{-1}\|^q<C,$$

其中 θ 是任意停时, 进一步我们有

$$\mathbb{P}\left(x(t)\in\mathbb{R}_+^d,\forall t\in[0,T]\right)=1.$$

证 在本文证明过程中, 令 C 表示一般正常数 (其具体数值在不同位置可能有所不同). 设 k 为正整数, 并令

$$\tilde{\pi}_k(x_i)=k^{-1}I_{\{x_i<k^{-1}\}}+x_iI_{\{k^{-1}\leqslant x_i\leqslant k\}}+kI_{\{k<x_i\}},$$

以及

$$\tilde{\pi}_k(x)=\left(\tilde{\pi}_k(x_1),\ldots,\tilde{\pi}_k(x_d)\right)^T,$$
$$\tilde{f}_k(x)=f\left(\tilde{\pi}_k(x)\right) \quad 和 \quad \tilde{g}_k(x)=g\left(\tilde{\pi}_k(x)\right).$$

则由注释 2.1, 有

$$\|\tilde{f}_k(x)-\tilde{f}_k(y)\|\leqslant L\|x-y\| \quad 和 \quad \|\tilde{f}_k(x)\|^2\vee\|\tilde{g}_k(x)\|^2\leqslant C(1+\|x\|^2),$$

因此 $\tilde{f}_k(x)$ 和 $\tilde{g}_k(x)$ 满足全局 Lipschitz 以及线性增长. 那么, 由 [19] 中 2.3 节可得方程

$$d\tilde{x}_k(t)=\tilde{f}_k\left(\tilde{x}_k(t)\right)dt+\tilde{g}_k\left(\tilde{x}_k(t)\right)dB(t)$$

在 $[0,T]$ 存在唯一解. 现在定义停时

$$\tau_k=\inf\{t\in[0,T]\mid\tilde{x}_k(t)\notin(\frac{1}{k},k)^d\}.$$

由于 $\tilde{x}_k(t)$ 的唯一性, 那么在 $t\in[0,T\wedge\tau_n]$ 时 $\tilde{x}_m(t)=\tilde{x}_n(t)$, 其中 $m>n$ 且足够大. 那么, τ_n 非减, 同时定义 $\tau_\infty=\lim_{n\to\infty}\tau_n$.

设 $\omega\in\Omega$. 对任意 $t<\tau_n(\omega)$, 存在 $k(\omega)>0$ 使得 $t<\tau_k(\omega)\leqslant\tau_\infty(\omega)$. 现在定义 $x(t,\omega)=\tilde{x}_k(t,\omega)$, 对于一个固定的正整数 n 和任意的 $t\in[0,T]$, 有

$$\begin{aligned}x(t\wedge\tau_n)=&\tilde{x}_n(t\wedge\tau_n)\\=&x_0+\int_0^{t\wedge\tau_n}\tilde{f}_n(\tilde{x}_n(s))ds+\int_0^{t\wedge\tau_n}\tilde{g}_n(\tilde{x}_n(s))dB(s)\\=&x_0+\int_0^{t\wedge\tau_n}f(x(s))ds+\int_0^{t\wedge\tau_n}g(x(s))dB(s).\end{aligned}$$

应用 Itô 公式、(A_2) 以及注释 2.1, 对任意 $t\in[0,T]$ 有

$$\begin{aligned}\mathbb{E}\|x(t\wedge\tau_n)\|^p\leqslant&\|x_0\|^p+p\mathbb{E}\int_0^{t\wedge\tau_n}\|x(s)\|^{p-2}\left(x(s)^Tf(x(s))+\frac{p-1}{2}\|g(x(s))\|^2\right)ds\\\leqslant&\|x_0\|^p+C\mathbb{E}\int_0^{t\wedge\tau_n}\|x(s)\|^{p-2}\left(1+\|x(s)\|^2\right)ds\\\leqslant&\|x_0\|^p+C\mathbb{E}\int_0^{t}\left(1+\|x(s\wedge\tau_n)\|^p\right)ds,\end{aligned}$$

结合 Gronwall 不等式, 存在常数 C 使得

$$\sup_{t\in[0,T]} \mathbb{E}\left(\|x(t\wedge\tau_n)\|^p\right) < C.$$

因此, 对于 $i=1,\dots,d$, 有

$$\sup_{t\in[0,T]} \mathbb{E}\left(x_i(t\wedge\tau_n)^p\right) < C.$$

同理, 对 $i=1,\dots,d$, 使用上述估计, 当 $t\in[0,T]$ 有

$$\begin{aligned}
&\mathbb{E}\left(x_i(t\wedge\tau_n)^{-q}\right)\\
=&x_i(0)^{-q} - q\mathbb{E}\int_0^{t\wedge\tau_n} x_i(s)^{-(q+2)}\left(x_i(s)f_i(x(s)) - \frac{q+1}{2}\|g_i(x(s))\|^2\right)ds\\
\leqslant&x_i(0)^{-q} + C\mathbb{E}\int_0^{t\wedge\tau_n} x_i(s)^{-(q+2)}\left(1+\|x(s)\|^{(\alpha+2)\vee(\beta+2)} + \left\|x(s)^{-1}\right\|^{\beta+2}\right)I_{\{x_i(s)\geqslant x^*\}}ds\\
\leqslant&C + C\mathbb{E}\int_0^{t\wedge\tau_n}\left(1+\|x(s)\|^{(\alpha+2)\vee(\beta+2)} + \left\|x(s)^{-1}\right\|^{\beta+2}\right)ds\\
\leqslant&C + C\mathbb{E}\int_0^{t\wedge\tau_n}\|x(s)\|^{(\alpha+2)\vee(\beta+2)}ds + C\mathbb{E}\int_0^{t\wedge\tau_n}\left\|x(s)^{-1}\right\|^{\beta+2}ds\\
\leqslant&C + C\mathbb{E}\int_0^{t\wedge\tau_n}\sum_{j=1}^d x_j(s)^{-(\beta+2)}ds.
\end{aligned}$$

那么,

$$\mathbb{E}\sum_{i=1}^d\left(x_i(t\wedge\tau_n)^{-q}\right) \leqslant C + C\mathbb{E}\int_0^{t\wedge\tau_n}\sum_{j=1}^d x_j(s)^{-(\beta+2)}ds \leqslant C + C\mathbb{E}\int_0^{t\wedge\tau_n}\sum_{j=1}^d x_j(s)^{-q}ds.$$

应用 Gronwall 不等式, 存在常数 M 使得

$$\sup_{t\in[0,T]}\sum_{i=1}^d \mathbb{E}\left(x_i(t\wedge\tau_n)^{-q}\right) < M.$$

进一步, 存在常数 C 使得

$$\sup_{t\in[0,T]} \mathbb{E}\left\|x(t\wedge\tau_n)^{-1}\right\|^q < C.$$

因此

$$\sup_{t\in[0,T]} \mathbb{E}\left(x_i(t\wedge\tau_n)^p + x_i(t\wedge\tau_n)^{-q}\right) < C.$$

同样地, 如果 $\mathbb{P}(\tau_\infty\leqslant T)>0$, 则使 n 足够大时

$$\sum_{i=1}^d \mathbb{E}\left(x_i(T\wedge\tau_n)^p + x_i(T\wedge\tau_n)^{-q}\right) \geqslant n^{p\wedge q}\mathbb{P}(\tau_\infty\leqslant T),$$

无界. 从而矛盾. 因此 $\mathbb{P}(\tau_\infty > T) = 1$. 这意味这 SDE (2.1) 在 $[0,T]$ 上有唯一强解并且

$$\mathbb{P}\left(x(t) \in \mathbb{R}_+^d, \forall t \in [0,T]\right) = 1.$$

按照上述类似的讨论, 存在常数 C 使得

$$\sup_{t\in[0,T]} \mathbb{E}\|x(t\wedge\theta)\|^p < C \quad 和 \quad \sup_{t\in[0,T]} \mathbb{E}\|x(t\wedge\theta)^{-1}\|^q < C,$$

其中 θ 表示任意停时.

3 对数截断 EM 方法

为了定义对数截断 EM 数值解, 我们首先对于每个分量进行对数变换, 定义 $y = (\ln x_1, \ldots, \ln x_d)^T \triangleq \ln x, x \in \mathbb{R}_+^d$. 另外使用逆变换, 定义 $x = (e^{y_1}, \ldots, e^{y_d})^T$. 应用 Itô 公式, 得到新的 SDE

$$dy(t) = F(y)dt + G(y)dB(t),$$

其中

$$F(y) = \mathrm{diag}\left(e^{-y_1}, \ldots, e^{-y_d}\right) f(e^y) - \frac{1}{2}\mathrm{diag}\left(e^{-2y_1}, \ldots, e^{-2y_d}\right)\left(\|g_1(e^y)\|^2, \ldots, \|g_d(e^y)\|^2\right)^T,$$

且

$$G(y) = \mathrm{diag}\left(e^{-y_1}, \ldots, e^{-y_d}\right) g(e^y),$$

对任意 $y \in \mathbb{R}^d$, 由注释 2.1 有

$$\begin{aligned}
\|F(y)\| \leqslant & C_1 \left(\left\|e^{-y}\right\| \cdot \|f(e^y)\| + \left\|e^{-2y}\right\| \cdot \|g(e^y)\|^2\right) \\
\leqslant & C_2 \left(\left\|e^{-y}\right\| \cdot \|f(e^y)\| + \left\|e^{-y}\right\|^2 \cdot \|g(e^y)\|^2\right) \\
\leqslant & C_3 \left\|e^{-y}\right\| \left(1 + \|e^y\|^{(\alpha\vee\beta)+1} + \left\|e^{-y}\right\|^{\beta+1}\right) + C_3 \left\|e^{-y}\right\|^2 \left(1 + \|e^y\|^{(\alpha\vee\beta)+2} + \left\|e^{-y}\right\|^{\beta+2}\right) \\
\leqslant & C_4 \left(1 + \|e^y\|^{(\alpha\vee\beta)+4} + \left\|e^{-y}\right\|^{(\alpha\vee\beta)+4}\right) \\
\leqslant & C_5 \left(1 + e^{((\alpha\vee\beta)+4)\|y\|}\right).
\end{aligned}$$

同样地,

$$\|G(y)\|^2 \leqslant C_6 \left(1 + e^{((\alpha\vee\beta)+4)\|y\|}\right).$$

因此, 存在某个常数 $C_0 > 1$ 及与参数 α 、β 相关的常数 $C(\alpha,\beta) \leqslant (\alpha\vee\beta)+4$, 使得

$$\|F(y)\| \vee \|G(y)\|^2 \leqslant C_0(1 + e^{C(\alpha,\beta)\|y\|}), \tag{3.1}$$

现引入严格递增的连续函数 $\varphi(r) = C_0(2 + e^{C(\alpha,\beta)r})$, 使得

$$\sup_{\|y\|<r} \|F(y)\| \vee \|G(y)\|^2 \leqslant \varphi(r), \quad \forall r > 0.$$

记 φ 的反函数为 φ^{-1} ，显然 $\varphi^{-1}:[3C_0,\infty)\to\bar{\mathbb{R}}_+$ 也是一个严格递增的连续函数. 此外, 我们引入一个严格递减函数 $h:(0,1]\to[1,\infty)$ ，该函数对 $\forall\Delta\in(0,1]$ 满足

$$\lim_{\Delta\to 0} h(\Delta)=+\infty,\quad \Delta^{\frac{1}{3}}h(\Delta)\leqslant 4C_0\vee\varphi(\|\ln x_0\|)\quad 和 \quad h(1)>3C_0\vee\varphi(\|\ln x_0\|). \tag{3.2}$$

对于给定的步长 $\Delta\in(0,1]$, 我们定义截断映射 $\pi_\Delta:\mathbb{R}\to\mathbb{R}$ 为

$$\pi_\Delta(y)=\left(\|y\|\wedge\varphi^{-1}(h(\Delta))\right)\frac{y}{\|y\|},$$

其中约定当 $y=0$ 时, $\frac{y}{\|y\|}=0$. 我们进一步定义截断函数

$$F_\Delta(y)=F(\pi_\Delta(y))\quad 和 \quad G_\Delta(y)=G(\pi_\Delta(y)),$$

其中 $y\in\mathbb{R}^d$. 由此可得以下一致界：

$$\|F_\Delta(y)\|\vee\|G_\Delta(y)\|^2\leqslant\varphi\left(\varphi^{-1}(h(\Delta))\right)=h(\Delta), \tag{3.3}$$

该不等式对所有 $y\in\mathbb{R}^d$ 均成立.

