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LOCAL SS-QUASINORMALITY OF SOME SUBGROUPS OF FINITE GROUPS

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Abstract: This paper investigates the influence of local SS-quasinormal maximal subgroups of Sylow subgroups on the structure of finite groups. We present several new criteria on p-nilpotency of finite groups by utilizing a small quantity of local SS-quasinormal maximal subgroups of Sylow p-subgroups. As applications, we obtain some sufficient conditions for a finite group to be in a saturated formation containing the class of supersolvable groups.

Keywords: finite group; maximal subgroup; p-nilpotent group; SS-quasinormal group

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1 Introduction

All groups considered in this paper are finite. Let G be a group, by $\pi(G)$, we denote the set of all prime divisors of $|G|$. Following [1], by $\mathcal{M}_d(P) = \{P_1, \ldots, P_d\}$, we denote a set of maximal subgroups of a p-group P such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$, where d and $\Phi(P)$ are the minimum number of generators and the Frattini subgroup of P, respectively. Other notations and terminology are mostly standard and can be found in [2, 3].

Let G be a group. Two subgroups H and K of G are said to permute if $HK = KH$, that is, HK is a subgroup of G. According to Kegel [4], a subgroup H of a group G is called an S-quasinormal (S-permutable or π-quasinormal) in G if H permutes with every Sylow subgroup of G. Later, Ballester-Bolinches and Pedraza-Aguilera [5] introduced the concept of S-quasinormally embedded subgroups. Let H be a subgroup of a group G, H is said to S-quasinormally embedded in G if for each prime p dividing $|G|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of S-quasinormal subgroup of G. In 2008, Li et al. [1] extended S-quasinormal subgroup to SS-quasinormal subgroup. A subgroup H of a group G is called an SS-quasinormal (Supplement-Sylow-quasinormal) subgroup of G if there exists $B \leq G$ such that $G = HB$ and H permutes with every Sylow subgroup of B. It is obvious that

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every S-quasinormal subgroup of G is an SS-quasinormal subgroup of G, but the converse is not true (see [1, 6]).

Some authors have been studied the structure of finite groups under assumptions that some subgroups are SS-quasinormal. For example, Li et al. in [1] gave a sufficient condition for a finite group G to be p-nilpotent with $d(P)$ SS-quasinormal maximal subgroups of P, where $P \in \mathrm{Syl}_p(G)$ and $d(P)$ is the minimum number of generators of P. Li et al. in [7] obtained some sufficient and necessary conditions for a finite group to be in a saturated formation with some SS-quasinormal subgroups of the generalized Fitting subgroup.

We remark that the SS-quasinormality mentioned above are global. As we all know, it is significant to characterize the global structure of a group by using some local properties of subgroups. From this perspective, the present paper investigates the influence of local SS-quasinormal maximal subgroups of Sylow subgroups on the structure of finite groups. In detail, we study the relationship between $d(P)$ local SS-quasinormal maximal subgroups of some Sylow p-subgroup P and the structure of a given group G. In this work, we derive certain sufficient conditions for p-nilpotency of a group. As applications, we establish some sufficient or necessary conditions for a group to be in a saturated formation containing the class of supersolvable groups.

2 Preliminaries

The following Lemmas are crucial to the proofs of main results in Section 3.

Lemma 2.1 [1, Lemma 2.1] Let H be a SS-quasinormal subgroup of a group G, $K \leq G$ and $N \trianglelefteq G$. Then

(1) If $H \leq K \leq G$, then H is SS-quasinormal in K.

(2) HN/N is SS-quasinormal in G/N.

(3) If $N \leq K$ and K/N is SS-quasinormal in G/N, then K is SS-quasinormal in G.

Lemma 2.2 Let G be a group, $H \leq G$, $L \leq G$ and $H \leq \Phi(L)$. If H is SS-quasinormal in G, then H is S-quasinormal in G.

Proof By the hypotheses, H is SS-quasinormal in G, there exists a supplement B of H to G satisfying the condition that H permutes with each Sylow subgroup of B. Hence $HX = XH$ for all Sylow subgroups X of B. Then $L = H(B \cap L)$. Since $H \leq \Phi(L)$, we have $L = B \cap L$ and $L \leq B$. It follows that $G = B$ and so H permutes with every Sylow subgroup of G. Hence H is S-quasinormal in G.

Lemma 2.3 [8, I, Satz 17.4] Let N be a normal abelian subgroup of a group G and let $N \leq M \leq G$ such that $(|N|, |G:M|) = 1$. If a complement subgroup of N in M exists, then N possesses a complement subgroup in G.

Lemma 2.4 [9, Lemma 2.2] If A is a subnormal subgroup of a group G and A is a π-group, then $A \leq O_\pi(G)$.

Lemma 2.5 [10, Lemma 2.5] Let G be a group, K an S-quasinormal subgroup of G and P a Sylow p-subgroup of K, where p is a prime. If either $P \leq O_p(G)$ or $K_G = 1$, then P is S-quasinormal in G.

