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TWO GEOMETRY INQUEALITY IN RIEMANNIAN MANIFOLD WITH NONNEGATIVE RICCI CURVATURE AND CONVEX BOUNDARY

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Abstract: In this paper we investigates the problem of inequalities on Riemannian manifolds with nonnegative Ricci curvature. By employing the method of Jia-Wang-Xia-Zhang, two types of results on geometric inequalities are obtained, generalizing the results of Jia-Wang-Xia-Zhang.

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1 Introduction

1.1 Heintze-Karcher Type Inequality

The Heintze-Karcher inequality states that for a close embedding mean convex hypersurface Σ in Riemannian manifold M^{n+1} with $Ric \geq 0$,assume Σ enclose a bounded domain Ω,it holds that

$$\int_\Sigma \frac{1}{H} dA \geq \frac{n+1}{n}|\Omega|$$

and above equality holds if and only if Σ is a geodesic sphere.Where H is the mean curvature of Σ and Σ is said to be mean convex if $H > 0$. It was first proved by Heintze-Karcher [1] and Ros gives a new proof [2] and its new proof is relatived to Minkowski-Husing formula and Alexandrov's theorem we refer to [4] and references therein about more history.Recently,there are many extension about Heintze-Karcher inequality refer to [4] in close anisotropic cases in Euclidren space , [4] in Euclidren half space for capillary setting and in the spaces forms and more generally,in warped product manifolds,[5] (also [6]).

Recently,Jia-Wang-Xia-Zhang have studied the Heintze-Karcher-type inequality for hypersurfaces with boundary in Euclidren convex domains [7].Precisely,they prove the following result.

Theorem 1.1 Let $K \subset \mathbb{R}^{n+1}$ be a convex domain with boundary ∂K and $\Sigma \subset \overline{K}$ an embedded compact C^2-hypersurface with boundary $\partial\Sigma \subset \mathrm{Reg}(\partial K)$ intersecting ∂K

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Biography: Lu Chenghang(2001–), male, born at Nanping Fujian, postgraduate, major in: differential geometry. E-mail: chenghanglu@stu.xmu.edu.cn transversally such that for $x \in \partial\Sigma$, $\langle v(x), \bar{N}(x)\rangle \leq 0$. Let Ω denote the domain enclosed by Σ and ∂K. Assume the mean curvature H of Σ is positive. Then we have $\displaystyle\int_\Sigma \frac{1}{H} dA \geq \frac{n+1}{n}|\Omega|$.

Moreover, equality in holds if and only if Σ is a free boundary sphere cap and $\partial K \cap \overline{\Omega}$ is a part of the boundary of the convex cone C with a smooth sector that is determined by Σ.

A hypersurface Σ ,which intersects ∂K transversally ,is called free boundary if $\langle \nu(x), \bar{N}\rangle = 0$ for all $x \in \Sigma$,here ν is the normal vector field with respect to Ω,the sphere cap is part of a sphere.In facts,they study more general Heintze-Karcher inequality in anisotropic case.

In this paper,we general to this result on manifold with nonnegetive Ricci curvature and convex boundary.Precisely,we have

Theorem 1.2 Let (K^{n+1}, g) be a Riemannian manifold with convex boundary ∂K and $Ric \geq 0$, and $\Sigma \subset \overline{K}$ an embedded compact C^2-hypersurface with boundary $\partial\Sigma \subset \mathrm{Reg}(\partial K)$ intersecting ∂K transversally such that for $x \in \partial\Sigma$,

$$\langle v(x), \overline{N}(x)\rangle \leq 0.$$

Let Ω denote the domain enclosed by Σ and ∂K. Assume the mean curvature H of Σ is positive. Then we have

$$\int_\Sigma \frac{1}{H} dA \geq \frac{n+1}{n}|\Omega|. \tag{1.4}$$

Moreover, if equality holds then $\Omega \cong (0, r_0] \times \Sigma, g = ds^2 + s^2(\frac{H}{n})^2 g_\Sigma$.

