Vol. 46 (2026)
No. 1

数 学 杂 志
J. of Math. (PRC)

λ-BIHARMONIC HYPERSURFACES IN 6-DIMENSIONAL PSEUDO-RIEMANNIAN SPACE FORMS

DUAN Zhen-ping, YANG Chao, LIU Jian-cheng, CHEN Jia-rui
(*College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China*)

Abstract: In this paper, we study λ-biharmonic hypersurfaces M_r^5 of 6-dimensional pseudo-Riemannian space form $N_p^6(c)$ with the indexs $0 \leq p \leq 6$, $r = p-1$ or p, and constant curvature c. It was proved that if the shape operator of M_r^5 is diagonalizable, then the mean curvature is a constant. As an application, we find some types of biharmonic hypersurfaces of $N_p^6(c)$ are minimal.

Keywords: λ-biharmonic hypersurface; pseudo-Riemannian space form; constant mean curvature; shape operator; minimal

2010 MR Subject Classification: 53C50; 53C40

Document code: A **Article ID:** 0255-7797(2026)01-019-19

1 Introduction

Let $N_p^{n+1}(c)$ be the $(n+1)$-dimensional pseudo-Riemannian space form of constant sectional curvature c with index s $(0 \leq s \leq n+1)$, and $\varphi : M_r^n \to N_p^{n+1}(c)$ be an isometric immersion from a pseudo-Riemannian mainfold M_r^n with index r into $N_p^{n+1}(c)$. The hypersurface M_r^n is called λ-biharmonic if the immersion φ is a critical point of the following functional (cf. [1–3]),

$$E_{2,\lambda}(\varphi) = E_2(\varphi) + \lambda E(\varphi), \qquad \lambda \in R,$$

where $E(\varphi)$ and $E_2(\varphi)$ are the energy and bienergy functionals. The Euler-Lagrange equation of $E_{2,\lambda}(\varphi)$ gives the λ-biharmonic equation (cf. [1])

$$\tau_2(\varphi) = \lambda\tau(\varphi),$$

where $\tau(\varphi) := \mathrm{trace}(\nabla \mathrm{d}\varphi)$ and $\tau_2(\varphi) := -\Delta\tau(\varphi) - \mathrm{trace}\tilde{R}(\mathrm{d}\varphi, \tau(\varphi))\mathrm{d}\varphi$ are the tension and bitension fields of φ, and $\tilde{R}$ is the curvature tensor of $N_p^{n+1}(c)$. Specially, when $\lambda = 0$, the hypersurface M_r^n is called biharmonic hypersurface (cf. [4, 5]).

---

* **Received date:** 2025-07-03 **Accepted date:** 2025-10-16

Foundation item: Supported by National Natural Science Foundation of China (12161078), Foundation for Innovative Fundamental Research Group Project of Gansu Province (24JRRA778) and Project of Northwest Normal University (20240010).

Biography: Yang Chao (1989–), female, born at Jinchang, Gansu, associated professor, major in differential geometry, E-mail: yangch@nwnu.edu.cn.

In 1988, Chen Bang-yen [6] initiated the study of λ-biharmonic hypersurface M_r^n of $\mathbb{E}_s^{n+1}$, and proved that the surface M^2 in $\mathbb{E}^3$ (i.e. $\mathbb{E}_0^3$) is minimal, or an open part of a circular cylinder. And then, Ferrández A and Lucus P [7] classified such surfaces with $s=1$. For $n=3$, it has been proved that the hypersurface M_r^3 has constant mean curvature (cf. [8] with $s=0$, [9] with $s=1$, and [10] with $s=2$). Based on these results, Arvanitoyeorgos A and Kaimakamis G [10] conjectured that any λ-biharmonic hypersurface in $\mathbb{E}_s^{n+1}$ has constant mean curvature. For $n=4$, Fu Yu and Zhan Xin [11] gave an affirmative answer to this conjecture with $s=0$. Afterwards, Yang Chao, Liu Jiancheng and Du Li [12] showed this conjecture is also true for $n=4$ and $s>0$ and extended this result to hypersurfaces of non-flat pseudo-Riemannian space forms.

In this paper, we investigate the 5-dimensional λ-biharmonic hypersurface M_r^5 of $N_p^6(c)$, and prove that the mean curvature is a constant under the assumption that M_r^5 has diagonalizable shape operator. Applying this result, we show that some types of biharmonic hypersurfaces of $N_p^6(c)$ are minimal.

2 Some Equations and Lemmas

Let M_r^5 be a λ-biharmonic hypersurface of $N_p^6(c)$ with diagonalizable shape operator A. In this section, we give some important equations and lemmas about the hypersurface M_r^5 under the assumption that the mean curvature H is not a constant.

According to [13] and [14], the hypersurface M_r^5 satisfies

$$\begin{cases}\Delta H+\varepsilon H\mathrm{tr}A^2-(5c-\lambda)H=0,\\ 2A\nabla H+5\varepsilon H\nabla H=0,\end{cases}\tag{2.1}$$

where $\Delta H=\mathrm{div}(\nabla H)$ and $\varepsilon=\langle\xi,\xi\rangle$, with ξ a unit normal vector field on M_r^5. The assumption that H is not a constant tells us that $\nabla H\neq 0$ on some open subset. Then, we learn from the second equation of (2.1) that ∇H is an eigenvector of the shape operator A, with corresponding eigenvalue $-\frac{5}{2}\varepsilon H$. Considering that A is diagonalizable, we can choose a local orthonormal frame $\{e_i\}_{i=1}^5$ with $\langle e_i,e_i\rangle=\varepsilon_i=\pm1$, such that ∇H is parallel to e_5 and $A(e_i)=\mu_ie_i$ with $i=1,2,\cdots,5$. Here $\mu_5=-\frac{5}{2}\varepsilon H$.