基于上述构造, 我们可以建立变换后 SDE 的离散时间对数截断 EM 数值解. 设时间节点 $t_k=k\Delta$ 处的数值解为 $Y_\Delta(t_k)\approx y(t_k)$, 其计算过程如下：首先取初始值 $Y_\Delta(0)=y_0=\ln(x_0)$ 然后通过递推公式

$$Y_\Delta(t_{k+1})=Y_\Delta(t_k)+F_\Delta\left(Y_\Delta(t_k)\right)\Delta+G_\Delta\left(Y_\Delta(t_k)\right)\Delta B_k,$$

进行计算, 其中 $k=0,1,\ldots$, 且 $\Delta B_k=B(t_{k+1})-B(t_k)$ 表示布朗运动增量. 在此基础上, 我们将构造变换后 SDE 的两种连续时间对数截断 EM 数值解格式. 第一种格式定义为：

$$\bar{y}_\Delta(t)=\sum_{k=0}^{\infty}Y_\Delta(t_k)I_{[t_k,t_{k+1})}(t),\quad t\geqslant 0.$$

这是一个简单过程, 其样本路径不连续. 我们将其称为连续时间过程对数截断 EM 解. 另一种格式定义为：

$$y_\Delta(t)=y_0+\int_0^t F_\Delta\left(\bar{y}_\Delta(s)\right)ds+\int_0^t G_\Delta\left(\bar{y}_\Delta(s)\right)dB(s),\quad t\geqslant 0.$$

我们将其称为连续时间连续样本对数截断 EM 解. 对于所有 $k\geqslant 0$, 都有 $y_\Delta(t_k)=\bar{y}_\Delta(t_k)=Y_\Delta(t_k)$ 成立. 此外, $y_\Delta(t_k)$ 是一个 Itô过程, 其微分形式为

$$dy_\Delta(t)=F_\Delta\left(\bar{y}_\Delta(t)\right)dt+G_\Delta\left(\bar{y}_\Delta(t)\right)dB(t).$$

最后, 通过变换

$$\bar{x}_\Delta(t)=\left(e^{\bar{y}_{\Delta,1}(t)},\ldots,e^{\bar{y}_{\Delta,d}(t)}\right)^T\triangleq e^{\bar{y}_\Delta(t)},\quad x_\Delta(t)=\left(e^{y_{\Delta,1}(t)},\ldots,e^{y_{\Delta,d}(t)}\right)^T\triangleq e^{y_\Delta(t)}$$

我们得到原随机微分方程的数值解 $\bar{x}_\Delta(t)$ 和 $x_\Delta(t)$.

引理 3.1 设 $Z_1,\dots,Z_m \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$, $S_m \triangleq \sum_{i=1}^m Z_i^2 \sim \chi_m^2$ 为自由度 m 的卡方变量. 则对任意 $\beta>0$ 有,

$$\mathbb{E}\exp(\beta\|\Delta B_k\|) \leqslant 2^{\frac{m}{2}}\exp\left(\beta^2\Delta\right).$$

证 利用 Young 不等式和卡方矩母函数有

$$\begin{aligned}\mathbb{E}\exp(\beta\|\Delta B_k\|) =&\mathbb{E}\exp\left(\beta\sqrt{\Delta}\sqrt{S_m}\right)\\ \leqslant&\mathbb{E}\exp\left(\beta^2\Delta+\frac{1}{4}S_m\right)\\ =&\exp(\beta^2\Delta)(1-\frac{1}{2})^{-\frac{m}{2}}\\ =&2^{\frac{m}{2}}\exp\left(\beta^2\Delta\right).\end{aligned}$$

为简洁起见, 我们作如下定义:

$$\operatorname{diag}\left(\frac{x_{\Delta,1}(t)}{\bar{x}_{\Delta,1}(t)},\dots,\frac{x_{\Delta,d}(t)}{\bar{x}_{\Delta,d}(t)}\right) \triangleq \operatorname{diag}\left(\frac{x_\Delta(t)}{\bar{x}_\Delta(t)}\right).$$

引理 3.2 对于任意实数 $\bar{p}$, 存在依赖于 $\bar{p}$ 的常数 $C_1(\bar{p})$ 使得

$$\sup_{\Delta\in(0,1]}\sup_{t\in[0,T]}\mathbb{E}\left(\frac{\|x_\Delta(t)\|}{\|\bar{x}_\Delta(t)\|}\right)^{\bar{p}} \leqslant C_1(\bar{p}). \tag{3.4}$$

当 $\bar{p}\geqslant 2$ 且满足上述条件时, 存在依赖于 $\bar{p}$ 的常数 $C_2(\bar{p})$ 满足对 $\Delta\in(0,1]$ 有

$$\sup_{t\in[0,T]}\mathbb{E}\left\|\operatorname{diag}\left(\frac{x_\Delta(t)}{\bar{x}_\Delta(t)}\right)-E\right\|^{\bar{p}} \leqslant C_2(\bar{p})\Delta^{\frac{\bar{p}}{2}}h(\Delta)^{\frac{\bar{p}}{2}}. \tag{3.5}$$

其中 E 表示单位矩阵.

证 证明过程中, 约定用 $C_1(\bar{p})$ 和 $C_2(\bar{p})$ 表示仅依赖于 $\bar{p}$ 的正实数, 这些常数与 Δ 和 k 无关, 且在不同位置可能出现不同的取值. 根据 $x_\Delta(t)$ 和 $y_\Delta(t)$, 的定义, 可得:

$$\|x_\Delta(t)\| = \left\|\begin{pmatrix}\bar{x}_{\Delta,1}(t)\exp\left(F_{\Delta,1}(\bar{y}_\Delta(t))(t-t_k)+G_{\Delta,1}(\bar{y}_\Delta(t))(B(t)-B(t_k))\right)\\ \vdots\\ \bar{x}_{\Delta,d}(t)\exp\left(F_{\Delta,d}(\bar{y}_\Delta(t))(t-t_k)+G_{\Delta,d}(\bar{y}_\Delta(t))(B(t)-B(t_k))\right)\end{pmatrix}\right\|,$$

由此可得以下估计式:

$$\begin{aligned}\|x_\Delta(t)\| =& \left(\sum_{1=1}^d \bar{x}_{\Delta,i}^2(t)\exp\left(2F_{\Delta,i}(\bar{y}_\Delta(t))(t-t_k)+2G_{\Delta,i}(\bar{y}_\Delta(t))(B(t)-B(t_k))\right)\right)^{\frac{1}{2}}\\ \leqslant&\left(\sum_{1=1}^d \bar{x}_{\Delta,i}^2(t)\right)^{\frac{1}{2}}\exp\left(\max\left(F_{\Delta,i}(\bar{y}_\Delta(t))(t-t_k)+G_{\Delta,i}(\bar{y}_\Delta(t))(B(t)-B(t_k))\right)\right)\\ \leqslant&\|\bar{x}_\Delta(t)\|\exp\left(\|F_\Delta(\bar{y}_\Delta(t))(t-t_k)+G_\Delta(\bar{y}_\Delta(t))(B(t)-B(t_k))\|\right),\end{aligned}$$

以及

$$\begin{aligned}\|x_\Delta(t)\| \geqslant& \left(\sum_{1=1}^{d}\bar{x}^2_{\Delta,i}(t)\right)^{\frac{1}{2}} \exp\left(\min\left(F_{\Delta,i}(\bar{y}_\Delta(t))(t-t_k)+G_{\Delta,i}(\bar{y}_\Delta(t))(B(t)-B(t_k))\right)\right)\\ \geqslant& \|\bar{x}_\Delta(t)\|\exp\left(-\|F_\Delta(\bar{y}_\Delta(t))(t-t_k)+G_\Delta(\bar{y}_\Delta(t))(B(t)-B(t_k))\|\right),\end{aligned}$$

其中 $t\in[t_k,t_{k+1})$. 结合引理 3.1 可得对 $t\in[t_k,t_{k+1})$ 有

$$\begin{aligned}\mathbb{E}\left(\frac{\|x_\Delta(t)\|}{\|\bar{x}_\Delta(t)\|}\right)^{\bar{p}} \leqslant& \mathbb{E}\left(\exp\|F_\Delta(\bar{y}_\Delta(t))(t-t_k)+G_\Delta(\bar{y}_\Delta(t))(B(t)-B(t_k))\|\right)^{|\bar{p}|}\\ \leqslant& \mathbb{E}\exp\left(|\bar{p}|\cdot\|F_\Delta(\bar{y}_\Delta(t))(t-t_k)\|+|\bar{p}|\cdot\|G_\Delta(\bar{y}_\Delta(t))(B(t)-B(t_k))\|\right)\\ \leqslant& \mathbb{E}\exp\left(|\bar{p}|h(\Delta)\Delta+|\bar{p}|h(\Delta)^{\frac{1}{2}}\|B(t)-B(t_k)\|\right)\\ \leqslant& 2^{\frac{m}{2}}\mathbb{E}\exp\left(|\bar{p}|h(\Delta)\Delta+\bar{p}^2h(\Delta)\Delta\right),\end{aligned}$$

由于 $\Delta^{\frac{1}{3}}h(\Delta)\leqslant 4C_0\vee\varphi(\|\ln x_0\|)$, 因此存在仅依赖于 $\bar{p}$ 的常数 $C_1(\bar{p})$, 使得对所有 $\Delta\in(0,1]$ 都有 $\mathbb{E}\left(\frac{\|x_\Delta(t)\|}{\|\bar{x}_\Delta(t)\|}\right)^{\bar{p}}\leqslant C_1(\bar{p})$, 类似地, 对于每个分量 $i=1,\dots,d$, 我们可推得

$$\begin{aligned}\mathbb{E}\left(\frac{x_{\Delta,i}(t)}{\bar{x}_{\Delta,i}(t)}\right)^{\bar{p}} \leqslant&\mathbb{E}\exp\left(|\bar{p}|\cdot|F_{\Delta,i}(\bar{y}_\Delta(t))(t-t_k)|+|\bar{p}|\cdot\|G_{\Delta,i}(\bar{y}_\Delta(t))(B(t)-B(t_k))\|\right)\\ \leqslant&\mathbb{E}\exp\left(|\bar{p}|h(\Delta)\Delta+|\bar{p}|h(\Delta)^{\frac{1}{2}}\|B(t)-B(t_k)\|\right)\\ \leqslant&C_1(\bar{p}).\end{aligned}$$