Lemma 2.6 [11, Proposition B] If H is a nilpotent subgroup of a group G, then the following two statements are equivalent:

(1) H is S-quasinormal in G.

(2) The Sylow subgroups of H are S-quasinormal in G.

Lemma 2.7 [11, Lemma A] Let G be a group and P a S-quasinormal p-subgroup of G, where $p \in \pi(G)$. Then $O^p(G) \leq N_G(P)$.

Lemma 2.8 [12, Corollary] Let G be a group, $P \in \mathrm{Syl}_p(G)$ and N is a normal subgroup of G such that $P \cap N \leq \Phi(P)$, then N is p-nilpotent.

Lemma 2.9 [10, Lemma 2.8] Let G be a group and $p \in \pi(G)$ with $(|G|, p-1) = 1$. Then

(1) If N is normal in G of order p, then N lies in $Z(G)$.

(2) If G has cyclic Sylow p-subgroups, then G is p-nilpotent.

(3) If M is a subgroup of G with index p, then M is normal in G.

By Lemma 2.9 and with similar arguments as in the proof of [1, Theorem 1.1], we obtain Lemma 2.10.

Lemma 2.10 Let G be a group and $P \in \mathrm{Syl}_p(G)$, where $p \in \pi(G)$ with $(|G|, p-1) = 1$. If every member of some fixed $\mathcal{M}_d(P)$ is SS-quasinormal in G, then G is p-nilpotent.

By Lemma 2.10 and with similar arguments as in the proof of [13, Corollary 3.5], we obtain Lemma 2.11.

Lemma 2.11 Let H be a normal subgroup of a group G such that G/H is p-nilpotent, $P \in \mathrm{Syl}_p(H)$, where $p \in \pi(G)$ with $(|G|, p-1) = 1$. If every member of some fixed $\mathcal{M}_d(P)$ is SS-quasinormal in G, then G is p-nilpotent.

Lemma 2.12 [1, Theorem 1.3] Let G be a p-solvable group for a prime p, $P \in \mathrm{Syl}_p(G)$. If every member of some fixed $\mathcal{M}_d(P)$ is SS-quasinormal in G, then G is p-supersolvable.

Lemma 2.13 [7, Theorem 3.3] Let G be a group and let $\mathcal{F}$ be a saturated formation containing the class of supersolvable groups $\mathcal{U}$. Then $G \in \mathcal{F}$ if and only if there exists $H \trianglelefteq G$ such that $G/H \in \mathcal{F}$, and for every Sylow subgroup P of $F^*(H)$, all maximal subgroups of P are SS-quasinormal in G.

Lemma 2.14 [14, Theorem 3.1] Let H be a solvable normal subgroup of a group G such that $G/H \in \mathcal{F}$, where $\mathcal{F}$ is a saturated formation containing $\mathcal{U}$. If for every maximal subgroup M of G, either $F(H) \leq M$ or $F(H) \cap M$ is maximal in $F(H)$, then $G \in \mathcal{F}$. The converse also holds, in the case where $\mathcal{F} = \mathcal{U}$.

Recall that the generalized Fitting subgroup $F^*(G)$ is the unique maximal normal quasinilpotent subgroup of G (see [15]). For a prime $p \in \pi(G)$, *the generalized p-Fitting subgroup* $F_p^*(G)$ is defined to be as the normal subgroup of G such that $F_p^*(G)/O_{p'}(G) = F^*(G/O_{p'}(G))$. In the following, we would like to give some basic properties of $F^*(G)$ and $F_p^*(G)$.

Lemma 2.15 [14, Theorem 3.1] Let G be a group and N a subgroup of G. Then

(1) If N is normal in G, then $F^*(N) \leq F^*(G)$.

(2) $F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = Soc(F(G)C_G(F(G))/F(G))$.

(3) $F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$.
(4) $C_G(F^*(G)) \leq F(G)$.
(5) If $N \leq Z(G)$, then $F^*(G/N) = F^*(G)/N$.
(6) If $N \leq O_p(G)$ for some $p \in \pi(G)$, then $F^*(G/\Phi(N)) = F^*(G)/\Phi(N)$.

Lemma 2.16 [15, Lemma 2.3] Let G be a group. Then
(1) $Soc(G) \leq F_p^*(G)$.
(2) $O_{p'}(G) \leq F_p^*(G)$. In fact, $F^*(G/O_{p'}(G)) = F_p^*(G/O_{p'}(G)) = F_p^*(G)/O_{p'}(G)$.
(3) If $F_p^*(G)$ is p-solvable, then $F_p^*(G) = F_p(G)$.
(4) If $C = C_G(F_p(G)/O_{p'}(G))$, then $F_p^*(G)/F_p(G) = Soc(CF_p(G)/F_p(G))$.

3 Main Results

Theorem 3.1 Let H be a normal subgroup of a group G such that G/H is p-nilpotent and $P \in \mathrm{Syl}_p(H)$, where $p \in \pi(G)$ with $(|G|, p-1) = 1$. If every member of some fixed $\mathcal{M}_d(P)$ is SS-quasinormal in $N_G(P)$ and Ψ is SS-quasinormal in G for some $P' \leq \Psi \leq \Phi(N_H(P))$, then G is p-nilpotent.