To prove the theorem 1.2,we use the method by Jia-Wang-Xia-Zhang [4][7],to construct geodesic parallel hypersurfaces to show the surjectivity of the inward normal flow,under the small changes and run the same argument to prove Heintze-Karcher inequality and use the standard rigidity argument by [9].

1.2 Willmore-type Inequality

The classical Willmore inequality [9] states for a immersed closed hypersurface Σ in $\mathbb{R}^{n+1}$ then

$$\int_\Sigma \frac{|H|}{n} \geq |\mathbb{S}^n|$$

and equality holds then Σ is a round sphere,here H is the mean curvature of Σ and $|\mathbb{S}^n|$ is the area of round sphere of $\mathbb{S}^n$.

Recently,Agostiniani-Fogagnolo-Mazzieri [10] and Wang [8] generalize Willmore inequality to Riemannian manifold with nonnegative Ricci curvature,Precisely

Theorem 1.3 Let (M, g) be a complete non-compact Riemannian manifold with $\mathrm{Ric} \geq 0$ and Euclidean volume growth. Let $\Omega \subset M$ be a bounded domain with $\partial\Omega = \Sigma$. Then

$$\int_\Sigma \left(\frac{|H|}{n}\right)^n dA \geq \mathrm{AVR}(g)|S^n|.$$

Equality holds if and only if

$$(M \setminus \Omega, g) \simeq \left([r_0, \infty) \times \Sigma, dr^2 + \frac{r^2}{r_0^2} g_\Sigma \right), \quad r_0^n = \frac{|\Sigma|}{\mathrm{AVR}(g)|S^n|}.$$

Here AVR(g) is the asymptotic volume ratio of M i.e.

$$\mathrm{AVR}(g) := \lim_{R \to \infty} \frac{|B_R(p)|}{|B^{n+1}|R^{n+1}}.$$

See section 4 to more detail.

Willmore's original proof used the Gauss map to demonstrate that it is surjective on the set of positive Gaussian curvature. Clearly, this method fails on general manifolds.Agostiniani-Fogagnolo-Mazzieri proves the above theorem is highly nontrivial,they use capcitary function in $M \backslash \Omega$ and construct monotone quantity over level sets.Wang gave a short proof bases on the Heintez-Karcher's comparison theorem in Riemannian Geometry.

The above Willmore type inequality can be applied to prove reltative isoperimetric inequality in manifolds of non-nagative Ricci curvature when $n+1 \leq 7$ we refer [10][11] for more detail,Brendle [12] used ABP method can generalize any dimension.

Recently,Jia-Wang-Xia-Zhang study a similar problem in Euclidren unbounded closed convex sets[13]，they also solves this problem in anisotropic cases.

In this paper, we use Wang's and Jia-Wang-Xia-Zhang method to extend the Willmore-type inequality to Riemannian manifolds with convex boundaries,and non-negative Ricci curvature. Precisely,we have the following result.

Theorem 1.4 Let M be a non-compact manifold with non-negative Ricci curvature and convex boundary. Let $\Sigma \subset M$ be a compact properly embedded hypersurface with boundary $\partial\Sigma \subset \partial M$ intersects ∂M such that the contact angle $\theta(x) \geq \frac{\pi}{2}$ along $\partial\Sigma$. Let Ω be the bounded domain enclosed by Σ and ∂M. Then

$$\int_\Sigma \left(\frac{|H|}{n} \right)^n dA \geq \mathrm{AVR}(M)|\mathbb{S}^n|.$$

Moreover, equality holds if

$$M \setminus \Omega \simeq [r_0, \infty) \times \Sigma, \quad g = dr^2 + \left(\frac{r}{r_0} \right)^2 g_\Sigma, \quad r_0^n = \frac{|\Sigma|}{\mathrm{AVR}(g)|\mathbb{S}^n|}.$$

We prove the above theorem by constructing an outward normal flow, while the proof of Theorem 1 relies on an inward normal flow. Therefore, the Heintze-Karcher inequality can be regarded as the "dual" version of the Willmore inequality.