对 $e^{y_\Delta(t)}$ 的各个分量应用 Itô 公式, 我们得到如下分量表达式：对于 $i=1,\dots,d$ 且 $t\in[t_k,t_{k+1})$,

$$x_{\Delta,i}(t)=\bar{x}_{\Delta,i}(t)+\int_{t_k}^{t}x_{\Delta,i}(s)\left(F_{\Delta,i}(\bar{y}_\Delta(s))+\frac{1}{2}\|G_{\Delta,i}(\bar{y}_\Delta(s))\|^2\right)ds+\int_{t_k}^{t}x_{\Delta,i}(s)G_{\Delta,i}(\bar{y}_\Delta(s))dB(s),$$

现考虑 $\bar{p}\geqslant 2$ 的情形. 由于 $x_\Delta(t),\bar{x}_\Delta(t)\in\mathbb{R}^d_+$, 注意到当 $t\in[t_k,t_{k+1})$ 时 $\bar{x}_{\Delta,i}(t)=\bar{x}_{\Delta,i}(t_k)$, 结合式 (3.2)、(3.3)、Hölder 不等式以及文献 [19] 中的定理 7.1, 可推导得

$$\begin{aligned}&\mathbb{E}\left|\frac{x_{\Delta,i}(t)}{\bar{x}_{\Delta,i}(t)}-1\right|^{\bar{p}}\\ =&\mathbb{E}\left|\int_{t_k}^{t}\frac{x_{\Delta,i}(s)}{\bar{x}_{\Delta,i}(s)}\left(F_{\Delta,i}(\bar{y}_\Delta(s))+\frac{1}{2}\|G_{\Delta,i}(\bar{y}_\Delta(s))\|^2\right)ds+\int_{t_k}^{t}\frac{x_{\Delta,i}(s)}{\bar{x}_{\Delta,i}(s)}G_{\Delta,i}(\bar{y}_\Delta(s))dB(s)\right|^{\bar{p}}\\ \leqslant&C_2(\bar{p})(t-t_k)^{\bar{p}-1}\mathbb{E}\int_{t_k}^{t}\left|\frac{x_{\Delta,i}(s)}{\bar{x}_{\Delta,i}(s)}\right|^{\bar{p}}\left|F_{\Delta,i}(\bar{y}_\Delta(s))+\frac{1}{2}\|G_{\Delta,i}(\bar{y}_\Delta(s))\|^2\right|^{\bar{p}}ds\\ &+C_2(\bar{p})(t-t_k)^{\frac{\bar{p}}{2}-1}\mathbb{E}\int_{t_k}^{t}\left|\frac{x_{\Delta,i}(s)}{\bar{x}_{\Delta,i}(s)}\right|^{\bar{p}}\|G_{\Delta,i}(\bar{y}_\Delta(s))\|^{\bar{p}}ds\\ \leqslant&C_2(\bar{p})\Delta^{\frac{\bar{p}}{2}-1}(\Delta^{\frac{\bar{p}}{2}}h(\Delta)^{\bar{p}}+h(\Delta)^{\frac{\bar{p}}{2}})\mathbb{E}\int_{t_k}^{t}\left|\frac{x_{\Delta,i}(s)}{\bar{x}_{\Delta,i}(s)}\right|^{\bar{p}}ds\\ \leqslant&C_2(\bar{p})\Delta^{\frac{\bar{p}}{2}}h(\Delta)^{\frac{\bar{p}}{2}},\end{aligned}$$

其中 $t\in[t_k,t_{k+1})$. 由注释 2.1 即可得所需结论 (3.5).

下文将采用以下记法

$$\frac{1}{x_\Delta(t)}=\left(\frac{1}{x_{\Delta,1}(t)},\ldots,\frac{1}{x_{\Delta,d}(t)}\right)^T,\quad \frac{1}{\bar{x}_\Delta(t)}=\left(\frac{1}{\bar{x}_{\Delta,1}(t)},\ldots,\frac{1}{\bar{x}_{\Delta,d}(t)}\right)^T$$

以及分量平方差向量

$$x_\Delta(t)^2-\bar{x}_\Delta(t)^2=\left(x_{\Delta,1}(t)^2-\bar{x}_{\Delta,1}(t)^2,\ldots,x_{\Delta,d}(t)^2-\bar{x}_{\Delta,d}(t)^2\right)^T.$$

引理 3.3 在假设条件 (A_1)、(A_2)、(A_3) 成立且满足 $(\alpha+2)\vee(\beta+2)<p\wedge q$ 以及 $p\geqslant 4$ 的条件下, 对于任意停时 θ , 存在常数 C 使得下列矩估计一致成立

$$\sup_{\Delta\in(0,1]}\sup_{t\in[0,T]}\mathbb{E}\|x_\Delta(t\wedge\theta)\|^p<C\quad 和\quad \sup_{\Delta\in(0,1]}\sup_{t\in[0,T]}\mathbb{E}\left\|\frac{1}{x_\Delta(t\wedge\theta)}\right\|^q<C.$$

证 在下述证明过程中, 我们约定用 C 表示与步长 Δ 无关的通用正常数, 其具体值在不同位置可能不同. 取步长 $\Delta\in(0,1]$ 并定义停时 $\tau_n=\inf\{t\in[0,T]|x_\Delta(t)\notin(\frac{1}{n},n)^d\}$ 应用 Itô 公式并取数学期望, 可得

$$\begin{aligned}
\mathbb{E}\left\|x_\Delta(t\wedge\tau_n\wedge\theta)\right\|^p\leqslant{}&\|x_0\|^p+\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}p\left\|x_\Delta(s)\right\|^{p-2}\left(x_{\Delta,1}(s)^2,\ldots,x_{\Delta,d}(s)^2\right)\\
&\times\left(F_\Delta(\bar{y}_\Delta(s))+\frac{1}{2}\left(\left\|G_{\Delta,1}(\bar{y}_\Delta(s))\right\|^2,\ldots,\left\|G_{\Delta,d}(\bar{y}_\Delta(s))\right\|^2\right)^T\right)ds\\
&+\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}\frac{1}{2}p(p-1)\left\|x_\Delta(s)\right\|^{p-2}\left\|\mathrm{diag}\left(x_{\Delta,1}(s),\ldots,x_{\Delta,d}(s)\right)G_\Delta(\bar{y}_\Delta(s))\right\|^2ds\\
\leqslant{}&\|x_0\|^p+\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}p\left\|x_\Delta(s)\right\|^{p-2}\left(x_{\Delta,1}(s)^2,\ldots,x_{\Delta,d}(s)^2\right)\\
&\times\left(F_\Delta(\bar{y}_\Delta(s))+\frac{p}{2}\left(\left\|G_{\Delta,1}(\bar{y}_\Delta(s))\right\|^2,\ldots,\left\|G_{\Delta,d}(\bar{y}_\Delta(s))\right\|^2\right)^T\right)ds\\
={}&\|x_0\|^p+\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}p\left\|x_\Delta(s)\right\|^{p-2}\left(x_\Delta(s)^2-\bar{x}_\Delta(s)^2\right)^T\\
&\times\left(F_\Delta(\bar{y}_\Delta(s))+\frac{p}{2}\left(\left\|G_{\Delta,1}(\bar{y}_\Delta(s))\right\|^2,\ldots,\left\|G_{\Delta,d}(\bar{y}_\Delta(s))\right\|^2\right)^T\right)ds\\
&+\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}p\left\|x_\Delta(s)\right\|^{p-2}\left(\bar{x}_{\Delta,1}(s)^2,\ldots,\bar{x}_{\Delta,d}(s)^2\right)\\
&\times\left(F_\Delta(\bar{y}_\Delta(s))+\frac{p}{2}\left(\left\|G_{\Delta,1}(\bar{y}_\Delta(s))\right\|^2,\ldots,\left\|G_{\Delta,d}(\bar{y}_\Delta(s))\right\|^2\right)^T\right)ds\\
\triangleq{}&\|x_0\|^p+A+B.
\end{aligned}$$

由 Itô公式可得:

$$\begin{aligned}
x_{\Delta,i}(t)^2-\bar{x}_{\Delta,i}(t)^2={}&\int_{t_k}^t 2x_{\Delta,i}(s)^2\left(F_{\Delta,i}(\bar{y}_\Delta(s))+\|G_{\Delta,i}(\bar{y}_\Delta(s))\|^2\right)ds\\
&+\int_{t_k}^t 2x_{\Delta,i}(s)^2G_{\Delta,i}(\bar{y}_\Delta(s))dB(s).
\end{aligned}$$

于是,

$$
\begin{aligned}
&x_\Delta(t)^2-\bar{x}_\Delta(t)^2\\
=&2\int_{t_k}^{t}\mathrm{diag}(x_{\Delta,1}(s)^2,\ldots,x_{\Delta,d}(s)^2)\left(F_\Delta(\bar{y}_\Delta(s))+\left(\|G_{\Delta,1}(\bar{y}_\Delta(s))\|^2,\ldots,\|G_{\Delta,d}(\bar{y}_\Delta(s))\|^2\right)^T\right)ds\\
&+2\int_{t_k}^{t}\mathrm{diag}(x_{\Delta,1}(s)^2,\ldots,x_{\Delta,d}(s)^2)G_\Delta(\bar{y}_\Delta(s))dB(s).
\end{aligned}
$$

综合应用 Itô 公式、式 (3.2)、(3.3)、Hölder 不等式以及文献 [19] 中定理 7.1, 可推得以下估计：对于 $t\in[t_k,t_{k+1})$,

$$
\begin{aligned}
&\mathbb{E}\left\|x_\Delta(t)^2-\bar{x}_\Delta(t)^2\right\|^{\frac{p}{2}}\\
\leqslant&C\mathbb{E}\Big\|\int_{t_k}^{t}\mathrm{diag}(x_{\Delta,1}(s)^2,\ldots,x_{\Delta,d}(s)^2)\\
&\times\left(F_\Delta(\bar{y}_\Delta(s))+\left(\|G_{\Delta,1}(\bar{y}_\Delta(s))\|^2,\ldots,\|G_{\Delta,d}(\bar{y}_\Delta(s))\|^2\right)^T\right)ds\Big\|^{\frac{p}{2}}\\
&+C\mathbb{E}\left\|\int_{t_k}^{t}\mathrm{diag}(x_{\Delta,1}(s)^2,\ldots,x_{\Delta,d}(s)^2)G_\Delta(\bar{y}_\Delta(s))dB(s)\right\|^{\frac{p}{2}}\\
\leqslant&C(t-t_k)^{\frac{p}{2}-1}\mathbb{E}\int_{t_k}^{t}\|x_\Delta(s)\|^p\left\|F_\Delta(\bar{y}_\Delta(s))+\left(\|G_{\Delta,1}(\bar{y}_\Delta(s))\|^2,\ldots,\|G_{\Delta,d}(\bar{y}_\Delta(s))\|^2\right)^T\right\|^{\frac{p}{2}}ds\\
&+C(t-t_k)^{\frac{p}{4}-1}\mathbb{E}\int_{t_k}^{t}\|x_\Delta(s)\|^p\left\|G_\Delta(\bar{y}_\Delta(s))\right\|^{\frac{p}{2}}ds\\
\leqslant&C\Delta^{\frac{p}{4}-1}(\Delta^{\frac{p}{4}}h(\Delta)^{\frac{p}{2}}+h(\Delta)^{\frac{p}{4}})\mathbb{E}\int_{t_k}^{t}\|x_\Delta(s)\|^pds\\
\leqslant&C\Delta^{\frac{p}{4}-1}h(\Delta)^{\frac{p}{4}}\mathbb{E}\int_{t_k}^{t}\|x_\Delta(s)\|^pds,
\end{aligned}
$$