Proof Suppose otherwise and let G be a minimal counterexample. Write $\mathcal{M}_d(P) = \{P_1, \ldots, P_d\}$, where P_i is maximal in P with $\bigcap\limits_{i=1}^{d} P_i = \Phi(P)$. Then

(1) $N_H(P) < H$.

If not, then $P \trianglelefteq G$. We have $N_G(P) = G$. It follows from Lemma 2.11 that G is p-nilpotent.

(2) $O_{p'}(G) = 1$.

Write $N = N_G(P)$ and $T = O_{p'}(G)$. If $T \neq 1$, we consider the quotient group $\overline{G} = G/T$. Clearly, $\overline{H} = HT/T \trianglelefteq \overline{G}$ and $\overline{G}/\overline{H} \cong G/HT$, hence $\overline{G}/\overline{H}$ is p-nilpotent. It is obvious that $P \cong \overline{P} = PT/T \in \mathrm{Syl}_p(\overline{H})$, so PT/T has the same smallest generator number as P, i.e., d, and $\mathcal{M}_d(PT/T) = \{P_1T/T, \ldots, P_dT/T\}$ with $\bigcap\limits_{i=1}^{d} \overline{P_i} = \Phi(\overline{P})$. Meanwhile,

$$N_{\overline{G}}(\overline{P}) = N_{G/T}(PT/T) = N_G(P)T/T = NT/T = \overline{N}.$$

By the hypotheses, P_i is SS-quasinormal in N, there exists a supplement B_i of P_i to N such that P_i permutes with each Sylow subgroup of B_i. Hence $P_iX = XP_i$ for all Sylow subgroups X of B_i. Moreover,

$$NT/T = (P_iB_i)T/T = (P_iT/T)(B_iT/T).$$

For any $q \in \pi(G)$, by [8, VI. 4.7], there exist respectively Sylow q-subgroup $(B_i)_q$ of B_i and T_q of T such that $Y = (B_i)_qT_q$ is a Sylow q-subgroup of B_iT. We may assume that X is a Sylow q-subgroup of B_i. Since both XT/T and YT/T are Sylow q-subgroups of B_iT/T, by Sylow's theorem, we have

$$XT/T = (YT/T)^{bT} = ((B_i)_qT/T)^{bT} = (B_i)_q^bT/T$$

for some $b \in B_i$. Under the assumptions of the theorem, P_i permutes with every Sylow q-subgroup of B_i, $P_i(B_i)_q^b = (B_i)_q^b P_i$, $b \in B_i$. We obtain

$$P_iT/T \cdot XT/T = P_i(XT)/T = P_i((B_i)_q^b T)/T = (B_i)_q^b P_iT/T = XT/T \cdot P_iT/T,$$

which shows that P_iT/T is SS-quasinormal in NT/T. On the other hand, since Ψ is SS-quasinormal in G, by Lemma 2.1(2), $\Psi T/T$ is SS-quasinormal in G/T. Meanwhile,

$$(PT/T)^{'} \leq \Psi T/T \leq \Phi(N_H(P))T/T \leq \Phi(N_H(P)T/T) \leq \Phi(N_{HT/T}(PT/T)).$$

Hence G/T satisfies the conditions of the theorem. By the minimality of G, G/T is p-nilpotent, and therefore G is p-nilpotent, a contradiction.

(3) $H = G$.

If not, $H < G$. Since H satisfies the hypotheses of the theorem, H is p-nilpotent. Suppose that K is a normal p-complement of H. Then $K \trianglelefteq G$, hence $K = 1$ by (2) and $H = P \trianglelefteq G$, which is contrary to (1).

(4) $\Phi(P)_G = 1$.

Write $N = \Phi(P)_G$. If $N \neq 1$, we consider the quotient group $\overline{G} = G/N$. Clearly, $N_{G/N}(P/N) = N_G(P)/N$ and $\mathcal{M}_d(P/N) = \{P_1/N, \dots, P_d/N\}$ with $\bigcap\limits_{i=1}^{d} \overline{P_i} = \Phi(\overline{P})$. Since P_i is SS-quasinormal in $N_G(P)$ and Ψ is SS-quasinormal in G, in view of Lemma 2.1(2), P_i/N is SS-quasinormal in $N_{G/N}(P/N)$ and $\Psi N/N$ is SS-quasinormal in G/N. Meanwhile,

$$(P/N)' = P'N/N \leq \Psi N/N \leq \Phi(N_G(P))N/N \leq \Phi(N_G(P)/N) = \Phi(N_{G/N}(P/N)).$$

By the minimality of G, G/N is p-nilpotent. Since $N \leq \Phi(P)$, $N \leq \Phi(G)$ by [8, III, 3.3], it follows that G is p-nilpotent, which is impossible.

(5) $N_G(P)$ is p-nilpotent and $\Psi \cap \Phi(P) \neq 1$.