1.3 Organization of This Paper

The rest of the paper is organized as follows.In section 2,we review some notations in Submanifold Geometry.In section 3 we prove the Heintze-Karcher type inequality in theorem 1.2 and 1.3.In section 4 we prove the Willmore type inequality in theorem 1.4.

2 Preliminaries

If (M^{n+1}, g) is a Riiemannian manifoldn,we denote the connection by ∇ on M, Riemann curvature tensor with four agument by $R(\cdot,\cdot,\cdot,\cdot)$ the secitonal curvature by sec,the Ricci curvature by Ric,and the scalar curvature by R.In paricualr,if$\{e_1, e_2 \ldots, e_n\}$ is an orthonormal basis for T_pM,then $sec(\Pi) = R(e_1, e_2, e_1, e_2), Ric(e_1) = \sum_{i=2}^{n}(e_1, e_i, e_1, e_i)$,here Π is a two plane span by $\{e_1, e_2\}$.

Let $x : (\Sigma^n, g_\Sigma) \to (M^{n+1}, g)$ be an isometric embedded of an orientable n-dimensional compact hypersurface Σ with boundary $\partial\Sigma$ satisfying $x_{|\partial\Sigma} : \partial\Sigma \to \partial M$. Such an immersion is called an embedded supported on ∂M. We emphasize that in this paper any hypersurfaces we consider is embedded and its boundary map $x_{|\partial\Sigma} : \partial\Sigma \to \partial M$ is embedded. However, for the convenience of notation, we do not distinguish M with its image $x(M)$through all computations are in fact carried out on Σ by using the pull-back of x. We choose one of the unit normal vector field along x and denote it by ν, and We denote by $\bar{N}$ the unit outward normal to $\partial\Sigma$ in ∂M. Denote by h and H the second fundamental form and the mean curvature of the embedding x respectively. Precisely,for $X, Y \in \Gamma(T\Sigma), h(X, Y) = \langle \nabla_X \nu, Y\rangle$ and $H = trh$.

3 Proof of theorem 1.2

Proof of Theorem 1.2 For any $x \in \Sigma$,let $e_i(x)$ be the corresponding principal vector of Σ at x with respect to principal curvature $\kappa_i(x)$,suth that $|e_1 \wedge e_2 \wedge \cdots \wedge e_n = 1|$. Since Σ is stritly mean convex ,

Define

$$Z = \{(x, t) \in \Sigma \times \mathbb{R} : 0 < t \leq \frac{n}{H}\}$$

and

$$\zeta : Z \to K, \zeta(x, t) = \exp_x(-t\nu(x)).$$

Claim.$\Omega \subset \zeta(Z)$

For any $y \in \Omega$ we consider geodesic ball $B_r(y)$, such that $B_r(y)$ touches Σ for the first time from the interior at $x_y \in \Sigma$.

Case1.$x_y \in \mathring{\Sigma}$.

In this case,the geodesic ball is tangent to Σ at x_y, and hence

$$\nu(x_y) = \nu^{B_r(y)}(x_y).$$

Moreover,since $B_r(y)$ touches Σ from the interior ,we easily infer

$$d\nu \leq d\nu_r^B(y),$$

hence,by mean curvature comparsion theorem (Racall $Ric \geq 0$)

$$H(x) \leq H^{B_r(y)}(x) \leq \frac{n}{r},$$

hence we find

$$r \le \frac{n}{H}.$$

Invoking the definition of Z and ζ,we get that $y \in \zeta(Z)$.

Case2. $x_y \in \partial\Sigma$.