基于 Young 不等式, 我们进一步得到

$$
\begin{aligned}
A\leqslant&\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}p\|x_\Delta(s)\|^{p-2}\left\|x_\Delta(s)^2-\bar{x}_\Delta(s)^2\right\|\\
&\times\left\|F_\Delta(\bar{y}_\Delta(s))+\frac{p}{2}\left(\|G_{\Delta,1}(\bar{y}_\Delta(s))\|^2,\ldots,\|G_{\Delta,d}(\bar{y}_\Delta(s))\|^2\right)^T\right\|ds\\
\leqslant&\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}(p-2)\|x_\Delta(s)\|^pds+\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}\frac{2}{p}\left\|x_\Delta(s)^2-\bar{x}_\Delta(s)^2\right\|^{\frac{p}{2}}\\
&\times\left\|F_\Delta(\bar{y}_\Delta(s))+\frac{p}{2}\left(\|G_{\Delta,1}(\bar{y}_\Delta(s))\|^2,\ldots,\|G_{\Delta,d}(\bar{y}_\Delta(s))\|^2\right)^T\right\|^{\frac{p}{2}}ds\\
\leqslant&C\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}\|x_\Delta(s)\|^pds+C\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}\left\|x_\Delta(s)^2-\bar{x}_\Delta(s)^2\right\|^{\frac{p}{2}}h(\Delta)^{\frac{p}{2}}ds\\
\leqslant&C\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}\|x_\Delta(s)\|^pds+C\int_0^{t\wedge\tau_n\wedge\theta}\Delta^{\frac{p}{4}-1}h(\Delta)^{\frac{p}{2}}h(\Delta)^{\frac{p}{4}}\int_{t_k}^{s}\mathbb{E}\|x_\Delta(u)\|^pduds
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}\|x_\Delta(s)\|^p ds + C\int_0^{t\wedge\tau_n\wedge\theta}\Delta^{\frac{p}{4}}h(\Delta)^{\frac{3p}{4}}\sup_{r\in[0,s]}\mathbb{E}\|x_\Delta(r)\|^p ds\\
&\leqslant C\int_0^{t\wedge\tau_n\wedge\theta}\sup_{r\in[0,s]}\mathbb{E}\|x_\Delta(r)\|^p ds,
\end{aligned}
$$

其中 $s\in[t_k,t_{k+1})$. 根据假设条件 (A_2) 和引理 3.2, 我们可得到以下估计式

$$
\begin{aligned}
B =&\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta} p\,\|x_\Delta(s)\|^{p-2}\left(\bar{x}_{\Delta,1}(s)^2,\ldots,\bar{x}_{\Delta,d}(s)^2\right)\\
&\times\left(F_\Delta(\bar{y}_\Delta(s))+\frac{p}{2}\left(\|G_{\Delta,1}(\bar{y}_\Delta(s))\|^2,\ldots,\|G_{\Delta,d}(\bar{y}_\Delta(s))\|^2\right)^T\right)ds\\
=&\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta} p\,\|x_\Delta(s)\|^{p-2}\left(\bar{x}_\Delta(s)^T f_\Delta(\bar{x}_\Delta(s))+\frac{p-1}{2}\|g_\Delta(\bar{x}_\Delta(s))\|^2\right)ds\\
\leqslant& C\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}\|x_\Delta(s)\|^{p-2}(1+\|\bar{x}_\Delta(s)\|^2)ds\\
\leqslant& C+C\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}\|x_\Delta(s)\|^p ds + C\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}\|\bar{x}_\Delta(s)\|^p ds\\
\leqslant& C+C\int_0^{t\wedge\tau_n\wedge\theta}\left(\sup_{r\in[0,s]}\mathbb{E}\|x_\Delta(r)\|^p\right)ds.
\end{aligned}
$$

由此可得存在常数 C 使得对所有 $r\in[0,T]$ 成立:

$$
\sup_{t\in[0,r]}\mathbb{E}\|x_\Delta(t\wedge\tau_n\wedge\theta)\|^p\leqslant C+C\int_0^r\sup_{t\in[0,s]}\mathbb{E}\|x_\Delta(t\wedge\tau_n\wedge\theta)\|^p ds,
$$

应用 Gronwall 不等式可知存在常数 C 满足:

$$
\sup_{t\in[0,T]}\mathbb{E}\|x_\Delta(t\wedge\tau_n\wedge\theta)\|^p<C.
$$

类似地, 结合注释 2.1、引理 3.2 和条件 $(\alpha+2)\vee(\beta+2)<p\wedge q$, 可选取充分小的 $\varepsilon>0$ 使得 $((\alpha\vee\beta)+2)(1+\varepsilon)<p\wedge q$, 从而对每个分量 $i=1,\ldots,d$ 有

$$
\begin{aligned}
&\mathbb{E}(x_{\Delta,i}(t\wedge\tau_n\wedge\theta))^{-q}\\
=&x_{\Delta,i}(0)^{-q}-q\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}x_{\Delta,i}(s)^{-q}\left(F_{\Delta,i}(\bar{y}_\Delta(s))-\frac{q}{2}\|G_{\Delta,i}(\bar{y}_\Delta(s))\|^2\right)ds\\
=&x_{\Delta,i}(0)^{-q}-q\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}x_{\Delta,i}(s)^{-q}\left(\frac{\bar{x}_{\Delta,i}(s)f_{\Delta,i}(\bar{x}_\Delta(s))-\frac{q+1}{2}\|g_{\Delta,i}(\bar{x}_\Delta(s))\|^2}{\bar{x}_{\Delta,i}(s)^2}\right)ds\\
\leqslant& C+C\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}x_{\Delta,i}(s)^{-q}\left(\frac{|\bar{x}_{\Delta,i}(s)||f_{\Delta,i}(\bar{x}_\Delta(s))|+\|g_{\Delta,i}(\bar{x}_\Delta(s))\|^2}{\bar{x}_{\Delta,i}(s)^2}\right)I_{\{\bar{x}_{\Delta,i}(s)\geqslant x^*\}}ds\\
\leqslant& C+C\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}\left(\frac{x_{\Delta,i}(s)}{\bar{x}_{\Delta,i}(s)}\right)^{-q}\bar{x}_{\Delta,i}(s)^{-q-2}\\
&\times(1+\|\bar{x}_\Delta(s)\|^{(\alpha+2)\vee(\beta+2)}+\left\|\bar{x}_\Delta(s)^{-1}\right\|^{\beta+2})I_{\{\bar{x}_{\Delta,i}(s)\geqslant x^*\}}ds
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C + C\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}\left(\frac{x_{\Delta,i}(s)}{\bar{x}_{\Delta,i}(s)}\right)^{-q}(1+\|\bar{x}_\Delta(s)\|^{(\alpha+2)\vee(\beta+2)}+\left\|\bar{x}_\Delta(s)^{-1}\right\|^{\beta+2})ds\\
&\leqslant C + C\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}\left(\frac{x_{\Delta,i}(s)}{\bar{x}_{\Delta,i}(s)}\right)^{-q}ds + C\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}\left(\frac{x_{\Delta,i}(s)}{\bar{x}_{\Delta,i}(s)}\right)^{-q(1+\varepsilon^{-1})}\\
&\quad + \|\bar{x}_\Delta(s)\|^{((\alpha\vee\beta)+2)(1+\varepsilon)} + \left\|\bar{x}_\Delta(s)^{-1}\right\|^{(\beta+2)(1+\varepsilon)}ds\\
&\leqslant C + C\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}\left\|\bar{x}_\Delta(s)^{-1}\right\|^{(\beta+2)(1+\varepsilon)}ds\\
&\leqslant C + C\mathbb{E}\int_0^{t\wedge\tau_n\wedge\theta}\sum_{i=1}^d \bar{x}_{\Delta,i}(s)^{-(\beta+2)(1+\varepsilon)}ds,
\end{aligned}
$$

因此有

$$
\sup_{t\in[0,r]}\mathbb{E}\sum_{i=1}^d(x_{\Delta,i}(t\wedge\tau_n\wedge\theta))^{-q}\leqslant C + C\int_0^r\sup_{t\in[0,s]}\mathbb{E}\sum_{i=1}^d(x_{\Delta,i}(t\wedge\tau_n\wedge\theta))^{-q}ds.
$$

再次应用 Gronwall 不等式即得

$$
\sup_{t\in[0,T]}\mathbb{E}\sum_{i=1}^d(x_{\Delta,i}(t\wedge\tau_n\wedge\theta))^{-q}<C.
$$

进一步有

$$
\sup_{t\in[0,T]}\mathbb{E}\left\|x_\Delta(t\wedge\tau_n\wedge\theta)^{-1}\right\|^q<C.
$$

最终令 $n\to\infty$ 即得所需结论.

在下文中, 我们记 $e_\Delta(t)=x(t)-x_\Delta(t)$ 并取实数 $R>\|\ln x_0\|$. 定义两个停时

$$
\tau_R=\inf\{t\in[0,T]\mid \|y(t)\|\geqslant R\}\quad 和 \quad \tau_R^\Delta=\inf\{t\in[0,T]\mid \|y_\Delta(t)\|\geqslant R\}, \tag{3.6}
$$

其中 $y(t)=\ln(x(t))$. 进一步设 $\tau=\tau_R\wedge\tau_R^\Delta$. 在后续推导中, 我们用 C 表示依赖于 x_0,T 等参数, 但与步长 Δ 和 R 无关的通用正常数, 其具体值在不同位置可能不同.