For simplicity, we write $-\frac{5}{2}\varepsilon H$ as μ. And then, we have

$$S:=\mathrm{tr}A^2=\sum_{i=1}^4\mu_i^2+\mu^2\tag{2.2}$$

and

$$\sum_{i=1}^4\mu_i=-3\mu\tag{2.3}$$

by $\mathrm{tr}A=5\varepsilon H$. Since e_5 is parallel to ∇H, we get

$$e_5(\mu)\neq 0\text{ and }e_i(\mu)=0,\ 1\leq i\leq 4.\tag{2.4}$$

Let ∇ be the Levi-Civita connection of M^5_r, and $\nabla_{e_i}e_j = \sum\limits_{k=1}^{5}\Gamma^k_{ij}e_k$ with $1 \leq i,j \leq 5$. By compatibility and symmetry of the connection ∇, we obtain

$$\Gamma^i_{ki} = 0, \quad \Gamma^j_{ki} = -\varepsilon_i\varepsilon_j\Gamma^i_{kj}, \ 1 \leq i,j,k \leq 5, \tag{2.5}$$

and

$$\Gamma^5_{ij} = \Gamma^5_{ji}, \ 1 \leq i,j \leq 4. \tag{2.6}$$

Combining (2.4), (2.5) and (2.6), we deduce from the Codazzi equation $\langle(\nabla_{e_i}A)e_j, e_k\rangle = \langle(\nabla_{e_j}A)e_i, e_k\rangle$ that

$$\begin{cases}\Gamma^5_{5i} = \Gamma^5_{ij} = 0,\\ e_i(\mu_j) = (\mu_i - \mu_j)\Gamma^j_{ji},\\ (\mu_i - \mu_j)\Gamma^j_{ki} = (\mu_k - \mu_j)\Gamma^j_{ik},\end{cases} \tag{2.7}$$

for distinct i,j,k. From (2.4) and the second equation of (2.7), we find $\mu_j \neq \mu$, for $1 \leq j \leq 4$. By using (2.5) and the third equation of (2.7), we obtain that

$$\varepsilon_k(\mu_j - \mu_k)\Gamma^k_{ij} = \varepsilon_k(\mu_i - \mu_k)\Gamma^k_{ji} = \varepsilon_i(\mu_j - \mu_i)\Gamma^i_{kj}, \tag{2.8}$$

for distinct i,j,k and $1 \leq i,j,k \leq 4$, which together with (2.5) implies that

$$(\mu_i - \mu_k)(\mu_j - \mu_k)(\mu_i - \mu_j)(\Gamma^k_{ij}\Gamma^k_{ji} + \Gamma^i_{jk}\Gamma^i_{kj} + \Gamma^j_{ik}\Gamma^j_{ki}) = 0,$$

i.e.

$$\Gamma^k_{ij}\Gamma^k_{ji} + \Gamma^i_{jk}\Gamma^i_{kj} + \Gamma^j_{ik}\Gamma^j_{ki} = 0, \tag{2.9}$$

for distinct μ_i, μ_j, μ_k and $1 \leq i,j,k \leq 4$. Applying Gauss equation for $\langle R(e_i,e_j)e_k, e_5\rangle$ with distinct i,j,k and $1 \leq i,j,k \leq 4$, combining (2.5) and (2.7), we get

$$\varepsilon_k(\Gamma^j_{j5} - \Gamma^k_{k5})\Gamma^k_{ij} = \varepsilon_k(\Gamma^i_{i5} - \Gamma^k_{k5})\Gamma^k_{ji} = \varepsilon_i(\Gamma^j_{j5} - \Gamma^i_{i5})\Gamma^i_{kj}. \tag{2.10}$$

Let f be a smooth function on M^5_r, and denote by f', f'' and $f^{(k)}$ (the index $k \geq 3$) the first, second and k-th derivatives of f along e_5. It follows from the first equation of (2.1) and the second equation of (2.7) that

$$\begin{cases}\mu'' = -\mu'(\sum_{i=1}^4 \Gamma^i_{i5}) + \varepsilon_5\mu(\varepsilon S - 5c + \lambda),\\ \mu'_i = (\mu - \mu_i)\Gamma^i_{i5}, \quad 1 \leq i \leq 4.\end{cases} \tag{2.11}$$

By using the Gauss equation

$$R(e_5,e_i)e_5 = c(\langle e_i,e_5\rangle e_5 - \langle e_5,e_5\rangle e_i) + \varepsilon\langle A(e_i),e_5\rangle A(e_5) - \varepsilon\langle A(e_5),e_5\rangle A(e_i),$$

combining (2.5), (2.6) and the first equation of (2.7), we derive that

$$(\Gamma^i_{i5})' = -(\Gamma^i_{i5})^2 - (\varepsilon\mu\mu_i + c)\varepsilon_5. \tag{2.12}$$

Applying (2.11) and (2.12), $\sum_{i=1}^{4}(\Gamma_{i5}^{i})^{k}$ with the index $1 \leq k \leq 7$ can be expressed by μ, $\sum_{i=1}^{4}\Gamma_{i5}^{i}$ and $\sum_{i=1}^{4}\mu_i^3$ and their derivatives (see Lemma 2.2). By substituting these expressions into the Murnaghan-Nakayama type formula (c.f. [15]) and employing a complex elimination process, we can demonstrate that $e_j(\sum_{i=1}^{4}\Gamma_{i5}^{i}) = 0$ for $j = 1, 2, 3, 4$ (see Lemma 2.3). Furthermore, it can be proved that $e_j(\mu_i) = 0$ for $1 \leq i \leq 5$ and $1 \leq j \leq 4$ (see Lemma 2.4). This result will play a crucial role in the subsequent proof of our main theorems.

Let $F_{r,0} = \sum_{i=1}^{4}\mu_i^r$, $F_{0,s} = \sum_{i=1}^{4}(\Gamma_{i5}^{i})^s$ and $F_{r,s} = \sum_{i=1}^{4}\mu_i^r(\Gamma_{i5}^{i})^s$ with $r, s = 1, 2, \cdots$. Differentiating $F_{r,0}$, $F_{0,s}$, and $F_{r,s}$ with respect to e_5, combining the second equation of (2.11) and (2.12), we derive the recurrence formulas as Lemma 2.1.