Since the geodesic ball $B_r(y)$ touches Σ from the interior,we find

$$0 \ge \langle \nu(x_y), \bar{N}(x_y)\rangle \ge \langle \nu^{B_r(y)}(x_y), \bar{N}(x_y)\rangle.$$

On the other hand,since K is convex and $\partial\Sigma \subset \mathrm{Reg}(\partial K)$,it means that K lies completely on the one side of tangent space $T_{x_y}\partial K$.Moreover,since $y \in \Omega \subset K$,we get

$$\langle \nu^{B_r(y)}(x_y), \bar{N}(x_y)\rangle > 0,$$

that contradicts,so it can not be that $x_y \in \partial\Sigma$, we complete the proof of **Claim**.

By the Laplace comparison and area formula,

$$|\Omega| \le |\zeta(Z)| \le \int_\Sigma \int_0^{\frac{n}{H(p)}} \mathrm{Jac}(\zeta)(p,t)dt\, dA(p) \le= \int_\Sigma \int_0^{\frac{n}{H(p)}} \prod_i (1 - t\kappa_i(p))dt\, dA(p).$$

By the AM-GM inequality,we obtain

$$|\Omega| \le \int_\Sigma \int_0^{\frac{n}{H(p)}} \left(1 - t\frac{H(p)}{n}\right)^n dt\, dA(p) = \frac{n}{n+1}\int_\Sigma \frac{1}{H}\, dA.$$

This is the inequality we need.

From the proof of the Heintze-Karcher inequality, equality holds, then from the above argument,we have $\kappa_1(x) = \kappa_2(x) = \cdots = \kappa_n(x) = \kappa$, $H = n\kappa$ for all $x \in \Sigma$. The following argument we follow by [8]. Define the distance function $r(x) =$ distance from x to Σ for $x \in \Omega$. On Σ, $r = 0$ Along the geodesic $\gamma_p(t)$, we have $r(\gamma_p(t)) = t$.

From the Jacobian equality, the mean curvature of the level sets $\Sigma_t = \{x \in \Omega : r(x) = t\}$ is given by

$$H(t) = -\frac{d}{dt}\log \mathrm{Jac}(\zeta)(p,t) = \frac{H}{1 - (H/n)t}.$$

The second fundamental form of Σ_t satisfies

$$\Pi_t = \frac{H(t)}{n} g_{\Sigma_t} = \frac{H}{n(1 - (H/n)t)} g_{\Sigma_t},$$

which implies $\nabla^2 r(Y,Y) = -\frac{H}{n(1-(H/n)r)}$ for unit vectors $\langle Y, \nabla r\rangle = 0$. Taking the trace, we get the Laplacian

$$\Delta r = -n \cdot \frac{H}{n(1 - (H/n)r)} = -\frac{H}{1 - (H/n)r}.$$

Therefore, we have

$$\nabla^2 r = \frac{\Delta r}{n}(g - dr^2).$$

- $|\nabla^2 r|^2 = \frac{(\Delta r)^2}{n}$,
- $\langle \nabla \Delta r, \nabla r\rangle = \frac{d}{dr}\Delta r = \frac{d}{dr}\left(-\frac{H}{1-(H/n)r}\right) = -\frac{H^2/n}{(1-(H/n)r)^2} = -\frac{(\Delta r)^2}{n}$.

Substituting into Bochner formula

$$0 = \frac{(\Delta r)^2}{n} + \mathrm{Ric}(\nabla r, \nabla r) - \frac{(\Delta r)^2}{n} = \mathrm{Ric}(\nabla r, \nabla r).$$