By Lemma 2.11, $N_G(P)$ satisfies the conditions of the theorem. By the choice of G, $N_G(P)$ is p-nilpotent. Let K be the normal p-complement of $N_G(P)$, then $N_G(P) = P \times K$. Write $N = N_G(P)$, we have $\Phi(N) = \Phi(P) \times \Phi(K)$. From this equation, we have $\Psi = (\Psi \cap \Phi(P)) \times (\Psi \cap \Phi(K))$. If $\Psi \cap \Phi(P) = 1$, then $P^{'} = 1$ as $P^{'} \leq \Psi \cap \Phi(P)$, which implies that P is an abelian group so that $N_G(P) = C_G(P)$ and G is p-nilpotent by Burnside's Theorem, a contradiction.

(6) The final contradiction.

By Lemma 2.2, we see that Ψ is S-quasinormal in G. From the proof of (5), we obtain $\Psi_G = (\Psi_G \cap \Phi(P)) \times (\Psi_G \cap \Phi(K))$, hence both $\Psi_G \cap \Phi(P)$ and $\Psi_G \cap \Phi(K)$ are normal in G, it follows from (2) and (4) that $\Psi_G = 1$. By Lemma 2.5 and Lemma 2.7, we have $O^p(G) \leq N_G(\Psi \cap \Phi(P))$. However, $P^{'} \leq \Psi \cap \Phi(P)$, hence $P \leq N_G(\Psi \cap \Phi(P))$, i.e., $\Psi \cap \Phi(P)$ is normal in G. But $\Psi \cap \Phi(P) \neq 1$ by (5) and this is contrary to (4). The proof is complete.

Theorem 3.2 Let H be a subnormal subgroup of a group G containing $F_p^*(G)$ and let $P \in \mathrm{Syl}_p(H)$, where $p \in \pi(G)$ with $(|G|, p-1) = 1$. If every member of some fixed $\mathcal{M}_d(P)$

is SS-quasinormal in $N_G(P)$ and Ψ is SS-quasinormal in G for some $P' \le \Psi \le \Phi(N_H(P))$, then G is p-nilpotent.

Proof Let G be a minimal counterexample. Then

(1) $O_{p'}(G) = 1$.

If not, then $T = O_{p'}(G) > 1$. We consider the quotient group $\overline{G} = G/T$. In fact,

$$F_p^*(\overline{G}) = F^*(\overline{G}) = F_p^*(G)/T \le H/T = \overline{H}.$$

Clearly, $\overline{H} \lhd \lhd \overline{G}$ and $\overline{P} = PT/T \in \mathrm{Syl}_p(\overline{H})$. Noticing that $P \cong \overline{P}$, hence $\overline{P}$ has the same smallest generator number, i.e., d, and $\mathcal{M}_d(\overline{P}) = \mathcal{M}_d(PT/T) = \{P_1T/T, \ldots, P_dT/T\}$ with $\bigcap\limits_{i=1}^{d} \overline{P_i} = \Phi(\overline{P})$. Furthermore,

$$N_{\overline{G}}(\overline{P}) = N_{G/T}(PT/T) = N_G(P)T/T.$$

Since P_i is SS-quasinormal in $N_G(P)$, with similar arguments as in the proof of Theorem 3.1, we know that P_iT/T is SS-quasinormal in $N_G(P)T/T$. Moreover, $\Psi T/T$ is SS-quasinormal in G/T by Lemma 2.1(2). At the same time,

$$(PT/T)' \le \Psi T/T \le \Phi(N_H(P))T/T \le \Phi(N_H(P)T/T) \le \Phi(N_{HT/T}(PT/T)).$$

Thereby G/T satisfies the conditions of the theorem. By the minimality of G, G/T is p-nilpotent, hence G is p-nilpotent, a contradiction.

(2) $H = P = O_p(G) = F^*(G)$.

By Theorem 3.1 and Lemma 2.1(1), we have that H is p-nilpotent. Hence H has a normal p-complement subgroup $H_{p'}$. Clearly, $H_{p'} \lhd \lhd G$. By Lemma 2.4, we obtain $H_{p'} \le O_{p'}(G) = 1$. Hence $H_{p'} = 1$ and $H = P$. Consequently, $H = P \le O_p(G)$. Moreover, $O_p(G) \le F_p(G) = F_p^*(G) = F^*(G) \le H$, so $H = P = O_p(G) = F^*(G)$.

(3) The final contradiction.

Let Q be a Sylow q-subgroup of G with $q \ne p$. We consider $H_1 = PQ$. By Lemma 2.1(1), every member of $\mathcal{M}_d(P)$ is SS-quasinormal in H_1 and Ψ is SS-quasinormal in H_1. Hence H_1 is p-nilpotent by Theorem 3.1. From Lemma 2.15(4), we have

$$Q \le C_G(P) = C_G(F^*(G)) \le F(G) = P.$$

This is the final contradiction and the proof is complete.

When p is not a special prime, we may omit $d(P)$ SS-quasinormal maximal subgroups of P but need to employ the p-nilpotency of $N_G(P)$.