引理 3.4 在假设条件 (A_1)、(A_2) 和 (A_3) 成立且满足参数约束 $\frac{pr^*}{p-r^*}<\frac{p\wedge q}{(\alpha+2)\vee(\beta+2)}$ 的前提下, 给定阈值 $R>\|\ln x_0\|$, 设 τ 为式 (3.6) 定义的停时. 当步长 Δ 充分小使得 $\varphi^{-1}(h(\Delta))\geqslant R$ 时, 对任意 $2\leqslant r<r^*$, 存在与步长 Δ 无关的常数 C 使得

$$
\sup_{t\in[0,T]}\mathbb{E}\|e_\Delta(t\wedge\tau)\|^r<C\Delta^{\frac{r}{2}}h(\Delta)^{\frac{r}{2}}.
$$

证 首先观察到, 对于所有 $s\in[0,T\wedge\tau]$, 有 $\|y_\Delta(s)\|<R$. 根据假设条件 $\varphi^{-1}(h(\Delta))\geqslant R$, 可以推得在 $s\in[0,T\wedge\tau]$ 范围内, $F_\Delta(\bar{y}_\Delta(s))=F(\bar{y}_\Delta(s))$ 和 $G_\Delta(\bar{y}_\Delta(s))=G(\bar{y}_\Delta(s))$ 成立.

对 $e^{y_\Delta(t)}$ 应用 Itô 公式, 得到

$$
\begin{aligned}
&x_\Delta(t\wedge\tau)\\
=&x_0+\int_0^{t\wedge\tau}\operatorname{diag}(x_{\Delta,1}(s),\ldots,x_{\Delta,d}(s))\left(F_\Delta(\bar{y}_\Delta(s))+\frac{1}{2}\left(\|G_{\Delta,1}(\bar{y}_\Delta(s))\|^2,\ldots,\|G_{\Delta,d}(\bar{y}_\Delta(s))\|^2\right)^T\right)ds\\
&+\int_0^{t\wedge\tau}\operatorname{diag}(x_{\Delta,1}(s),\ldots,x_{\Delta,d}(s))\,G_\Delta(\bar{y}_\Delta(s))dB(s).
\end{aligned}
$$

进一步地

$$\begin{aligned}
&e_\Delta(t\wedge\tau)\\
=&\int_0^{t\wedge\tau} f(x(s))\\
&-\operatorname{diag}\left(x_{\Delta,1}(s),\ldots,x_{\Delta,d}(s)\right)\left(F_\Delta(\bar{y}_\Delta(s))+\frac{1}{2}\left(\|G_{\Delta,1}(\bar{y}_\Delta(s))\|^2,\ldots,\|G_{\Delta,d}(\bar{y}_\Delta(s))\|^2\right)^T\right)ds\\
&+\int_0^{t\wedge\tau} g(x(s))-\operatorname{diag}\left(x_{\Delta,1}(s),\ldots,x_{\Delta,d}(s)\right)G_\Delta(\bar{y}_\Delta(s))dB(s)\\
=&\int_0^{t\wedge\tau} f(x(s))-\operatorname{diag}\left(\frac{x_{\Delta,1}(s)}{\bar{x}_{\Delta,1}(s)},\ldots,\frac{x_{\Delta,d}(s)}{\bar{x}_{\Delta,d}(s)}\right)f(\bar{x}_\Delta(s))ds\\
&+\int_0^{t\wedge\tau} g(x(s))-\operatorname{diag}\left(\frac{x_{\Delta,1}(s)}{\bar{x}_{\Delta,1}(s)},\ldots,\frac{x_{\Delta,d}(s)}{\bar{x}_{\Delta,d}(s)}\right)g(\bar{x}_\Delta(s))dB(s).
\end{aligned}$$

由 Young 不等式

$$2ab\leqslant\frac{2\varepsilon}{2}a^2+\frac{2}{2\varepsilon}b^2=\varepsilon a^2+\frac{1}{\varepsilon}b^2,\quad a,b>0,$$

取 $\varepsilon=\frac{r^*-r}{r-1}$, 对 $\|x(t)-x_\Delta(t)\|^r$ 应用 Itô 公式, 则可得到以下估计式:

$$\begin{aligned}
\mathbb{E}\|e_\Delta(t\wedge\tau)\|^r\leqslant&\mathbb{E}\int_0^{t\wedge\tau} r\|e_\Delta(s)\|^{r-2}e_\Delta(s)^T\left(f(x(s))-\operatorname{diag}\left(\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right)f(\bar{x}_\Delta(s))\right)ds\\
&+\mathbb{E}\int_0^{t\wedge\tau}\frac{r(r-1)}{2}\|e_\Delta(s)\|^{r-2}\left\|g(x(s))-\operatorname{diag}\left(\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right)g(\bar{x}(s))\right\|^2ds\\
\leqslant&J_1+J_2,
\end{aligned}$$

其中

$$J_1=r\mathbb{E}\int_0^{t\wedge\tau}\|e_\Delta(s)\|^{r-2}\left(e_\Delta(s)^T\left(f(x(s))-f(x_\Delta(s))\right)+\frac{r^*-1}{2}\|g(x(s))-g(x_\Delta(s))\|^2\right)ds$$

以及

$$\begin{aligned}
J_2=&r\mathbb{E}\int_0^{t\wedge\tau}\|e_\Delta(s)\|^{r-2}e_\Delta(s)^T\left(f(x_\Delta(s))-\operatorname{diag}\left(\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right)f(\bar{x}_\Delta(s))\right)ds\\
&+\frac{r(r-1)(r^*-1)}{2(r^*-r)}\mathbb{E}\int_0^{t\wedge\tau}\|e_\Delta(s)\|^{r-2}\left\|g(x_\Delta(s))-\operatorname{diag}\left(\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right)g(\bar{x}_\Delta(s))\right\|^2ds.
\end{aligned}$$

根据假设条件 (A_3), 我们首先得到: $J_1\leqslant rH\mathbb{E}\int_0^{t\wedge\tau}\|e_\Delta(s)\|^r ds$. 进一步应用 Young 不等式, 可推导出以下估计:

$$\begin{aligned}
J_2\leqslant&C\mathbb{E}\int_0^{t\wedge\tau}\|e_\Delta(s)\|^{r-1}\left\|f(x_\Delta(s))-f(\bar{x}_\Delta(s))+f(\bar{x}_\Delta(s))-\operatorname{diag}\left(\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right)f(\bar{x}_\Delta(s))\right\|ds\\
&+C\mathbb{E}\int_0^{t\wedge\tau}\|e_\Delta(s)\|^{r-2}\left\|g(x_\Delta(s))-g(\bar{x}_\Delta(s))+g(\bar{x}_\Delta(s))-\operatorname{diag}\left(\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right)g(\bar{x}_\Delta(s))\right\|^2ds
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\mathbb{E}\int_0^{t\wedge\tau} \|e_\Delta(s)\|^r + \left\|f(x_\Delta(s)) - f(\bar{x}_\Delta(s)) + f(\bar{x}_\Delta(s)) - \mathrm{diag}\left(\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right) f(\bar{x}_\Delta(s))\right\|^r ds \\
&\quad + C\mathbb{E}\int_0^{t\wedge\tau} \|e_\Delta(s)\|^r + \left\|g(x_\Delta(s)) - g(\bar{x}_\Delta(s)) + g(\bar{x}_\Delta(s)) - \mathrm{diag}\left(\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right) g(\bar{x}_\Delta(s))\right\|^r ds \\
&\leqslant C\mathbb{E}\int_0^{t\wedge\tau} \|e_\Delta(s)\|^r ds \\
&\quad + C\mathbb{E}\int_0^{t\wedge\tau} \|f(x_\Delta(s)) - f(\bar{x}_\Delta(s))\|^r ds + C\mathbb{E}\int_0^{t\wedge\tau} \left\|E - \mathrm{diag}\left(\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right)\right\|^r \|f(\bar{x}_\Delta(s))\|^r ds \\
&\quad + C\mathbb{E}\int_0^{t\wedge\tau} \|g(x_\Delta(s)) - g(\bar{x}_\Delta(s))\|^r ds + C\mathbb{E}\int_0^{t\wedge\tau} \left\|E - \mathrm{diag}\left(\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right)\right\|^r \|g(\bar{x}_\Delta(s))\|^r ds.
\end{aligned}
$$

基于假设条件 (A_1)、注释 2.1 以及 Hölder 不等式, 可得以下估计式:

$$
\begin{aligned}
J_2 \leqslant & C\mathbb{E}\int_0^{t\wedge\tau} \|e_\Delta(s)\|^r ds + C\mathbb{E}\int_0^{t\wedge\tau} \bigg(\mathbb{E}\Big(1 + \|x_\Delta(s)\|^{(1+\varepsilon)\alpha r} + \left\|x_\Delta(s)^{-1}\right\|^{(1+\varepsilon)\beta r} + \|\bar{x}_\Delta(s)\|^{(1+\varepsilon)\alpha r} \\
& + \left\|\bar{x}_\Delta(s)^{-1}\right\|^{(1+\varepsilon)\beta r}\Big)\bigg)^{\frac{1}{1+\varepsilon}} \left(\mathbb{E}\|x_\Delta(s) - \bar{x}_\Delta(s)\|^{\frac{(1+\varepsilon)r}{\varepsilon}}\right)^{\frac{\varepsilon}{1+\varepsilon}} ds \\
& + C\mathbb{E}\int_0^{t\wedge\tau} \bigg(\mathbb{E}\Big(1 + \|x_\Delta(s)\|^{(1+\varepsilon)\alpha r/2} + \left\|x_\Delta(s)^{-1}\right\|^{(1+\varepsilon)\beta r/2} + \|\bar{x}_\Delta(s)\|^{(1+\varepsilon)\alpha r/2} \\
& + \left\|\bar{x}_\Delta(s)^{-1}\right\|^{(1+\varepsilon)\beta r/2}\Big)\bigg)^{\frac{1}{1+\varepsilon}} \left(\mathbb{E}\|x_\Delta(s) - \bar{x}_\Delta(s)\|^{\frac{(1+\varepsilon)r}{\varepsilon}}\right)^{\frac{\varepsilon}{1+\varepsilon}} ds \\
& + C\mathbb{E}\int_0^{t\wedge\tau} \left(\mathbb{E}\left\|E - \mathrm{diag}\left(\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right)\right\|^{\frac{(1+\varepsilon)r}{\varepsilon}}\right)^{\frac{\varepsilon}{1+\varepsilon}} \\
& \times \left(\mathbb{E}\left(1 + \|\bar{x}_\Delta(s)\|^{(1+\varepsilon)((\alpha\vee\beta)+1)r} + \left\|\bar{x}_\Delta(s)^{-1}\right\|^{(1+\varepsilon)(\beta+1)r}\right)\right)^{\frac{1}{1+\varepsilon}} ds. \\
& + C\mathbb{E}\int_0^{t\wedge\tau} \left(\mathbb{E}\left\|E - \mathrm{diag}\left(\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right)\right\|^{\frac{(1+\varepsilon)r}{\varepsilon}}\right)^{\frac{\varepsilon}{1+\varepsilon}} \\
& \times \left(\mathbb{E}\left(1 + \|\bar{x}_\Delta(s)\|^{(1+\varepsilon)((\alpha\vee\beta)+2)r/2} + \left\|\bar{x}_\Delta(s)^{-1}\right\|^{(1+\varepsilon)(\beta+2)r/2}\right)\right)^{\frac{1}{1+\varepsilon}} ds.
\end{aligned}
$$

在 $\frac{pr^*}{p-r^*} < \frac{p}{(\alpha+2)\vee(\beta+2)} \wedge \frac{q}{(\alpha+2)\vee(\beta+2)}$ 成立的条件下, 存在 $\varepsilon > 0$ 使得

$$
\frac{p}{(\alpha+2)\vee(\beta+2)} \wedge \frac{q}{(\alpha+2)\vee(\beta+2)} > (1+\varepsilon)r^* > \frac{pr^*}{p-r^*}.
$$