Thus, $\mathrm{Ric}(\nabla r, \nabla r) = 0$. Since $\mathrm{Ric} \geq 0$, we conclude $\mathrm{Ric}(\nabla r, \cdot) = 0$. On Σ, consider an orthonormal frame $\{e_0 = \nu, e_1, \ldots, e_n\}$, where $\nu = \nabla r$. Since Σ is umbilical (from the equality condition), the second fundamental form is $\mathrm{II} = \frac{H}{n} g_\Sigma$. By the Codazzi equation

$$R(e_k, e_j, e_i, \nu) = \mathrm{II}_{ij,k} - \mathrm{II}_{ik,j} = \frac{1}{n}(H_k \delta_{ij} - H_j \delta_{ik}),$$

where H_k is the covariant derivative of H. Taking the trace over i and k

$$-\frac{n-1}{n} H_j = \mathrm{Ric}(e_j, \nu) = 0,$$

so $H_j = 0$ for all j, meaning H is constant on Σ. Also, the pullback metric $\zeta^* g$ takes the warped product form $g = dr^2 + h_r$, where h_r is a family of metrics on Σ with $h_0 = g_\Sigma$. From the equation $D^2 r = \frac{\Delta r}{n}(g - dr^2)$ and $\Delta r = -\frac{H}{1-(H/n)r}$, we have

$$\frac{1}{2}\frac{\partial}{\partial r} h_{ij} = \frac{H}{n}\frac{1}{1-(H/n)r} h_{ij}.$$

Integrating this equation yields

$$h_r = \left(1 - \frac{H}{n} r\right)^2 g_\Sigma.$$

Make the change of coordinates $s = r_0 - r$, then $dr = -ds$ and

$$g = ds^2 + \left(\frac{s}{r_0}\right)^2 g_\Sigma.$$

We finish the proof.

Remark 1 In fact, the characterization of the equality here may be incomplete. Indeed, we hope that the enclosed region Ω is precisely the Euclidean hemisphere, and the equality holds if and only if it is achieved on the Euclidean hemisphere.

4 Proof of Willmore type inequality

Proposition 4.1 [13] Let M be a non-compact $(n+1)$-manifold of non-negative Ricci curvature and with convex boundary ∂M (here convexity means geodesical convexity). Then for any $p_0 \in \partial M$ and any $0 \leq r < +\infty$,

$$\frac{|M \cap B_{r+R}(p_0)|}{R^{n+1}}$$

is non-increasing about R.

Proof of Proposition 4.1 By using the co-area formula, we have

$$\frac{d}{dR}\frac{|M\cap B_{r+R}(p_0)|}{R^{n+1}} = \frac{1}{R^{n+1}}\left(|M\cap\partial B_{r+R}(p_0)| - \frac{n+1}{R}|M\cap B_{r+R}(p_0)|\right).$$

On the other hand, by the Laplace comparison theorem, $\Delta r \leq \frac{n}{r}$, where r is the distance on M with respect to the fixed point p_0. This is equivalent to $\mathrm{div}(r\partial_r) \leq n+1$.

Thus, by the divergence theorem, we have

$$\begin{aligned}(n+1)|M\cap B_{r+R}(p_0)| &\geq \int_{M\cap B_{r+R}(p_0)} \mathrm{div}(r\partial_r)\\ &= \int_{M\cap\partial B_{r+R}(p_0)} \langle R\partial_r, \partial_r\rangle + \int_{\partial M\cap B_{r+R}(p_0)} \langle r\partial_r, \bar{N}\rangle(x)\\ &\geq R|M\cap\partial B_{r+R}(p_0)|.\end{aligned}$$

In the last inequality we used $\langle r\partial_r, \bar{N}\rangle \geq 0$ due to the convexity of ∂M.

It follows that

$$\frac{d}{dR}\frac{|M\cap B_{r+R}(p_0)|}{R^{n+1}} \leq 0.$$

A direct consequence is that one can define the asymptotic volume ratio for complete Ricci nonnegative manifold (M,g) by

$$\mathrm{AVR}(g) = \lim_{R\to\infty}\frac{|M\cap B_R(p_0)|}{\omega_{n+1}R^{n+1}}.$$

It is easy to see that the limit does not depend on the choice of $p_0 \in \partial M$, and from the Bishop-Gromov comparison theorem, in conjunction with Proposition 1.1, one sees that $\mathrm{AVR}(g) \in [0,1]$.