Theorem 3.3 Let H be a normal group of a group G such that G/H is p-nilpotent and $P \in \mathrm{Syl}_p(H)$, where $p \in \pi(G)$. Suppose that $N_G(P)$ is p-nilpotent and Ψ is SS-quasinormal in G for some $P' \le \Psi \le \Phi(N_H(P))$, then G is p-nilpotent.

Proof Suppose otherwise and let G be a minimal counterexample.

(1) $O_{p'}(G) = 1$.

Denote $T = O_{p'}(G)$. If $T \neq 1$, we consider the quotient group $\overline{G} = G/T$. Clearly, $\overline{P} = PT/T \in \mathrm{Syl}_p(\overline{H})$, $\overline{H} = HT/T \trianglelefteq \overline{G}$ and $\overline{G}/\overline{H} \cong G/HT$, hence $\overline{G}/\overline{H}$ is p-nilpotent. Noticing that $N_{\overline{G}}(\overline{P}) = N_G(P)T/T$, hence $N_{\overline{G}}(\overline{P})$ is p-nilpotent. On the other hand, by the hypotheses and Lemma 2.1(2), $\Psi T/T$ is SS-quasinormal in G/T. Meanwhile,

$$(PT/T)^{'} \leq \Psi T/T \leq \Phi(N_H(P))T/T \leq \Phi(N_H(P)T/T) \leq \Phi(N_{HT/T}(PT/T)).$$

By the minimality of G, G/T is p-nilpotent, hence G is p-nilpotent, a contradiction.

(2) $\Phi(P)_G = 1$.

Write $N = \Phi(P)_G$. If $N \neq 1$, we consider the quotient group $\overline{G} = G/N$. Clearly, $N_{H/N}(P/N) = N_H(P)/N$. By the hypotheses and Lemma 2.1(2), $\Psi N/N$ is SS-quasinormal in G/N. Moreover,

$$(P/N)' = P'N/N \leq \Psi N/N \leq \Phi(N_H(P))N/N \leq \Phi(N_H(P)/N) = \Phi(N_{H/N}(P/N)).$$

By the choice of G, G/N is p-nilpotent. Since $N \leq \Phi(P)$, $N \leq \Phi(G)$ by [8, III, 3.3], it follows that G is p-nilpotent, which is impossible.

(3) $H = G$.

If not, $H < G$. Since H satisfies the conditions of the theorem, H is p-nilpotent. Suppose that K is a normal p-complement of H. Then $K \trianglelefteq G$, hence $K = 1$ by (1) and $H = P \trianglelefteq G$ so that $G = N_G(P)$ is p-nilpotent, a contradiction.

(4) The final contradiction.

By the hypotheses, $N_H(P)$ is p-nilpotent. Let K be the normal p-complement of $N_H(P)$, then $N_H(P) = P \times K$. Furthermore, $\Phi(N_H(P)) = \Phi(P) \times \Phi(K)$. From this equation, we obtain $\Psi = (\Psi \cap \Phi(P)) \times (\Psi \cap \Phi(K))$. Since $\Psi \leq \Phi(N_H(P))$ and Ψ is SS-quasinormal in G, by Lemma 2.2, we see that Ψ is S-quasinormal in G. Clearly, $\Psi_G = (\Psi_G \cap \Phi(P)) \times (\Psi_G \cap \Phi(K))$ and $\Psi_G \cap \Phi(P) \trianglelefteq G, \Psi_G \cap \Phi(K) \trianglelefteq G$, hence $\Psi_G = 1$ by (1) and (2). By Lemma 2.5 and Lemma 2.7, $\Psi \cap \Phi(P)$ is S-quasinormal in G, thereby $O^p(G) \leq N_G(\Psi \cap \Phi(P))$. However, $P^{'} \leq \Psi \cap \Phi(P)$, hence $P \leq N_G(\Psi \cap \Phi(P))$, i.e., $\Psi \cap \Phi(P) \trianglelefteq G$. If $\Psi \cap \Phi(P) = 1$, then $P^{'} = 1$ and P is an abelian group, we have $N_G(P) = C_G(P)$. By Burnside's Theorem, we know that G is p-nilpotent, a contradition. So $\Psi \cap \Phi(P) \neq 1$. This is contrary to (2) and the proof is complete.

In the following, we give some applications of Theorem 3.1 and Theorem 3.2.

Theorem 3.4 Let G be a group. If for every $p \in \pi(G)$, there exists a subnormal subgroup H of G containing $F_p^*(G)$ and some $P \in \mathrm{Syl}_p(H)$ such that every member of some fixed $\mathcal{M}_d(P)$ is SS-quasinormal in $N_G(P)$ and Ψ is SS-quasinormal in G for some $P' \leq \Psi \leq \Phi(N_H(P) \cap P^H)$, then G is supersolvable.

Proof Suppose otherwise, let G be a minimal counterexample. Then

(1) G has a Sylow tower of supersolvable type.