Lemma 2.1 We have

$$\begin{cases} rF_{r,1} = -F'_{r,0} + r\mu F_{r-1,1}, \\ sF_{0,s+1} = -F'_{0,s} - s\varepsilon\varepsilon_5\mu F_{1,s-1} - s\varepsilon_5 cF_{0,s-1}, \\ (r+s)F_{r,s+1} = -F'_{r,s} + r\mu F_{r-1,s+1} - s\varepsilon\varepsilon_5\mu F_{r+1,s-1} - s\varepsilon_5 cF_{r,s-1}, \end{cases} \tag{2.13}$$

for positive integers r and s, where $F_{0,0} = 4$.

Lemma 2.2 Denote $T := F_{0,1}$, then we have

$$\begin{cases} F_{0,2} = -T' + 3\varepsilon\varepsilon_5\mu^2 - 4\varepsilon_5 c, \\ F_{0,3} = \frac{1}{2}T'' - 6\varepsilon\varepsilon_5\mu\mu' - \varepsilon_5(\varepsilon\mu^2 + c)T, \\ F_{0,4} = -\frac{1}{6}T^{(3)} + \frac{4}{3}\varepsilon_5(\varepsilon\mu^2 + c)T' + \frac{5}{3}\varepsilon\varepsilon_5\mu\mu' T + A_0, \\ F_{0,5} = \frac{1}{24}T^{(4)} - \frac{5}{6}\varepsilon_5(\varepsilon\mu^2 + c)T'' - \frac{25}{12}\varepsilon\varepsilon_5\mu\mu' T' + A_1 T + A_2, \\ F_{0,6} = -\frac{1}{4}\varepsilon\varepsilon_5\mu^3 F_{3,0} - \frac{1}{120}T^{(5)} + \frac{1}{3}\varepsilon_5(\varepsilon\mu^2 + c)T^{(3)} + \frac{5}{4}\varepsilon\varepsilon_5\mu\mu' T'' + A_3 T' + A_4 T + A_5, \\ F_{0,7} = \frac{7}{24}\varepsilon\varepsilon_5(\mu^2\mu' F_{3,0} + \mu^3 F'_{3,0}) + \frac{1}{720}T^{(6)} - \frac{7}{72}\varepsilon_5(\mu^2\varepsilon + c)T^{(4)} - \frac{35}{72}\varepsilon\varepsilon_5\mu\mu' T^{(3)} \\ \qquad +A_6 T'' + A_7 T' + A_8 T + A_9, \end{cases} \tag{2.14}$$

where the expressions for $A_0, A_1, \cdots, A_9$ can be found in (2.21), (2.24), (2.26) and (2.28).

Proof Since $F_{0,0} = 4$ and $F_{1,0} = -3\mu$, it follows from the second equation of (2.13) with $s = 1$ that

$$F_{0,2} = -T' + 3\varepsilon\varepsilon_5\mu^2 - 4\varepsilon_5 c. \tag{2.15}$$

The first equation of (2.13) with $r = 1$ tells us that

$$F_{1,1} = \mu T + 3\mu'. \tag{2.16}$$

Substituting (2.15) and (2.16) into the second equation of (2.13) with $s = 2$, we have

$$F_{0,3} = \frac{1}{2}T'' - 6\varepsilon\varepsilon_5\mu\mu' - \varepsilon_5(\varepsilon\mu^2 + c)T. \tag{2.17}$$

We obtain from (2.2) and the first equation of (2.11) that

$$F_{2,0} = S - \mu^2 = \frac{\mu'' + \mu' T}{\varepsilon_5\varepsilon\mu} - \mu^2 + \frac{5c}{\varepsilon} - \frac{\lambda}{\varepsilon}. \tag{2.18}$$

Putting (2.15), (2.16) and (2.18) into the third equation of (2.13) with $r = s = 1$ gives that

$$F_{1,2} = -\mu T' - \mu' T - 2\mu'' + 2\varepsilon\varepsilon_5\mu^3 - 3\varepsilon_5\mu c + \frac{\varepsilon_5\lambda}{2}\mu. \tag{2.19}$$

Combining (2.15), (2.17) and (2.19), we get from the second equation of (2.13) with $s = 3$ that

$$F_{0,4} = -\frac{1}{6}T^{(3)} + \frac{4}{3}(\varepsilon\varepsilon_5\mu^2 + \varepsilon_5 c)T' + \frac{5}{3}\varepsilon\varepsilon_5\mu\mu' T + A_0, \tag{2.20}$$

where

$$A_0 = 2\varepsilon\varepsilon_5\mu'^2 + 4\varepsilon\varepsilon_5\mu\mu'' - 2\mu^4 + 4c^2 - \frac{\varepsilon\lambda}{2}\mu^2. \tag{2.21}$$

Since (2.16) and (2.18), it follows from the first equation of (2.13) with $r = 2$ that

$$F_{2,1} = -\frac{\mu'}{2\varepsilon\varepsilon_5\mu}T' + (\mu^2 - \frac{\mu''\mu - \mu'^2}{2\varepsilon\varepsilon_5\mu^2})T + 4\mu\mu' + \frac{\mu'\mu'' - \mu\mu^{(3)}}{2\varepsilon\varepsilon_5\mu^2}. \tag{2.22}$$

Combining (2.16), (2.17), (2.19) and (2.22), we derive from the third equation of (2.13) with $r = 1, s = 2$ that

$$\begin{aligned} F_{1,3} =& \frac{1}{2}\mu T'' + \mu' T' + \frac{T}{3}(2\mu'' - 3\varepsilon\varepsilon_5\mu^3 - \frac{\mu'^2}{\mu} - 3\varepsilon_5 c\mu) \\ &+ \mu^{(3)} - \frac{20}{3}\varepsilon\varepsilon_5\mu^2\mu' - \frac{\mu'\mu''}{3\mu} - \varepsilon_5 c\mu' - \frac{\varepsilon_5}{6}\mu'\lambda. \end{aligned} \tag{2.23}$$