Let Σ be a hypersurface in M, for each $p \in \Sigma$, we define $\nu(p)$ to be the outer unit normal to Σ with respect to g. Suppose that Σ meets ∂M transversally with contact angle $\theta(p) \geq \frac{\pi}{2}$, given by

$$-\cos\theta(p) := \langle \nu, \bar{N}\rangle(p), \quad \forall p \in \partial\Sigma,$$

where $\bar{N}$ the outer unit normal to ∂M with respect to g.

Let $\exp_p^g$ be the exponential map at p with respect to g and Φ the normal exponential map from Σ, given by

$$\Phi : \Sigma\times[0,\infty) \to M,$$

$$\Phi(p,t) = \exp_p^g(t\nu(p)).$$

Let $d_g(q,\Sigma)$ be the distance function on M at q from Σ with respect to the metric g and define

$$\tau(p) = \sup\{t>0 : d_g(\Phi(p,t),\Sigma) = t\}.$$

Denote $T = \max_{p\in\Sigma}\tau(p)$, and define the sets

$$A = \{(p,t) \in \Sigma\times[0,T) : d_g(\Phi(p,t),\Sigma) = t\},$$

$$A^* = \{(p,t) \in \Sigma \times [0,T) : (p, t+\delta) \in A \text{ for some } \delta > 0\}. \tag{4.1}$$

It is well-known that τ is a continuous function on Σ, A^* is an open set of $\Sigma \times [0,T)$, and the restriction $\Phi_{|A^*}$ is a diffeomorphism. Also, $\Phi(A)$ is a closed set and the focus locus $\Phi(A \setminus A^*)$ is a closed set of measure zero in M.

Let Ω be the region enclosed by Σ and ∂M. Define the distance function

$$d_g(q, \overline{\Omega}) = \inf_{r>0} \left\{ r : \overline{B_r(q)} \cap \overline{\Omega} \neq \emptyset \right\}, \quad \forall q \in M.$$

Proposition 4.2 1. For any $R > 0$, there holds

$$\left\{ q \in M : d_g(q, \overline{\Omega}) \leq R \right\} \subset \overline{\Omega} \cup \{\Phi(p,t) : p \in \Sigma, t \in [0, \min(R, \tau(p))]\}.$$

2. For any $r, R > 0$, there holds

$$\{q \in M : d_g(q, B_r(p_0)) \leq R\} = \overline{B_{r+R}(p_0)}. \tag{4.2}$$

3. The asymptotic volume ratio of the set is exactly $\mathrm{AVR}(g)$, i.e.,

$$\lim_{R \to +\infty} \frac{\mathrm{Vol}\left(\left\{ q \in M : d_g\left(q, \overline{\Omega}\right) \leq R \right\}\right)}{\omega_{n+1} R^{n+1}} = \mathrm{AVR}(g). \tag{4.3}$$

Proof of Proposition 4.2 From the definition of $d_g(\cdot, \overline{\Omega})$, it is clear that for any $q \in \overline{\Omega}$, $d_g(q, \overline{\Omega}) = 0$. For any $q \in M \setminus \overline{\Omega}$ satisfying $d_g(q, \overline{\Omega}) \leq R$, we use a family of closed, concentric, geodesic balls $\left\{ \overline{B_r(q)} \right\}_{r>0}$ to touch $\overline{\Omega}$. Since $d_g(q, \overline{\Omega}) \leq R$, we know there must exist $p \in \Sigma$ and $r_0 \in (0, R]$, such that $B_{r_0}(q)$ touches $\overline{\Omega}$ for the first time at the point p. Let ν_B be the outer unit normal of $B_{r_0}(q)$ at p. We consider the following two cases.

Case 1. $p \in \tilde{\Sigma}$.