由此可得以下参数约束关系:

$$
\frac{(1+\varepsilon)((\alpha\vee\beta)+2)r^*}{2} < (1+\varepsilon)((\alpha\vee\beta)+2)r^* < p\wedge q,
$$

由于 $r^* > r \geqslant 2$, 可得 $p > 4$, 结合 $(1+\varepsilon)(p-r^*) > p$, 有 $2 < \frac{(1+\varepsilon)r}{\varepsilon} < \frac{(1+\varepsilon)r^*}{\varepsilon} < p$. 综合应用

式 (3.2)、(3.3)、引理 3.2、Hölder 不等式以及文献 [19] 中的定理 7.1, 我们最终得到

$$
\begin{aligned}
&\mathbb{E}\left\|x_\Delta(t)-\bar{x}_\Delta(t)\right\|^{\frac{(1+\varepsilon)r}{\varepsilon}}\\
\leqslant &C\mathbb{E}\bigg\|\int_{t_k}^{t}\operatorname{diag}\left(x_{\Delta,1}(s),\ldots,x_{\Delta,d}(s)\right)\\
&\times\left(F_\Delta(\bar{y}_\Delta(s))+\frac{1}{2}\left(\|G_{\Delta,1}(\bar{y}_\Delta(s))\|^2,\ldots,\|G_{\Delta,d}(\bar{y}_\Delta(s))\|^2\right)^T\right)ds\bigg\|^{\frac{(1+\varepsilon)r}{\varepsilon}}\\
&+C\mathbb{E}\left\|\int_{t_k}^{t}\operatorname{diag}\left(x_{\Delta,1}(s),\ldots,x_{\Delta,d}(s)\right)G_\Delta(\bar{y}_\Delta(s))dB(s)\right\|^{\frac{(1+\varepsilon)r}{\varepsilon}}\\
\leqslant &C(t-t_k)^{\frac{(1+\varepsilon)r}{\varepsilon}-1}\\
&\times\mathbb{E}\int_{t_k}^{t}\|x_\Delta(s)\|^{\frac{(1+\varepsilon)r}{\varepsilon}}\left\|F_\Delta(\bar{y}_\Delta(s))+\frac{1}{2}\left(\|G_{\Delta,1}(\bar{y}_\Delta(s))\|^2,\ldots,\|G_{\Delta,d}(\bar{y}_\Delta(s))\|^2\right)^T\right\|^{\frac{(1+\varepsilon)r}{\varepsilon}}ds\\
&+C(t-t_k)^{\frac{(1+\varepsilon)r}{2\varepsilon}-1}\mathbb{E}\int_{t_k}^{t}\|x_\Delta(s)\|^{\frac{(1+\varepsilon)r}{\varepsilon}}\|G_\Delta(\bar{y}_\Delta(s))\|^{\frac{(1+\varepsilon)r}{\varepsilon}}ds\\
\leqslant &C\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}-1}\left(\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}}h(\Delta)^{\frac{(1+\varepsilon)r}{\varepsilon}}+h(\Delta)^{\frac{(1+\varepsilon)r}{2\varepsilon}}\right)\mathbb{E}\int_{t_k}^{t}\|x_\Delta(s)\|^{\frac{(1+\varepsilon)r}{\varepsilon}}ds\\
\leqslant &C\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}}h(\Delta)^{\frac{(1+\varepsilon)r}{2\varepsilon}}.
\end{aligned}
$$

综合引理 3.2 和 3.3 的结论, 我们得到以下估计：$J_2\leqslant C\int_0^{t\wedge\tau}\mathbb{E}\|e_\Delta(s)\|^r+C\Delta^{\frac{r}{2}}h(\Delta)^{\frac{r}{2}}$. 应用 Gronwall 不等式可得 $\sup_{t\in[0,T]}\mathbb{E}\|e_\Delta(t\wedge\tau)\|^r<C\Delta^{\frac{r}{2}}h(\Delta)^{\frac{r}{2}}$.

4 对数截断 EM 解的收敛性

现在, 我们给出关于收敛速率的主要结果.

定理 4.1 在假设条件 (A_1)、(A_2) 和 (A_3) 成立, 且参数满足 $\frac{pr^*}{p-r^*}<\frac{p\wedge q}{(\alpha+2)\vee(\beta+2)}$ 的前提下, 对于任意 $2\leqslant r<r^*$ 和步长 $\Delta\in(0,1]$, 存在常数 C 使得以下强收敛估计成立：

$$
\sup_{t\in[0,T]}\mathbb{E}\|e_\Delta(t)\|^r<C\Delta^{\frac{(p-r)(p\wedge q)r}{3\sqrt{d}(2(p-r)(p\wedge q)+C(\alpha,\beta)pr)}},
$$

其中 $h(\Delta)=(3.5C_0\vee\varphi(\|\ln x_0\|))\,\Delta^{-\frac{C(\alpha,\beta)pr}{3(2(p-r)(p\wedge q)+C(\alpha,\beta)pr)}}$.

证 设 $R=\varphi^{-1}(h(\Delta))$. 当参数满足 $\frac{pr^*}{p-r^*}<\frac{p\wedge q}{(\alpha+2)\vee(\beta+2)}$ 时, 可推得 $(\alpha\vee\beta)+2<p\wedge q$. 综合应用 Young 不等式、Markov 不等式及引理 3.3, 我们得到：

$$
\begin{aligned}
\sup_{t\in[0,T]}\mathbb{E}(\|e_\Delta(t)\|^rI_{\{\tau\leqslant T\}})=&\sup_{t\in[0,T]}\mathbb{E}(\|e_\Delta(t)\|^r\delta^{\frac{r}{p}}I_{\{\tau\leqslant T\}}\delta^{-\frac{r}{p}})\\
\leqslant&\frac{r}{p}\sup_{t\in[0,T]}\mathbb{E}\|e_\Delta(t)\|^p\delta+\frac{p-r}{p}\mathbb{E}(I_{\{\tau\leqslant T\}}^{\frac{p}{p-r}})\delta^{-\frac{r}{p-r}}\\
=&\frac{r}{p}\sup_{t\in[0,T]}\mathbb{E}\|e_\Delta(t)\|^p\delta+\frac{p-r}{p}\mathbb{P}(\tau\leqslant T)\delta^{-\frac{r}{p-r}}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C\delta + C\left(\frac{\sum_{i=1}^d \mathbb{E}[x_i^p(T\wedge\tau)+x_{\Delta,i}^p(T\wedge\tau)]}{e^{pR/\sqrt{d}}} + \frac{\sum_{i=1}^d \mathbb{E}[x_i^{-q}(T\wedge\tau)+x_{\Delta,i}^{-q}(T\wedge\tau)]}{e^{qR/\sqrt{d}}}\right)\delta^{-\frac{r}{p-r}}\\
&\leqslant C\delta + Ce^{-(p\wedge q)\varphi^{-1}(h(\Delta))/\sqrt{d}}\delta^{-\frac{r}{p-r}}.
\end{aligned}
$$

取 $\delta = e^{-\left((p-r)(p\wedge q)\varphi^{-1}(h(\Delta))\right)/\sqrt{d}p}$, 则可推得以下估计式:

$$
\begin{aligned}
\sup_{t\in[0,T]} \mathbb{E}(\|e_\Delta(t)\|^r I_{\{\tau\leqslant T\}}) &\leqslant Ce^{-\left((p-r)(p\wedge q)\varphi^{-1}(h(\Delta))\right)/\sqrt{d}p} + Ce^{-(p\wedge q)\varphi^{-1}(h(\Delta))/\sqrt{d}}e^{\left(r(p\wedge q)\varphi^{-1}(h(\Delta))\right)/\sqrt{d}p}\\
&\leqslant Ce^{-\left((p-r)(p\wedge q)\varphi^{-1}(h(\Delta))\right)/\sqrt{d}p}.
\end{aligned}
$$

综合上述结果与引理 3.4, 我们最终得到以下收敛性估计:

$$
\begin{aligned}
\sup_{t\in[0,T]} \mathbb{E}(\|e_\Delta(t)\|^r) &= \sup_{t\in[0,T]} \mathbb{E}(\|e_\Delta(t)\|^r I_{\{\tau>T\}}) + \sup_{t\in[0,T]} \mathbb{E}(\|e_\Delta(t)\|^r I_{\{\tau\leqslant T\}})\\
&= \sup_{t\in[0,T]} \mathbb{E}\|e_\Delta(t\wedge\tau)\|^r + \sup_{t\in[0,T]} \mathbb{E}(\|e_\Delta(t)\|^r I_{\{\tau\leqslant T\}})\\
&\leqslant C\Delta^{\frac{r}{2}} h(\Delta)^{\frac{r}{2}} + Ce^{-\left((p-r)(p\wedge q)\varphi^{-1}(h(\Delta))\right)/\sqrt{d}p}.
\end{aligned}
$$

由于 $h(\Delta)\geqslant h(1)>3C_0$ 且 $\varphi(r)=C_0(2+e^{C(\alpha,\beta)r})$ 有

$$
e^{-\left((p-r)(p\wedge q)\varphi^{-1}(h(\Delta))\right)/\sqrt{d}p} = \left(\frac{h(\Delta)}{C_0}-2\right)^{-\frac{(p-r)(p\wedge q)}{\sqrt{d}pC(\alpha,\beta)}} \leqslant \left(\frac{h(\Delta)}{3C_0}\right)^{-\frac{(p-r)(p\wedge q)}{\sqrt{d}pC(\alpha,\beta)}}.
$$

现取 $h(\Delta) = (3.5C_0\vee\varphi(\|\ln x_0\|))\,\Delta^{-\frac{C(\alpha,\beta)pr}{3(2(p-r)(p\wedge q)+C(\alpha,\beta)pr)}}$, 则存在常数 C 使得最终强收敛估计成立:

$$
\sup_{t\in[0,T]} \mathbb{E}\|e_\Delta(t)\|^r < C\Delta^{\frac{(p-r)(p\wedge q)r}{3\sqrt{d}(2(p-r)(p\wedge q)+C(\alpha,\beta)pr)}}.
$$

注释 4.1 在引理 3.4 中取定 $\varepsilon=1/2$. 若进一步假设参数满足 $1.5r^* < \frac{p\wedge q}{(\alpha+6)\vee(\beta+6)}$, 则可推得 $\frac{p\wedge q}{(\alpha\vee\beta)+6} > (1+\varepsilon)r^* = \frac{(1+\varepsilon)r^*}{2\varepsilon}$. 因此,

$$
(1+\varepsilon)((\alpha\vee\beta)+6)r^* < p\wedge q, \quad \frac{((\alpha\vee\beta)+6)(1+\varepsilon)r^*}{2\varepsilon} < p\wedge q.
$$