Substitute (2.17), (2.20) and (2.23) into the second equation of (2.13) with $s = 4$, we have

$$F_{0,5} = \frac{1}{24}T^{(4)} - \frac{5}{6}\varepsilon_5(\varepsilon\mu^2 + c)T'' - \frac{25}{12}\varepsilon\varepsilon_5\mu\mu' T' + A_1 T + A_2,$$

where

$$\begin{cases} A_1 = \mu^4 - \frac{13}{12}\varepsilon\varepsilon_5\mu\mu'' - \frac{1}{12}\varepsilon\varepsilon_5\mu'^2 + 2\varepsilon c\mu^2 + c^2, \\ A_2 = -2\varepsilon\varepsilon_5\mu\mu^{(3)} - \frac{5}{3}\varepsilon\varepsilon_5\mu'\mu'' + \frac{26}{3}\mu^3\mu' + 7\varepsilon c\mu\mu' + \frac{5}{12}\varepsilon\mu\mu'\lambda. \end{cases} \tag{2.24}$$

Putting (2.18), (2.19) and (2.22) into the third equation of (2.13) with $r = 2, s = 1$ gives that

$$\begin{aligned} F_{2,2} =& -\frac{1}{3}\varepsilon\varepsilon_5\mu F_{3,0} + \frac{1}{6\varepsilon\varepsilon_5\mu}\mu' T'' - (\mu^2 + \frac{\mu'^2 - \mu\mu''}{3\varepsilon\varepsilon_5\mu^2})T' \\ &- \frac{T}{6\varepsilon\varepsilon_5\mu^4}(8\mu^5\mu'\varepsilon\varepsilon_5 + 2\varepsilon_5 c\mu^3\mu' - \mu^3\mu^{(3)} + 3\mu^2\mu'\mu'' - 2\mu\mu'^3) \\ &+ \frac{1}{6\varepsilon\varepsilon_5\mu^3}(-16\varepsilon\varepsilon_5\mu^4\mu'' - 8\varepsilon\varepsilon_5\mu^3\mu'^2 + 8\mu^7 - 10\varepsilon c\mu^5 - 10c^2\mu^3 \\ &- 2\varepsilon_5 c\mu^2\mu'' + \mu^2\mu^{(4)} - \mu\mu''^2 - 2\mu\mu'\mu^{(3)} + 2\mu'^2\mu'') + \frac{\varepsilon_5}{3\varepsilon}(\varepsilon\mu^2\lambda + c\lambda). \end{aligned} \tag{2.25}$$

By use of (2.19), (2.20), (2.23), (2.25) and the third equation of (2.13) with $r = 1, s = 3$, we can express $F_{1,4}$ in terms of $F_{3,0}$, T and μ. And then, applying the second equation of (2.13) with $s = 5$, we can write $F_{0,6}$ as following:

$$F_{0,6} = -\frac{\mu^3}{4}\varepsilon\varepsilon_5 F_{3,0} - \frac{1}{120}T^{(5)} + \frac{1}{3}\varepsilon_5(\varepsilon\mu^2 + c)T^{(3)} + \frac{5\varepsilon\varepsilon_5}{4}\mu\mu' T'' + A_3 T' + A_4 T + A_5,$$

where

$$\begin{cases} A_3 = -\frac{1}{30}(46\mu^4 + 92\varepsilon c\mu^2 - 39\varepsilon\varepsilon_5\mu\mu'' - 3\varepsilon\varepsilon_5\mu'^2 + 46c^2), \\ A_4 = \frac{1}{120\mu}(61\varepsilon\varepsilon_5\mu\mu^{(3)} - 35\varepsilon\varepsilon_5\mu\mu'\mu'' - 356\mu^4\mu' - 446\varepsilon c\mu^2\mu' + 40\varepsilon\varepsilon_5\mu'^3), \\ A_5 = \frac{1}{120\mu}(93\varepsilon\varepsilon_5\mu^2\mu^{(4)} + 48\varepsilon\varepsilon_5\mu\mu'\mu^{(3)} + 15\varepsilon\varepsilon_5\mu\mu''^2 + 40\varepsilon\varepsilon_5\mu'^2\mu'' - 1204\mu^3\mu'^2 \\ \quad -888\varepsilon c\mu^2\mu'' - 768\mu^4\mu'' - 408\varepsilon c\mu\mu'^2 + 180\varepsilon\varepsilon_5\mu^7 + 270\varepsilon_5 c\mu^5 - 540\varepsilon\varepsilon_5 c^2\mu^3 \\ \quad -480\varepsilon_5 c^3\mu) + \frac{\lambda}{24}(9\varepsilon_5\mu^4 + 27\varepsilon\varepsilon_5\mu^2 - 3\varepsilon\mu\mu'' - 2\varepsilon\mu'^2). \end{cases} \tag{2.26}$$

Substituting (2.22) into the first equation (2.13) with $r = 3$ yields that

$$F_{3,1} = -\frac{1}{3}F'_{3,0} - \frac{\mu'}{2\varepsilon\varepsilon_5}T' + (\mu^3 + \frac{\mu'^2 - \mu\mu''}{2\varepsilon\varepsilon_5\mu})T + 4\mu^2\mu' + (\frac{\mu'\mu'' - \mu\mu^{(3)}}{2\varepsilon\varepsilon_5\mu}). \tag{2.27}$$

Combining (2.22), (2.23), (2.25) and (2.27), we can obtain the expression of $F_{2,3}$ from the third equation of (2.13) with $r = s = 2$. Then, the third equation of (2.13) with $r = 1, s = 4$ gives the expression of $F_{1,5}$. It follows from the second equation of (2.13) with $s = 6$ that

$$\begin{aligned} F_{0,7} =& \frac{7}{24}\varepsilon\varepsilon_5\mu^2\mu' F_{3,0} + \frac{7}{24}\varepsilon\varepsilon_5\mu^3 F'_{3,0} + \frac{1}{720}T^{(6)} - \frac{7\varepsilon_5(\mu^2\varepsilon + c)}{72}T^{(4)} \\ & - \frac{35\varepsilon\varepsilon_5\mu\mu'}{72}T^{(3)} + A_6 T'' + A_7 T' + A_8 T + A_9, \end{aligned}$$