In this case, thanks to the first touching property, it is easy to see that $\overline{\Omega}$ and $\overline{B_{r_0}(q)}$ are mutually tangent at p; in other words, $\nu_B(p) = -\nu(p)$ and $r_0 \leq \tau(p)$. Therefore, for any such q, it can be written as $\Phi(p, r_0)$ with $r_0 \leq \min[R, \tau(p)]$.

Case 2. $p \in \partial\Sigma$.

First, thanks to the first touching property, we have

$$\langle \nu_B(p), \tau \rangle = 0, \quad \forall \tau \in T_p(\partial\Sigma).$$

Namely, $\nu_B(p)$ lies in the normal bundle $N_p(\partial\Sigma) := \{\nu \in T_pM : \langle \nu, w \rangle = 0 \text{ for } w \in T_p(\partial\Sigma)\}$. Moreover, since $\overline{B_{r_0}(q)}$ touches $\overline{\Omega}$ from outside at $p \in \partial\Sigma$, we clearly have

$$\left\langle -\nu_B, \bar{N} \right\rangle(p) \geq \left\langle \nu, \bar{N} \right\rangle(p) \geq 0. \tag{4.4}$$

On the other hand, by definition we have $q = \exp_p^g(r_0(-\nu_B(p)))$; that is, $-\nu_B(p)$ points outside Ω, and hence from the convexity of ∂M we get

$$\left\langle -\nu_B, \bar{N} \right\rangle(p) \leq 0.$$

Taking (2.4) into account, we find $-\nu_B(p)$ coincides with $\nu(p)$; that is to say, $\overline{\Omega}$ and $\overline{B_{r_0}(q)}$. We can then argue as **Case 1** to conclude again that q can be written as $\Phi(p, r_0)$ with $r_0 \le \min[R, \tau(p)]$. (1) is thus proved.

(2) is a direct consequence of the triangle inequality of distance function.

To prove (3), note that since Ω is bounded, we could find $r_1, r_2 > 0$, such that $B_{r_1}(p_0) \subset \Omega \subset B_{r_2}(p_0)$. From the definition of $d_g(\cdot, \overline{\Omega})$, we find

$$d_g(q, \overline{B_{r_2}(p_0)}) \le d_g(q, \overline{\Omega}) \le d_g(q, \overline{B_{r_1}(p_0)}), \quad \forall q \in M,$$

and it follows that

$$\{q \in M : d_g(q, \overline{B_{r_1}(p_0)}) \le R\} \subset \{q \in M : d_g(q, \overline{\Omega}) \le R\} \subset \{q \in M : d_g(q, \overline{B_{r_2}(p_0)}) \le R\}. \tag{4.5}$$

By (4.2) and Proposition 4.1, we obtain the asymptotic volume ratios of the sets above. (4.3) then follows easily from (4.5).

Proof of Theorem 1.4 We follow the strategy of the proof in . Recall the definition of the sets A, A^* in (4.1) and that $\Phi_{|A^*}$ is a diffeomorphism. The pull back of the metric then takes of form $\mathcal{A}(r,p)\, dr\, d\sigma(p)$. Using the Ricci nonnegative condition, it is standard to deduce that (see)

$$\mathcal{A}(r,p) \le \left(1 + \frac{H(p)}{n} r\right)^n.$$

we divide Σ into two parts $\Sigma_+ = \{x \in \Sigma : H(x) > 0\}$

Taking also Proposition 4.2 into account, we deduce for any $R > 0$

$$\begin{aligned}
&\mathrm{Vol}\left(\{q \in M : d_g(q, \Omega) \le R\}\right) \\
&\le |\Omega| + \int_{\Sigma_+} \int_0^{\min(R,\tau(p))} \mathcal{A}(r,p)\, dr\, d\sigma(p) + \int_{\Sigma \setminus \Sigma_+} \int_0^{\min(R,\tau(p))} \mathcal{A}(r,p)\, dr\, d\sigma(p) \\
&\le \int_{\Sigma_+} \int_0^{\min(R,\tau(p))} \left(1 + \frac{H(p)}{n} r\right)^n dr\, d\sigma(p) + O(R) \\
&\le \int_{\Sigma_+} \int_0^{R} \left(1 + \frac{H(p)}{n} r\right)^n dr\, d\sigma(p) + O(R) \\
&= \frac{R^{n+1}}{n+1} \int_{\Sigma_+} \left(\frac{H(p)}{n}\right)^n d\sigma(p) + O(R^n). \\
&\le \frac{R^{n+1}}{n+1} \int_{\Sigma} \left(\frac{|H(p)|}{n}\right)^n d\sigma(p) + O(R^n).
\end{aligned}$$