Let $q = min\ \pi(G)$. By Theorem 3.2, G has a normal q-complement subgroup K. Obviously, G is a solvable group by Odd Order Theorem. Now let $p = min\ \pi(K)$. By Lemma 2.16, $F_p^*(K) = F_p(K)$. Clearly, $F_p(K) \leq F_p(G)$. By the hypotheses, there exists

$H \lhd \lhd G$ such that $F_p(G) = F_p^*(G) \leq H$ and $P \in \mathrm{Syl}_p(H)$ such that every member of some fixed $\mathcal{M}_d(P)$ is SS-quasinormal in $N_G(P)$. Furthermore,

$$F_p(K) \leq F_p(G) \cap K \leq H \cap K \lhd \lhd K.$$

Clearly, $P \leq K$. Since $P \leq H \cap K \trianglelefteq H$, we have $P^H \leq H \cap K$. Therefore,

$$\Phi(N_H(P) \cap P^H) = \Phi(N_H(P) \cap P^H \cap H \cap K) = \Phi(N_{H\cap K}(P) \cap P^H) \leq \Phi(N_{H\cap K}(P)).$$

By Lemma 2.1(1), every member of some fixed $\mathcal{M}_d(P)$ is SS-quasinormal in $N_K(P)$ and Ψ is SS-quasinormal in K for some $P' \leq \Psi \leq \Phi(N_{H\cap K}(P))$. Thus K is p-nilpotent by Theorem 3.2. In this way, we see that G has a Sylow tower of supersolvable type.

(2) Let $p = max\ \pi(G)$ and $P \in Syl_p(G)$. Then G/P is supersolvable.

By (1), $P \trianglelefteq G$. Clearly, $P \leq F_p(G) = F_p^*(G)$. For every $p \neq q \in \pi(G)$, we have

$$F_q^*(G/P) = F_q(G/P) = F_q(G)/P = F_q^*(G)/P.$$

Consider the quotient group G/P. By the hypotheses, there exist $H \lhd \lhd G$ such that $F_q^*(G) \leq H$ and $Q \in \mathrm{Syl}_q(H)$ such that every member of some fixed $\mathcal{M}_d(Q)$ is SS-quasinormal in $N_G(Q)$. Let $\mathcal{M}_d(Q) = \{Q_1, \ldots, Q_d\}$, where Q_i is maximal in Q with $\bigcap\limits_{i=1}^{d} Q_i = \Phi(Q)$, $d = d(Q)$. Clearly, $QP/P \cong Q$, so QP/P has the same smallest generator number as Q, i.e., d, and $\mathcal{M}_d(QP/P) = \{Q_1P/P, \ldots, Q_dP/P\}$ with $\bigcap\limits_{i=1}^{d} \overline{Q_i} = \Phi(\overline{Q})$. Also, $N_{G/P}(QP/P) = N_G(Q)P/P$. Since every member of $\mathcal{M}_d(Q)$ is SS-quasinormal in $N_G(Q)$, with similar arguments as in the proof of Theorem 3.1, we obtain every member of $\mathcal{M}_d(QP/P)$ is SS-quasinormal in $N_{G/P}(QP/P)$. By Lemma 2.1(2), $\Psi P/P$ is SS-quasinormal in G/P. Moreover,

$$(QP/P)^{'} \leq \Psi P/P \leq \Phi(N_H(Q) \cap Q^H)P/P \leq \Phi(N_{H/P}(QP/P) \cap (QP/P)^{H/P}).$$

Hence G/P satisfies the hypotheses of the theorem and G/P is supersolvable.

(3) $P \in Syl_p(H)$.

Since $P \trianglelefteq G$, we have $P \leq F_p(G) = F_p^*(G) \leq H$. Hence $P \in Syl_p(H)$.

(4) $\Phi(P) = 1$.

Write $T = \Phi(P)$. If $T \neq 1$, we consider the quotient group $\overline{G} = G/T$. Let $q \in \pi(G)$. Since

$$F_q^*(G/T) = F_q(G/T) = F_q(G)/T = F_q^*(G)/T$$

and

$$(P/T)^{'} \leq \Psi T/T \leq \Phi(N_H(P) \cap P^H)/T \leq \Phi(N_{H/T}(P/T) \cap (P/T)^{H/T}),$$

we see easily that $G/\Phi(P)$ satisfies the conditions of the theorem. By the minimality of G, $G/\Phi(P)$ is supersolvable. Since $\Phi(P) \leq \Phi(G)$ by [8, III, 3.3], it follows that G is supersolvable, a contradiction.

(5) Conclusion of the proof.