where

$$\begin{cases} A_6 = \frac{1}{360}(-273\varepsilon\varepsilon_5\mu\mu'' - 21\varepsilon\varepsilon_5\mu'^2 + 392\mu^4 + 784\varepsilon c\mu^2 + 392c^2), \\ A_7 = \frac{1}{720\mu}(-427\varepsilon\varepsilon_5\mu^2\mu^{(3)} + 245\varepsilon\varepsilon_5\mu\mu'\mu'' - 280\varepsilon\varepsilon_5\mu'^3 + 3192\mu^4\mu' + 3822\varepsilon c\mu^2\mu'), \\ A_8 = \frac{1}{720\mu^2}(-127\varepsilon\varepsilon_5\mu^3\mu^{(4)} + 148\varepsilon\varepsilon_5\mu^2\mu'\mu^{(3)} + 185\varepsilon\varepsilon_5\mu^2\mu''^2 - 630\varepsilon\varepsilon_5\mu\mu'^2\mu'' \\ \quad +1352\mu^5\mu'' + 1982\varepsilon c\mu^3\mu'' + 122\varepsilon c\mu^2\mu'^2 + 280\varepsilon\varepsilon_5\mu'^4 + 1656\mu^4\mu'^2 \\ \quad -720\varepsilon\varepsilon_5\mu^8 - 2160\varepsilon_5 c\mu^6 - 2160\varepsilon\varepsilon_5 c^2\mu^4 - 720\varepsilon_5 c^3\mu^2), \\ A_9 = \frac{1}{720\mu^2}(-171\varepsilon\varepsilon_5\mu^3\mu^{(5)} - 21\varepsilon\varepsilon_5\mu^2\mu'\mu^{(4)} + 126\varepsilon\varepsilon_5\mu^2\mu''\mu^{(3)} - 280\varepsilon\varepsilon_5\mu\mu'^2\mu^{(3)} \\ \quad +2544\mu^5\mu^{(3)} + 3384\varepsilon c\mu^3\mu^{(3)} + 2520\varepsilon c\mu^2\mu'\mu'' + 8216\mu^4\mu'\mu'' - 350\varepsilon\varepsilon_5\mu\mu'\mu''^2 \\ \quad +280\varepsilon\varepsilon_5\mu'^3\mu'' + 3104\mu^3\mu'^3 - 3888\varepsilon\varepsilon_5 c^2\mu^3\mu' - 13236\varepsilon_5 c\mu^5\mu' - 7248\varepsilon\varepsilon_5\mu^7\mu') \\ \quad +\frac{\lambda}{720}(21\varepsilon\mu\mu^{(3)} + 35\varepsilon\mu'\mu'' - 546\varepsilon_5\mu^3\mu' - 756\varepsilon\varepsilon_5 c\mu\mu'). \end{cases} \tag{2.28}$$

Lemma 2.3 For $i = 1, 2, 3, 4$, the function T satisfies $e_i(T) = 0$.

Proof Denote $F_k := F_{0,k}$, then Murnaghan-Nakayama type formula (c.f. [15]) yields

$$\begin{cases} 0 = F_1^5 - 10F_1^3F_2 + 20F_1^2F_3 + 15F_1F_2^2 - 30F_1F_4 - 20F_2F_3 + 24F_5, \\ 0 = F_1^6 - 9F_1^4F_2 + 16F_1^3F_3 + 9F_1^2F_2^2 - 18F_1^2F_4 + 3F_2^3 - 18F_2F_4 - 8F_3^2 + 24F_6, \\ 0 = F_1^7 - 7F_1^5F_2 + 14F_1^4F_3 - 7F_1^3F_2^2 - 14F_1^3F_4 + 28F_1^2F_2F_3 + 21F_1F_2^3 \\ \quad -42F_1F_2F_4 - 14F_2^2F_3 - 28F_3F_4 + 48F_7. \end{cases} \tag{2.29}$$

It follows from Lemma 2.2 and (2.29) that

$$\begin{aligned}
&T^{(4)} + 5TT^{(3)} + 10T'T'' + 10T^2T'' - 50\varepsilon\varepsilon_5\mu^2T'' + 20\varepsilon_5cT'' + 15TT'^2 + 10T^3T' \\
&- 150\varepsilon\varepsilon_5\mu^2TT' - 170\varepsilon\varepsilon_5\mu\mu'T' + 60\varepsilon_5cTT' + T^5 + 20\varepsilon_5cT^3 - 170\varepsilon\varepsilon_5\mu\mu'T^2 \\
&- 50\varepsilon\varepsilon_5\mu^2T^3 - 62\varepsilon\varepsilon_5\mu'^2T - 146\varepsilon\varepsilon_5\mu\mu''T - 332\varepsilon c\mu^2T + 279\mu^4T + 15\lambda\varepsilon\mu^2T \\
&+ 64c^2T - 48\varepsilon\varepsilon_5\mu\mu^{(3)} - 40\varepsilon\varepsilon_5\mu'\mu'' + 568\mu^3\mu' - 312\varepsilon c\mu\mu' + 10\varepsilon\lambda\mu\mu' = 0,
\end{aligned} \tag{2.30}$$