Dividing both sides by $\omega_{n+1} R^{n+1} = \frac{|\mathbb{S}^n|}{n+1} R^{n+1}$, then letting $R \to \infty$, we deduce from (4.3) that

$$\mathrm{AVR}(g) \le \frac{1}{|\mathbb{S}^n|} \int_{\Sigma_+} \left(\frac{H}{n}\right)^n d\sigma \le \frac{1}{|\mathbb{S}^n|} \int_{\Sigma} \left(\frac{|H|}{n}\right)^n d\sigma.$$

Chaim. If the equality in holds, then $H \geq 0$ on Σ, $\tau(x) = +\infty$ on Σ_+, and equality in holds on Σ_+. To prove the first assertion of the claim, we deduce from and the equality case of that

$$\frac{1}{n+1}\int_{\Sigma}|H|^n dA = \frac{1}{n+1}\int_{\Sigma_+}(H)^n\, dA,$$

which implies $H \geq 0$ on Σ.

To prove $\tau(x) = +\infty$ on Σ_+, we argue by contradiction and assume that there exists a point $x_0 \in \Sigma_+$ satisfying $\tau(x_0) < +\infty$. Since $\Sigma \in C^2$, we can find a neighborhood of x_0 in Σ_+, denoted by $U(x_0)$, such that $\tau(x) \leq 2\tau(x_0) < +\infty$ on $U(x_0)$. For any $R > 2\tau(x_0)$, we obtain from that

$$\begin{aligned}
&\mathrm{Vol}\left(\left\{y \in K : d\left(y, \overline{\Omega}\right) \leq R\right\}\right)\\
\leq&\int_{\Sigma_+}\int_0^{\min(R,\tau(x))}(1+H(x)t)^n\, dtdA + O(R)\\
\leq&\int_{\Sigma_+\setminus U(x_0)}\int_0^{R}(1+H(x)t)^n\, dtdA + \int_{U(x_0)}\int_0^{\tau(x)}(1+H(x)t)^n\, dtdA + O(R)\\
\leq&\int_{\Sigma_+\setminus U(x_0)}\int_0^{R}(1+H(x)t)^n\, dtdA + O(R)\\
=&\frac{R^{n+1}}{n+1}\int_{\Sigma_+\setminus U(x_0)}H^n dA + O(R^n).
\end{aligned}$$

Dividing both sides by R^{n+1} and letting $R \to +\infty$, it follows that

$$\mathrm{AVR}(g)|\mathbb{S}^n| \leq \frac{1}{n+1}\int_{\Sigma_+\setminus U(x_0)}H^n dA,$$

which is a contradiction to the assumption that equality holds. when equality holds,it follows that

$$\mathcal{A}(r,p) = (1+\frac{H(p)}{n})^n, \forall t > 0$$

by the same argue of [8],we get $\Sigma = \Sigma_+$ be an umbilical hypersurface,also

$$\zeta^* g = dt^2 + (\frac{r_o}{r})^2 g_\Sigma.$$

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非负Ricci曲率带边黎曼流形上的两类几何不等式

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摘要: 本文研究了非负Ricci的黎曼流形上不等式问题，利用了贾-王-夏-张的方法，获得了两类几何不等式的结果，推广了贾-王-夏-张的结果.

关键词: Heintze-Karcher不等式; Willmore不等式; 超曲面

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