基于式 (3.2)、(3.3)、注释 2.1、引理 3.2 和 3.3, 结合 Hölder 不等式及文献 [19] 中定理 7.1, 我们可建立如下估计:

$$
\begin{aligned}
&\mathbb{E}\left|\frac{x_{\Delta,i}(t)}{\bar{x}_{\Delta,i}(t)}-1\right|^{\frac{(1+\varepsilon)r}{\varepsilon}}\\
\leqslant& C(t-t_k)^{\frac{(1+\varepsilon)r}{\varepsilon}-1}\mathbb{E}\int_{t_k}^t \left|\frac{x_{\Delta,i}(s)}{\bar{x}_{\Delta,i}(s)}\right|^{\frac{(1+\varepsilon)r}{\varepsilon}} \left|F_{\Delta,i}(\bar{y}_\Delta(s))+\frac{1}{2}\|G_{\Delta,i}(\bar{y}_\Delta(s))\|^2\right|^{\frac{(1+\varepsilon)r}{\varepsilon}} ds\\
&+C(t-t_k)^{\frac{(1+\varepsilon)r}{2\varepsilon}-1}\mathbb{E}\int_{t_k}^t \left|\frac{x_{\Delta,i}(s)}{\bar{x}_{\Delta,i}(s)}\right|^{\frac{(1+\varepsilon)r}{\varepsilon}} \|G_\Delta(\bar{y}_\Delta(s))\|^{\frac{(1+\varepsilon)r}{\varepsilon}} ds\\
\leqslant& C\Delta^{\frac{(1+\varepsilon)r}{\varepsilon}-1}h(\Delta)^{\frac{(1+\varepsilon)r}{2\varepsilon}}\mathbb{E}\int_{t_k}^t \left|\frac{x_{\Delta,i}(s)}{\bar{x}_{\Delta,i}(s)}\right|^{\frac{(1+\varepsilon)r}{\varepsilon}} \left(\|F_\Delta(\bar{y}_\Delta(s))\|^{\frac{(1+\varepsilon)r}{2\varepsilon}} + \|G_\Delta(\bar{y}_\Delta(s))\|^{\frac{(1+\varepsilon)r}{\varepsilon}}\right) ds
\end{aligned}
$$

$$
\begin{aligned}
&+C\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}-1}\mathbb{E}\int_{t_k}^{t}\left|\frac{x_{\Delta,i}(s)}{\bar{x}_{\Delta,i}(s)}\right|^{\frac{(1+\varepsilon)r}{\varepsilon}}\|G_\Delta(\bar{y}_\Delta(s))\|^{\frac{(1+\varepsilon)r}{\varepsilon}}ds\\
\leqslant&C\Delta^{\frac{5(1+\varepsilon)r}{6\varepsilon}-1}\int_{t_k}^{t}\left(\mathbb{E}\left|\frac{x_{\Delta,i}(s)}{\bar{x}_{\Delta,i}(s)}\right|^{\frac{(1+\varepsilon)(1+\eta)r}{\varepsilon\eta}}\right)^{\frac{\eta}{1+\eta}}\\
&\times\left(\mathbb{E}\left(1+\|\bar{x}_\Delta(s)\|^{\frac{((\alpha\vee\beta)+4)(1+\varepsilon)(1+\eta)r}{2\varepsilon}}+\left\|\bar{x}_\Delta(s)^{-1}\right\|^{\frac{((\alpha\vee\beta)+4)(1+\varepsilon)(1+\eta)r}{2\varepsilon}}\right)\right)^{\frac{1}{1+\eta}}ds\\
&+C\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}-1}\int_{t_k}^{t}\left(\mathbb{E}\left|\frac{x_{\Delta,i}(s)}{\bar{x}_{\Delta,i}(s)}\right|^{\frac{(1+\varepsilon)(1+\eta)r}{\varepsilon\eta}}\right)^{\frac{\eta}{1+\eta}}\\
&\times\left(\mathbb{E}\left(1+\|\bar{x}_\Delta(s)\|^{\frac{((\alpha\vee\beta)+4)(1+\varepsilon)(1+\eta)r}{2\varepsilon}}+\left\|\bar{x}_\Delta(s)^{-1}\right\|^{\frac{((\alpha\vee\beta)+4)(1+\varepsilon)(1+\eta)r}{2\varepsilon}}\right)\right)^{\frac{1}{1+\eta}}ds\\
\leqslant&C\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}}.
\end{aligned}
$$

同样地,

$$
\begin{aligned}
&\mathbb{E}\,\|x_\Delta(t)-\bar{x}_\Delta(t)\|^{\frac{(1+\varepsilon)r}{\varepsilon}}\\
\leqslant&C(t-t_k)^{\frac{(1+\varepsilon)r}{\varepsilon}-1}\\
&\times\mathbb{E}\int_{t_k}^{t}\|x_\Delta(s)\|^{\frac{(1+\varepsilon)r}{\varepsilon}}\left\|F_\Delta(\bar{y}_\Delta(s))+\frac{1}{2}\left(\|G_{\Delta,1}(\bar{y}_\Delta(s))\|^2,\ldots,\|G_{\Delta,d}(\bar{y}_\Delta(s))\|^2\right)^T\right\|^{\frac{(1+\varepsilon)r}{\varepsilon}}ds\\
&+C(t-t_k)^{\frac{(1+\varepsilon)r}{2\varepsilon}-1}\mathbb{E}\int_{t_k}^{t}\|x_\Delta(s)\|^{\frac{(1+\varepsilon)r}{\varepsilon}}\|G_\Delta(\bar{y}_\Delta(s))\|^{\frac{(1+\varepsilon)r}{\varepsilon}}ds\\
\leqslant&C\Delta^{\frac{(1+\varepsilon)r}{\varepsilon}-1}h(\Delta)^{\frac{(1+\varepsilon)r}{2\varepsilon}}\\
&\times\mathbb{E}\int_{t_k}^{t}\|x_\Delta(s)\|^{\frac{(1+\varepsilon)r}{\varepsilon}}\left(1+\|\bar{x}_\Delta(s)\|^{\frac{((\alpha\vee\beta)+4)(1+\varepsilon)r}{2\varepsilon}}+\left\|\bar{x}_\Delta(s)^{-1}\right\|^{\frac{((\alpha\vee\beta)+4)(1+\varepsilon)r}{2\varepsilon}}\right)ds\\
&+C\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}-1}\mathbb{E}\int_{t_k}^{t}\|x_\Delta(s)\|^{\frac{(1+\varepsilon)r}{\varepsilon}}\left(1+\|\bar{x}_\Delta(s)\|^{\frac{((\alpha\vee\beta)+4)(1+\varepsilon)r}{2\varepsilon}}+\left\|\bar{x}_\Delta(s)^{-1}\right\|^{\frac{((\alpha\vee\beta)+4)(1+\varepsilon)r}{2\varepsilon}}\right)ds\\
\leqslant&C\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}-1}\mathbb{E}\int_{t_k}^{t}\|x_\Delta(s)\|^{\frac{(1+\varepsilon)r}{\varepsilon}}ds\\
&+C\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}-1}\int_{t_k}^{t}\left(\mathbb{E}\|x_\Delta(s)\|^{\frac{(1+\varepsilon)((\alpha\vee\beta)+6)r}{2\varepsilon}}\right)^{\frac{2}{(\alpha\vee\beta)+6}}\left(\mathbb{E}\,\|\bar{x}_\Delta(s)\|^{\frac{((\alpha\vee\beta)+6)(1+\varepsilon)r}{2\varepsilon}}\right)^{\frac{(\alpha\vee\beta)+4}{(\alpha\vee\beta)+6}}ds\\
&+C\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}-1}\int_{t_k}^{t}\left(\mathbb{E}\|x_\Delta(s)\|^{\frac{(1+\varepsilon)((\alpha\vee\beta)+6)r}{2\varepsilon}}\right)^{\frac{2}{(\alpha\vee\beta)+6}}\left(\mathbb{E}\left\|\bar{x}_\Delta(s)^{-1}\right\|^{\frac{((\alpha\vee\beta)+6)(1+\varepsilon)r}{2\varepsilon}}\right)^{\frac{(\alpha\vee\beta)+4}{(\alpha\vee\beta)+6}}ds\\
\leqslant&C\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}}.
\end{aligned}
$$

现设 $h(\Delta)=(4C_0\vee\varphi(\|\ln x_0\|))\,\Delta^{-\frac{1}{3}}$. 通过引理 3.4 和定理 4.1 的类似论证方法, 可得到以下强收敛估计:

$$
\sup_{t\in[0,T]}\mathbb{E}\|e_\Delta(t)\|^r<C\Delta^{\frac{r}{2}}+C\Delta^{\frac{(p-r)(p\wedge q)}{3\sqrt{d}C(\alpha,\beta)p}}<C\Delta^{\frac{r}{2}\wedge\frac{(p-r)(p\wedge q)}{3\sqrt{d}C(\alpha,\beta)p}}.
$$

5 数值算例

在第四节中, 我们建立了对数截断 EM 方法的一般收敛性理论, 所得收敛速率表达式较为复杂. 本节将通过一个典型算例具体展示该方法的收敛性能. 值得注意的是, 对于下列具体的随机微分方程模型, 该方法实际可获得略优于 1/2 阶的收敛速率.

例 5.1 考虑如下二维随机微分方程：

$$\begin{cases} dx_1 = (a_{11}x_1^{-1}(t) + a_{12}x_2(t) - a_{13}x_1^2(t))dt + \sigma_1 x_1(t)dB_1(t), \\ dx_2 = (a_{21}x_2^{-1}(t) + a_{22}x_1(t) - a_{23}x_2^2(t))dt + \sigma_2 x_2(t)dB_2(t). \end{cases} \tag{5.1}$$

因此, 其系数为

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} a_{11}x_1^{-1} + a_{12}x_2 - a_{13}x_1^2 \\ a_{21}x_2^{-1} + a_{22}x_1 - a_{23}x_2^2 \end{bmatrix} \quad \text{和} \quad g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \sigma_1 x_1 & 0 \\ 0 & \sigma_2 x_2 \end{bmatrix}.$$

另外,

$$x^T f(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11}x_1^{-1} + a_{12}x_2 - a_{13}x_1^2 \\ a_{21}x_2^{-1} + a_{22}x_1 - a_{23}x_2^2 \end{bmatrix} = a_{11} + a_{12}x_2x_1 - a_{13}x_1^3 + a_{21} + a_{22}x_1x_2 - a_{23}x_2^3,$$

$$\|g(x)\|^2 = \text{trace}\left(\begin{bmatrix} \sigma_1 x_1 & 0 \\ 0 & \sigma_2 x_2 \end{bmatrix} \begin{bmatrix} \sigma_1 x_1 & 0 \\ 0 & \sigma_2 x_2 \end{bmatrix}\right) = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2.$$

设 $r \geqslant 2$ 为给定实数. 对于任意 $x, y \in \mathbb{R}_+^2$, 存在常数 M, 使得

$$\begin{aligned} \|f(x) - f(y)\| =& \left\| \begin{pmatrix} a_{11}(x_1^{-1} - y_1^{-1}) + a_{12}(x_2 - y_2) - a_{13}(x_1^2 - y_1^2) \\ a_{21}(x_2^{-1} - y_2^{-1}) + a_{22}(x_1 - y_1) - a_{23}(x_2^2 - y_2^2) \end{pmatrix} \right\| \\ =& \Big((a_{11}(x_1^{-1} - y_1^{-1}) + a_{12}(x_2 - y_2) - a_{13}(x_1^2 - y_1^2))^2 \\ &+ (a_{21}(x_2^{-1} - y_2^{-1}) + a_{22}(x_1 - y_1) - a_{23}(x_2^2 - y_2^2))^2 \Big)^{\frac{1}{2}} \\ \leqslant& a_{11}|y_1 - x_1|(x_1^{-1} \cdot y_1^{-1}) + a_{12}|x_2 - y_2| + a_{13}|x_1 - y_1|(x_1 + y_1) \\ &+ a_{21}|y_2 - x_2|(x_2^{-1} \cdot y_2^{-1}) + a_{22}|x_1 - y_1| + a_{23}|x_2 - y_2|(x_2 + y_2) \\ \leqslant& \frac{1}{2}a_{11}|y_1 - x_1|(x_1^{-2} + y_1^{-2}) + a_{12}|x_2 - y_2| + a_{13}|x_1 - y_1|(x_1 + y_1) \\ &+ \frac{1}{2}a_{21}|y_2 - x_2|(x_2^{-2} + y_2^{-2}) + a_{22}|x_1 - y_1| + a_{23}|x_2 - y_2|(x_2 + y_2) \\ \leqslant& M\left(1 + \|x\| + \|y\| + \left\|x^{-1}\right\|^2 + \left\|y^{-1}\right\|^2\right)\|x - y\|, \end{aligned}$$

由此假设 (A_1) 成立, 参数取值为 $\alpha = 1$ 和 $\beta = 2$.