$$\begin{aligned}
&30\varepsilon\varepsilon_5\mu^4F_{3,0} + \mu T^{(5)} + 5(3T' - 3T^2 - 17\varepsilon\varepsilon_5\mu^2 + 4\varepsilon_5c)\mu T^{(3)} + 10\mu T''^2 - 10(4\varepsilon_5T^3T'' \\
&+ 4\varepsilon\mu^2T + 4cT + 39\varepsilon\mu\mu')\varepsilon_5\mu T'' + 15\mu T'^3 + 15(4\varepsilon_5c - 3T^2 - 17\varepsilon\varepsilon_5\mu^2)\mu T'^2 - 45\mu T^4T' \\
&+ 30(13\varepsilon\mu^2 - 8c)\varepsilon_5\mu T^2T' - 150\varepsilon\varepsilon_5\mu^2\mu'TT' + (1129\mu^4 - 516\varepsilon\varepsilon_5\mu\mu'' - 192\varepsilon\varepsilon_5\mu'^2 + 64c^2 \\
&- 832\varepsilon c\mu^2 + 45\varepsilon\lambda\mu^2)\mu T' - 5\mu T^6 + 5(43\varepsilon\mu^2 - 20c)\varepsilon_5\mu T^4 + 630\varepsilon\varepsilon_5\mu^2\mu'T^3 + 1286\mu^4\mu'T \\
&+ 5(72\varepsilon\varepsilon_5\mu\mu'' - 109\mu^4 - 9\varepsilon\lambda\mu^2 - 64c^2 + 36\varepsilon\varepsilon_5\mu'^2 + 232\varepsilon c\mu^2)\mu T^2 - 61\varepsilon\varepsilon_5\mu^2\mu^{(3)}T \\
&- 40\varepsilon\varepsilon_5\mu'^3T + 326\varepsilon c\mu^2\mu'T + 35\varepsilon\varepsilon_5\mu\mu'\mu''T - 93\varepsilon\varepsilon_5\mu^2\mu^{(4)} - 48\varepsilon\varepsilon_5\mu\mu'\mu^{(3)} - 15\varepsilon\varepsilon_5\mu\mu''^2 \\
&- 40\varepsilon\varepsilon_5\mu'^2\mu'' + 3(5\lambda - 184c)\varepsilon\mu^2\mu'' + 1848\mu^4\mu'' + 3184\mu^3\mu'^2 + 2(5\lambda - 156c)\varepsilon\mu\mu'^2 \\
&- 1125\varepsilon\varepsilon_5\mu^7 + 90(23c - 2\lambda)\varepsilon_5\mu^5 + 45(\lambda - 12c)\varepsilon\varepsilon_5c\mu^3 = 0,
\end{aligned} \tag{2.31}$$

and

$$\begin{aligned}
&70\varepsilon_5(\varepsilon\mu^2 + c)\mu^2T^{(4)} - 210\varepsilon\varepsilon_5\mu^5F'_{3,0} - 210\varepsilon\varepsilon_5\mu^4\mu'F_{3,0} - \mu^2T^{(6)} - 35(T'' - 3TT')\mu^2T^{(3)} \\
&- 35\varepsilon_5\mu^2(\varepsilon_5T^3 + 7\varepsilon\mu^2T - 13cT - 22\varepsilon\mu\mu')T^{(3)} + 7\mu^2(120\varepsilon_5cT^2 - 15T^4 - 90\varepsilon\varepsilon_5\mu^2T^2 \\
&- 37\mu^4 + 50\varepsilon\varepsilon_5\mu\mu'T - 15\varepsilon\lambda\mu^2 - 584\varepsilon c\mu^2 + 248c^2 + 66\varepsilon\varepsilon_5\mu'^2 + 198\varepsilon\varepsilon_5\mu\mu'')T'' + 35(6T^2 \\
&+ 3T' - 10\varepsilon\varepsilon_5\mu^2 + 32\varepsilon_5c)\mu^2T'T'' + (315T - 8144\mu)\mu^2T'^3 + 105(\varepsilon_5T^3 + 26cT - 37\varepsilon\mu^2T \\
&- 12\varepsilon\mu\mu')\varepsilon_5\mu^2T'^2 + 7(100\varepsilon_5c\mu T^3 - 15\mu T^5 - 110\varepsilon\varepsilon_5\mu^3T^3 - 510\varepsilon\varepsilon_5\mu^2\mu'T^2 + 1855\mu^5T \\
&+ 1000c^2\mu T - 3580\varepsilon c\mu^3T - 360\varepsilon\varepsilon_5\mu^2\mu''T - 180\varepsilon\varepsilon_5\mu\mu'^2T - 2466\varepsilon c\mu^2\mu' - 35\varepsilon\varepsilon_5\mu\mu'\mu'' \\
&+ 144\mu^4\mu' + 40\varepsilon\varepsilon_5\mu'^3 + 61\varepsilon\varepsilon_5\mu^2\mu^{(3)} + 45\varepsilon\lambda\mu^3T)\mu T' - 15\mu^2T^7 + 105\varepsilon_5(5\varepsilon\mu^2 - 2c)\mu^2T^5 \\
&+ 1610\varepsilon\varepsilon_5\mu^3\mu'T^4 + 105(17\mu^4 - \varepsilon\lambda\mu^2 - 28\varepsilon c\mu^2 + 8c^2 + 4\varepsilon\varepsilon_5\mu'^2 + 8\varepsilon\varepsilon_5\mu\mu'')\mu^2T^3 \\
&+ 70(143\mu^2 - 214\varepsilon c)\mu^3\mu'T^2 + 127\varepsilon\varepsilon_5\mu^3\mu^{(4)}T - 148\varepsilon\varepsilon_5\mu^2\mu'\mu^{(3)}T + 630\varepsilon\varepsilon_5\mu\mu'^2\mu''T \\
&+ 2(2264\mu^2 - 6871\varepsilon c)\mu^3\mu''T - 185\varepsilon\varepsilon_5\mu^2\mu''^2T - (280\varepsilon\varepsilon_5\mu'^2 + 6002\varepsilon c\mu^2 + 2916\mu^4)\mu'^2T \\
&- 15(841\varepsilon\mu^6 - 3014c\mu^4 + 49\lambda\mu^4 - 98\varepsilon c\lambda\mu^2 + 2376\varepsilon c^2\mu^2 - 960c^3)\varepsilon_5\mu^2T + 171\varepsilon\varepsilon_5\mu^3\mu^{(5)} \\
&+ 21\varepsilon\varepsilon_5\mu^2\mu'\mu^{(4)} - 126\varepsilon\varepsilon_5\mu^2\mu''\mu^{(3)} - 3384\varepsilon c\mu^3\mu^{(3)} + 280\varepsilon\varepsilon_5\mu\mu'^2\mu^{(3)} - 2520\varepsilon c\mu^2\mu'\mu'' \\
&- 2544\mu^5\mu^{(3)} - 21\varepsilon\lambda\mu^3\mu^{(3)} + 350\varepsilon\varepsilon_5\mu\mu'\mu''^2 - 280\varepsilon\varepsilon_5\mu'^3\mu'' - (18296\mu^2 + 35\varepsilon\lambda)\mu^2\mu'\mu'' \\
&+ 948\varepsilon\varepsilon_5\mu^7\mu' + (43476c + 1806\lambda)\varepsilon_5\mu^5\mu' - (26352c - 756\lambda)\varepsilon\varepsilon_5c\mu^3\mu' = 0.
\end{aligned} \tag{2.32}$$