First of all, we will show that every minimal subgroup N of G is of order p, where $p = max\ \pi(G)$. In fact, since G is solvable by (1), N is an elementary abelian q-group for some $q \in \pi(G)$. Suppose $q \neq p$. Clearly, $(G/N)/(PN/N) \cong G/PN$ is supersolvable by (2). Since every member of some fixed $\mathcal{M}_d(P)$ is SS-quasinormal in $N_G(P) = G$ and Ψ is SS-quasinormal in G for some $P' \leq \Psi \leq \Phi(P)$, every member of $\mathcal{M}_d(PN/N)$ is SS-quasinormal in G/N and $\Psi N/N$ is SS-quasinormal in G/N, where $(PN/N)' = P'N/N \leq \Psi N/N \leq \Phi(P)N/N \leq \Phi(PN/N)$. Hence G/N is supersolvable by the minimality of G. It follows from (2) that G is supersolvable, a contradiction. Hence $q = p$ and $N \leq P$. Since $P_i \leq P \trianglelefteq G$, we have $P_i \leq O_p(G)$, hence P_1 is S-quasinormal in G by [7, Lemma 2.2]. By Lemma 2.7, we have $O^p(G) \leq N_G(P_i)$. However, P_i is normalized by P. We have $G = PO^p(G) \leq N_G(P_i)$, that is, $P_i \trianglelefteq G$. Hence $N \cap P_i \trianglelefteq G$. The minimality of N implies that $N \cap P_i = 1$ or N. If $N \cap P_i = N$ for each i, then $N \leq P_i$. It follows that $N \leq \bigcap\limits_{i=1}^{d} P_i = \Phi(P) = 1$ and consequently $N = 1$, a contradiction. Hence $N \cap P_i = 1$ for some i. As $NP_i = P$, we obtain $|N| = p$. Now, it is easy to see that N is complemented in P, hence it is complemented in G by Lemma 2.3. Consequently, $P \cap \Phi(G) = 1$. By [10, Lemma 2.9], we obtain $P = N_1 \times N_2 \times \cdots \times N_s$, where N_i $(i = 1, \ldots, s)$ is minimal normal in G of order p. It follows from G/P is supersolvable that G is supersolvable. This is the final contradiction and the proof is complete.

In the following, we give some sufficient and necessary conditions for a finite group to be in a saturated formation.

Theorem 3.5 Let G be a group and let $\mathcal{F}$ be a saturated formation containing the class of supersolvable groups $\mathcal{U}$. Then the following two statements are equivalent:

(1) $G \in \mathcal{F}$.

(2) There exists $H \trianglelefteq G$ such that $G/H \in \mathcal{F}$, and for every Sylow subgroup P of F^*, all maximal subgroups of P are SS-quasinormal in $N_G(P)$ and Ψ is SS-quasinormal in G for some $P' \leq \Psi \leq \Phi(N_{F^*}(P))$, where $F^* = F^*(H)$.

Proof Clearly, (1) implies (2). We only need to prove that (2) implies (1). Assume that the theorem is false and let G be a minimal counterexample. Then

(1) $F^* = F(H)$.

Let $q = min\ \pi(F^*)$. It is clear that F^* is q-nilpotent by Theorem 3.1, hence F^* is solvable and $F^* = F(H)$ by Lemma 2.15(3).

(2) $F(H)$ is elementary abelian.

Let $P \in \mathrm{Syl}_p(F(H))$, where $p \in \pi(F(H))$. Clearly, $\Phi(P) \trianglelefteq G$ and $(P/\Phi(P))' = 1$. By Lemma 2.15(6), $F^*(H/\Phi(P)) = F^*/\Phi(P)$, so all maximal subgroups of Sylow subgroups of $F^*(H/\Phi(P))$ are SS-quasinormal in $G/\Phi(P)$ by Lemma 2.1(2). If $\Phi(P) \neq 1$, then $G/\Phi(P) \in \mathcal{F}$ by the choice of G. However, $\Phi(P) \leq \Phi(G)$, thereby $G \in \mathcal{F}$, a contradiction. So $\Phi(P) = 1$ and P is elementary abelian. This proves (2).

(3) H is solvable.

By Lemma 2.1(1) and Lemma 2.13, H is supersolvable, of course, H is solvable.

(4) Finish of the proof.

Let M be a maximal subgroup of G such that $F(H) \nsubseteq M$. Then there exists $P \in \mathrm{Syl}_p(F(H))$ such that $G = PM$. Set $M_p \in \mathrm{Syl}_p(M)$. Clearly, $G_p = PM_p \in \mathrm{Syl}_p(G)$. Take a maximal subgroup G_1 of G_p containing M_p and let $P_1 = P \cap G_1$. Then $G_1 = P_1 M_p$ and

$$P_1 \cap M = (P \cap M) \cap G_1 = P \cap M \trianglelefteq G,$$

hence $P \cap M \leq (P_1)_G$ and

$$|P : P_1| = |PM_p : P_1 M_p| = |G_p : G_1| = p,$$

that is, P_1 is maximal in P. Futhermore, $(P_1)_G M < G$ implies that $(P_1)_G \leq M$, thus $P \cap M = (P_1)_G$. Clearly, $P \leq O_p(G)$, hence $P_1 \leq O_p(G)$. By the hypotheses of the theorem and [7, Lemma 2.2], P_1 is S-quasinormal in G. Noticing that $P_1 = P \cap G_1$, thereby P_1 is normalized by G_p. It follows from $O^p(G) \leq N_G(P_1)$ that $P_1 \trianglelefteq G$. Consequently, $P_1 = (P_1)_G \leq M$ and therefore $|F(H) : F(H) \cap M| = |G : M| = p$. By Lemma 2.14, $G \in \mathcal{F}$, a final contradiction. This completes our proof.