对于 $p=13r^*, r^*=2r$ 和 $x\in\mathbb{R}_+^2$, 利用 Young 不等式可证存在常数 K 使得

$$\begin{aligned}
& x^T f(x)+\frac{p-1}{2}\|g(x)\|^2\\
=& a_{11}+a_{12}x_1x_2-a_{13}x_1^3+a_{12}+a_{22}x_1x_2-a_{23}x_1^3+\frac{p-1}{2}(\sigma_1^2x_1^2+\sigma_2^2x_2^2)\\
\leqslant& a_{11}+a_{12}+\frac{1}{2}a_{12}x_1^2+\frac{1}{2}a_{12}x_2^2+\frac{1}{2}a_{22}x_1^2+\frac{1}{2}a_{22}x_2^2+\frac{p-1}{2}(\sigma_1^2+\sigma_2^2)(x_1^2+x_2^2)\\
\leqslant& K(1+\|x\|^2).
\end{aligned}$$

取 $q=p$, 我们总能找到一个足够小的 $x^*>0$ 使得

$$x_1f_1(x)-\frac{q+1}{2}\|g_1(x)\|^2=a_{11}+a_{12}x_1x_2-a_{13}x_1^3-\frac{q+1}{2}\sigma_1^2x_1^2>0,\quad x_1\in(0,x^*), x_2\in\mathbb{R}_+$$
$$x_2f_2(x)-\frac{q+1}{2}\|g_2(x)\|^2=a_{12}+a_{22}x_1x_2-a_{23}x_2^3-\frac{q+1}{2}\sigma_2^2x_2^2>0,\quad x_2\in(0,x^*), x_1\in\mathbb{R}_+.$$

由此假设 (A_2) 成立.

设 $x,y\in\mathbb{R}_+^2$, 则

$$\begin{aligned}
& (x-y)^T\left(f(x)-f(y)\right)\\
=& (x-y)^T\begin{bmatrix} a_{11}(x_1^{-1}-y_1^{-1})+a_{12}(x_2-y_2)-a_{13}(x_1^2-y_1^2)\\ a_{21}(x_2^{-1}-y_2^{-1})+a_{22}(x_1-y_1)-a_{23}(x_2^2-y_2^2)\end{bmatrix}\\
=& a_{11}(x_1-y_1)(x_1^{-1}-y_1^{-1})+a_{12}(x_1-y_1)(x_2-y_2)-a_{13}(x_1-y_1)(x_1^2-y_1^2)\\
& +a_{21}(x_2-y_2)(x_2^{-1}-y_2^{-1})+a_{22}(x_1-y_1)(x_2-y_2)-a_{23}(x_2-y_2)(x_2^2-y_2^2)\\
\leqslant& (a_{12}+a_{22})(x_1-y_1)(x_2-y_2)\\
\leqslant& \frac{1}{2}(a_{12}+a_{22})\left((x_1-y_1)^2+(x_2-y_2)^2\right).
\end{aligned}$$

以及

$$\begin{aligned}
\|g(x)-g(y)\|^2=&\mathrm{trace}\left(\begin{bmatrix}\sigma_1(x_1-y_1) & 0\\ 0 & \sigma_2(x_2-y_2)\end{bmatrix}\begin{bmatrix}\sigma_1(x_1-y_1) & 0\\ 0 & \sigma_2(x_2-y_2)\end{bmatrix}\right)\\
\leqslant&(\sigma_1^2+\sigma_2^2)((x_1-y_1)^2+(x_2-y_2)^2).
\end{aligned}$$

因此, 存在常数 H 使得

$$(x-y)^T\left(f(x)-f(y)\right)+\frac{r^*-1}{2}\|g(x)-g(y)\|^2\leqslant H\|x-y\|^2.$$

因此, 假设 (A_3) 满足.

由 $1.5r^*<\frac{p\wedge q}{(\alpha+6)\vee(\beta+6)}$. 故注释 4.1 的条件也满足. 进一步

$$\frac{(p-r)(p\wedge q)}{3\sqrt{2}C(\alpha\vee\beta)p}>\frac{(p-r)(p\wedge q)}{3\sqrt{2}((\alpha\vee\beta)+6)p}=\frac{(26r-r)}{24\sqrt{2}}>\frac{r}{2}.$$

因此, 对于任意 $\Delta \in (0,1]$ 有

$$\sup_{t\in[0,T]} \mathbb{E}\|x(t)-x_\Delta(t)\|^r < C\Delta^{\frac{r}{2}}.$$

6 数值模拟

在本节中, 我们将对第 5 节中的模型进行数值模拟, 以支持我们的理论结果. 我们现在用 100 个步长为 $\Delta = 2^{-19}, 2^{-18}, \ldots, 2^{-12}$ 的样本路径进行数值模拟. 我们使用步长为 $\Delta = 2^{-20}$ 的数值解作为精确解的良好近似值.

考虑例 5.1 中参数取值为 $a_{11}=1, a_{12}=2, a_{13}=3, a_{21}=1, a_{22}=1, a_{23}=4, \sigma_1=2, \sigma_2=1$ 的情形, 有

$$\begin{aligned}F(y) &= \begin{bmatrix} e^{-y_1} & 0 \\ 0 & e^{-y_2}\end{bmatrix}\begin{bmatrix} e^{-y_1}+2e^{y_2}-3e^{2y_1} \\ e^{-y_2}+e^{y_1}-4e^{2y_2}\end{bmatrix} - \frac{1}{2}\begin{bmatrix} e^{-2y_1} & 0 \\ 0 & e^{-2y_2}\end{bmatrix}\begin{bmatrix} 4e^{2y_1} \\ e^{2y_2}\end{bmatrix} \\ &= \begin{bmatrix} e^{-2y_1}+2e^{y_2-y_1}-3e^{y_1}-2 \\ e^{-2y_2}+e^{y_1-y_2}-4e^{y_2}-\frac{1}{2}\end{bmatrix}\end{aligned}$$

以及

$$G(y) = \begin{bmatrix} e^{-y_1} & 0 \\ 0 & e^{-y_2}\end{bmatrix}\begin{bmatrix} 2e^{y_1} & 0 \\ 0 & e^{y_2}\end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1\end{bmatrix}.$$

进一步我们也有

$$\begin{aligned}\|F(y)\| &= \left(F_1(y)^2+F_2(y)^2\right)^{\frac{1}{2}} \\ &= \left((e^{-2y_1}+2e^{y_2-y_1}-3e^{y_1}-2)^2+(e^{-2y_2}+e^{y_1-y_2}-4e^{y_2}-\frac{1}{2})^2\right)^{\frac{1}{2}} \\ &\leqslant 2.5+5e^{2\|y\|}+7e^{\|y\|} \\ &\leqslant 8.5(1+e^{2\|y\|})\end{aligned}$$

以及

$$\|G(y)\|^2 = \sum_{i=1}^{2}\sum_{j=1}^{2} G_{i,j}^2(y) = 4+1 = 5.$$

设定终端时间 $T=1$ 且初始值 $x_0=(2,1)^T$, 然后设置 $C(\alpha,\beta)=2$, 和

$$\varphi(r) = 8.5(2+e^{2r}), \quad h(\Delta) = 51\Delta^{-\frac{1}{3}}.$$

数值实验结果 (见图 (6.1)) 显示, 二阶矩的强收敛误差阶数略高于 1 阶, 这与注释 4.1 的理论证明一致. 图 (6.2) 和 (6.3) 通过随机选取的数值样本路径, 直观展示了该方法生成的数值解始终保持严格正性.

本文对对数截断 EM 方法进行了深入研究, 通过改进约束条件使该方法可适用于多维随机微分方程. 我们不仅证明了该方法在 L^p 空间的具体收敛速率, 同时保证数值解的正性. 特别地, 对于某些特定随机微分方程模型, 其收敛阶数可达 1/2 阶甚至略高于 1/2 阶.

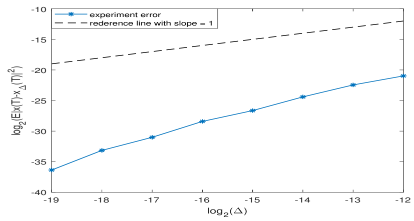

图 6.1 模型 (5.1) 的对数截断 EM 方法的 L^2 收敛阶.

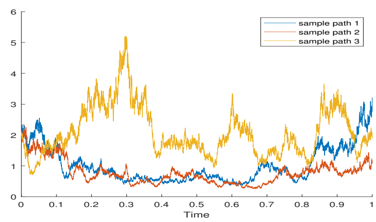

图 6.2 模型 (5.1) 中 x_1 的数值解样本路径.

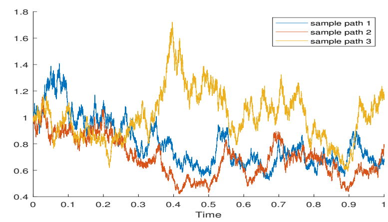

图 6.3 模型 (5.1) 中 x_2 的数值解样本路径.

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LOGARITHMIC TRUNCATED EM METHOD FOR A CLASS OF MULTI-DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract: This paper studies the problem of preserving positivity in the numerical solution of a class of multi-dimensional stochastic differential equations (SDEs). By modifying the assumptions in the original theory and combining the properties of logarithmic transformation, a new method for constructing truncation functions is developed. The strong convergence result of this numerical method is obtained, and it is proved that the L^p convergence of the numerical solution holds under certain conditions, with the strong convergence rate reaching the 1/2 order. The results of the original method are generalized, successfully expanding the application scope of the logarithmic truncation EM method from scalar SDEs to multi-dimensional SDEs.

Keywords: multi-dimensional stochastic differential equations; positivity preserving; the logarithmic truncated EM method; numerical simulation

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