Differentiating (2.31) along e_5, and combining (2.31) and (2.32), we can eliminate $F_{3,0}$ and

$F'_{3,0}$, and get

$$\begin{aligned}
&6\mu^2T^{(6)}-14\mu\mu'T^{(5)}+5(21T'-21T^2-105\varepsilon\varepsilon_5\mu^2+42\varepsilon_5c)\mu^2T^{(4)}+210\mu^2T''T^{(3)}\\
&-105(2\mu'+\mu T)\mu T'T^{(3)}+5\mathcal{A}_1T^{(3)}-140\mu\mu'T''^2+420\mu^2T'^2T''+14\mathcal{A}_2\mu T''+7\mathcal{A}_3T'\\
&-420(3T^2-4\varepsilon_5c+10\varepsilon\varepsilon_5\mu^2)\mu^2T'T''-105(3\mu T+2\mu')\mu T'^3+105\mathcal{A}_4T'^2+\mathcal{A}_5=0,
\end{aligned}\tag{2.33}$$

where

$$\begin{aligned}
\mathcal{A}_1&=-63\mu^2T^3+42\mu\mu'T^2-105\varepsilon\varepsilon_5\mu^4T+42\varepsilon_5c\mu^2T-392\varepsilon\varepsilon_5\mu^3\mu'-56\varepsilon_5c\mu\mu',\\
\mathcal{A}_2&=40\mu'T^3-30\mu T^4-60\varepsilon_5c\mu T^2+150\varepsilon\varepsilon_5\mu^3T^2-50\varepsilon\varepsilon_5\mu^2\mu'T+40\varepsilon_5c\mu'T-64c^2\mu'\\
&\quad+156c^2\mu-354\varepsilon\varepsilon_5\mu^2\mu''-708\varepsilon c\mu^3+132\varepsilon\varepsilon_5\mu\mu'^2+546\mu^5+15\varepsilon\lambda\mu^3,\\
\mathcal{A}_3&=90\mu\mu'T^4-45\mu^2T^5+750\varepsilon\varepsilon_5\mu^4T^3-300\varepsilon_5c\mu^2T^3+(480\varepsilon_5c\mu+1380\varepsilon\varepsilon_5\mu^3)\mu'T^2\\
&\quad+(210\varepsilon\varepsilon_5\mu^3\mu''+765\mu^6+330\varepsilon\varepsilon_5\mu^2\mu'^2-1260\varepsilon c\mu^4+360c^2\mu^2-45\varepsilon\lambda\mu^4)T\\
&\quad+3688\mu^5\mu'-516\varepsilon\varepsilon_5\mu^3\mu^{(3)}+384\varepsilon\varepsilon_5\mu\mu'^3+132\varepsilon\varepsilon_5\mu^2\mu'\mu''-2140\varepsilon c\mu^3\mu',\\
\mathcal{A}_4&=-11\mu^2T^3+15\varepsilon_5\varepsilon\mu^4T-6\varepsilon_5c\mu^2T-8\varepsilon_5c\mu\mu'T+6\mu\mu'T^2-22\varepsilon_5\varepsilon\mu^3\mu',\\
\mathcal{A}_5&=-15\mu^2T^7+70\mu\mu'T^6+(525\varepsilon\mu^2-210c)\varepsilon_5\mu^2T^5+1400\varepsilon_5c\mu\mu'T^4-2940\varepsilon c\mu^4T^3\\
&\quad+\cdots-30552\varepsilon\varepsilon_5\mu^7\mu'-26352\varepsilon\varepsilon_5c^2\mu^3\mu'+756\varepsilon\varepsilon_5\lambda c\mu^3\mu'-714\varepsilon_5\lambda\mu^5\mu'.
\end{aligned}$$

By applying (2.30), we eliminate $T^{(6)}, T^{(5)}$ gradually from (2.33) and derive

$$5\mu\mathcal{B}_1T^{(4)}+5\mu\mathcal{B}_2T^{(3)}+5\mu\mathcal{B}_3T''+225\mu^2TT'^3+5\mu\mathcal{B}_4T'^2+5\mu\mathcal{B}_5T'+\mathcal{B}_6=0,\tag{2.34}$$

where

$$\begin{aligned}
\mathcal{B}_1&=3\mu T'+3\mu T^2-14\mu'T+45\varepsilon\varepsilon_5\mu^3-18\varepsilon_5c\mu,\\
\mathcal{B}_2&=15\mu TT'+15\mu T^3-70\mu'T^2-90T\varepsilon_5c\mu+225\varepsilon\varepsilon_5\mu^3T+88\varepsilon\varepsilon_5\mu^2\mu',\\
\mathcal{B}_3&=30\mu T'^2-140\mu'T'T-120\mu\varepsilon_5cT'+60\mu T'T^2+300\varepsilon\varepsilon_5\mu^3T'+30\mu T^4-140\mu'T^3\\
&\quad+300\varepsilon\varepsilon_5\mu^3T^2-120\varepsilon_5c\mu T^2+1052\varepsilon\varepsilon_5T\mu^2\mu'-280\varepsilon_5c\mu'\mu T-216\varepsilon\varepsilon_5\mu\mu'^2\\
&\quad+1584\varepsilon c\mu^4-24\varepsilon\lambda\mu^3+720\varepsilon\varepsilon_5\mu^2\mu''-1194\mu^5-360c^2\mu,\\
\mathcal{B}_4&=75\mu T^3-210\mu'T^2+225\varepsilon\varepsilon_5\mu^3T-90\varepsilon_5c\mu T-246\varepsilon\varepsilon_5\mu^2\mu',\\
\mathcal{B}_5&=33\mu T^5-140\mu'T^4-150\varepsilon\varepsilon_5\mu^3T^3+60\varepsilon_5c\mu T^3+1608\varepsilon\varepsilon_5\mu^2\mu'T^2-840\varepsilon_5c\mu'T^2\\
&\quad+\cdots-560\varepsilon\varepsilon_5\mu\mu'\mu''+112\varepsilon\varepsilon_5\mu'^3+2332\varepsilon c\mu^2\mu'+30\varepsilon\lambda\mu^2\mu',\\
\mathcal{B}_6&=15\mu^2T^7-70\mu\mu'T^6-525\varepsilon\varepsilon_5\mu^4T^5+210\varepsilon\varepsilon_5c\mu^2T^5-1400\varepsilon_5c\mu'\mu T^4\\
&\quad+\cdots-336\varepsilon\varepsilon_5\mu^2\mu''\mu^{(3)}+560\varepsilon\varepsilon_5\mu\mu'\mu''^2+5600\varepsilon\varepsilon_5\mu\mu'^2\mu^{(3)}-560\varepsilon\varepsilon_5\mu'^3\mu''.
\end{aligned}$$