Theorem 3.6 Let G be a group and let $\mathcal{F}$ be a saturated formation containing the class of supersolvable groups $\mathcal{U}$. Then the following two statements are equivalent:

(1) $G \in \mathcal{F}$.

(2) There exists $H \trianglelefteq G$ such that $G/H \in \mathcal{F}$, and for every Sylow subgroup P of H, all maximal subgroups of P are SS-quasinormal in $N_G(P)$ and Ψ is SS-quasinormal in G for some $P' \leq \Psi \leq \Phi(N_H(P))$.

Proof Clearly, (1) implies (2). We only need to prove that (2) implies (1). We distinguish two cases:

Case 1 $H = G$.

In this case, we claim that G is supersolvable. Assume G is not supersolvable with minimal order. It is clear that G is q-nilpotent by Theorem 3.1, where $q = min\ \pi(G)$. Of course, G is solvable. Let N be a minimal normal subgroup of G. Clealy, N is an elementary p-group for some prime p. With similar arguments as in the proof of Theorem 3.1, we observe that G/N satisfies the conditions of the theorem. By the minimality of G, G/N is supersolvable. Now we may assume that N is the unique minimal normal subgroup of G and $\Phi(G) = 1$. Moreover, $N = F(G)$ and $C_G(N) = N$. Let $R \in \mathrm{Syl}_r(G)$, where $r = max\ \pi(G)$. Since G/N is supersolvable, $RN/N \trianglelefteq G/N$. If $p = r$, then we obtain $N \leq R$ and $R \trianglelefteq G$. Hence G/R is supersolvable. Noticing that $\Phi(R) \trianglelefteq G$ and $R^{'} \leq \Phi(R) \leq \Phi(G)$, by Theorem 3.5, G is supersolvable, a contradiction. Hence we may assume $p \neq r$. Let M/N be a minimal normal subgroup of G/N contained in RN/N. Then M/N is of order r, hence M has the form of $R_0 N$, where $R_0 \leq R$ and $|R_0| = r$. Thus $MP = R_0 P \leq G$, where $P \in \mathrm{Syl}_p(G)$.

Suppose $P^{'} = 1$. Then we see that $R_0 P$ satisfies the hypotheses of the theorem. If $R_0 P < G$, then $R_0 P$ is supersovable and consequently, $R_0 \trianglelefteq R_0 P$. This yields that $R_0 \leq$

$C_G(N) = N$, which is impossible. Hence $R_0P = G$ and $p = q$. It follows from G is q-nilpotent that $R_0 \trianglelefteq G$. We get a contradiction.

Now suppose $P^{'} \neq 1$. By the hypotheses, there exists $P' \leq \Psi \leq \Phi(N_G(P))$ such that Ψ is SS-quasinormal in G. Obviously, $\Psi \cap P \in \mathrm{Syl}_p(\Psi)$. By Lemma 2.2 and Lemma 2.6, $\Psi \cap P$ is S-quasinormal in G. It follows from Lemma 2.7 that $O^p(G) \leq N_G(\Psi \cap P)$. However, $P^{'} \leq \Psi \cap P$, hence $P \leq N_G(\Psi \cap P)$, i.e., $\Psi \cap P \trianglelefteq G$. We have $N \leq \Psi \cap P$. Furthermore, $N \leq \Psi \cap P \leq \Phi(N_G(P))$ and thereby $N \leq \Phi(G)$ [8, III, 3.3], contradicting $\Phi(G) = 1$. This shows that G must be supersolvable.

Case 2 $H \neq G$.

By **Case 1**, H is supersolvable. Let $P \in \mathrm{Syl}_p(H)$, where $p = max\ \pi(H)$. Then $P \trianglelefteq H$ so that $P \trianglelefteq G$. Since G/P satisfies the hypotheses of the theorem, $G/P \in \mathcal{F}$ by the choice of G. Thus $G \in \mathcal{F}$ by Theorem 3.5. This is the final contradiction and the proof is complete.

Remark (1) The condition that Ψ is SS-quasinormal in G in theorems cannot be removed. For example, let $G = PSL(2, 17)$ be the projective special group of degree 17 over a field of order 2 and $P \in \mathrm{Syl}_2(G)$. Then every maximal subgroup of P is normal in $N_G(P) = P$, however, G is a nonabelian simple group.

(2) The condition that all maximal subgroups of P in Theorem 3.6 cannot be replaced by d maximal subgroups in some fixed $\mathcal{M}_d(P)$ in general. The example can be seen in [1, Proof of Example 1.6].

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有限群某些子群的局部 SS- 拟正规性

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摘 要: 本文研究了局部 SS- 拟正规极大子群对有限群结构的影响. 利用 Sylow p- 子群的少量局部 SS- 拟正规极大子群, 给出了有限群 p- 幂零性的若干新判据. 作为应用，我们得到了有限群属于包含超可解群类的饱和群系的若干充分条件.

关键词: 有限群; 极大子群; p- 幂零群; SS- 拟正规子群

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