From (2.30) and (2.34), we may eliminate $T^{(4)}$ and obtain

$$-55\varepsilon\varepsilon_5\mu^3\mu'T^{(3)}+5\mathcal{C}_1\mu^2T''+5\mathcal{C}_2\mu T'+\mathcal{C}_3=0,\tag{2.35}$$

where

$$\begin{aligned}
\mathcal{C}_1 &= -44\varepsilon\varepsilon_5\mu\mu' T - 132\mu^4 + 27\varepsilon c\mu^2 + 3\lambda\varepsilon\mu^2 - 36\varepsilon\varepsilon_5\mu\mu'' + 27\varepsilon\varepsilon_5\mu'^2,\\
\mathcal{C}_2 &= -66\varepsilon\varepsilon_5\mu^2\mu' T^2 + (81\varepsilon c\mu^2 - 376\mu^4 + 9\lambda\varepsilon\mu^2 - 108\varepsilon\varepsilon_5\mu\mu'' + 81\varepsilon\varepsilon_5\mu'^2)\mu T\\
&\quad - 335\mu^4\mu' - 39\varepsilon\varepsilon_5\mu^2\mu^{(3)} - 26\varepsilon c\mu^2\mu' + 55\varepsilon\varepsilon_5\mu\mu'\mu'' - 14\varepsilon\varepsilon_5\mu'^3,\\
\mathcal{C}_3 &= -660\mu^6T^3 + 15\varepsilon\lambda\mu^4T^3 + 135\varepsilon c\mu^4T^3 - 180\varepsilon\varepsilon_5\mu^3\mu''T^3 + 135\varepsilon\varepsilon_5\mu^2\mu'^2T^3\\
&\quad + \cdots + 228\varepsilon c\mu^2\mu'\mu'' - 5\varepsilon\lambda\mu^2\mu'\mu'' - 477\mu^5\mu^{(3)} - 24\varepsilon\varepsilon_5\mu^3\mu^{(5)}.
\end{aligned}$$

Differentiating (2.35) and combining (2.30), we arrive at

$$5\mu^2\mathcal{D}_1T^{(3)} + 5\mu^2\mathcal{D}_2T'' + 5\mu^2\mathcal{D}_3T'^2 + \mathcal{D}_4T' + \mathcal{D}_5 = 0, \tag{2.36}$$

where

$$\begin{aligned}
\mathcal{D}_1 &= 3\varepsilon\lambda\mu^4 - 132\mu^4 + 27c\varepsilon\mu^2 + 11\varepsilon\varepsilon_5\mu\mu'T - 47\varepsilon\varepsilon_5\mu\mu'' - 6\varepsilon\varepsilon_5\mu'^2,\\
\mathcal{D}_2 &= 44\varepsilon\varepsilon_5\mu\mu'T^2 - 396\mu^4T + 81\varepsilon c\mu^2T + 9\varepsilon\lambda\mu^2T - 152\varepsilon\varepsilon_5\mu\mu''T - 1677\mu^3\mu'\\
&\quad + 81\varepsilon c\mu^2 + 302\varepsilon c\mu\mu' + 12\varepsilon\lambda\mu\mu' - 75\varepsilon\varepsilon_5\mu\mu^{(3)} + \varepsilon\varepsilon_5\mu'\mu'',\\
\mathcal{D}_3 &= 33\varepsilon\varepsilon_5\mu\mu'T - 396\mu^4 + 81\varepsilon c\mu^2 + 9\varepsilon\lambda\mu^2 - 141\varepsilon\varepsilon_5\mu\mu'' - 18\varepsilon\varepsilon_5\mu'^2,
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_4 &= 330\varepsilon\varepsilon_5\mu^3\mu'T^3 - 1980\mu^6T^2 + 45\varepsilon\lambda\mu^4T^2 + 405\varepsilon c\mu^4T^2 - 870\varepsilon\varepsilon_5\mu^3\mu''T^2\\
&\quad + \cdots - 20\varepsilon\lambda\mu^2\mu'^2 - 98\varepsilon\varepsilon_5\mu^2\mu'\mu^{(3)} + 130\varepsilon\varepsilon_5\mu\mu'^2\mu'' + 670\varepsilon\varepsilon_5\mu\mu'^3,\\
\mathcal{D}_5 &= -55\varepsilon\varepsilon_5\mu^3\mu''T^4 - 165\varepsilon\varepsilon_5\mu^2\mu'^2T^4 - 6710\mu^5\mu'T^3 + 60\varepsilon\lambda\mu^3\mu'T^3\\
&\quad + \cdots + 102\varepsilon\varepsilon_5\mu^2\mu''\mu^{(4)} - 70\varepsilon\varepsilon_5\mu\mu''^3 + 140\varepsilon\varepsilon_5\mu'^2\mu''^2.
\end{aligned}$$

Combining (2.35) and (2.36), we can eliminate $T^{(3)}$. Then, using the similar methods as the above, we can eliminate $T^{(2)}$ and derive

$$\begin{cases} K_1T' + K_2T^2 + K_3T + K_4 = 0,\\ (P_1T + P_2)T' + P_3T^3 + P_4T^2 + P_5T + P_6 = 0, \end{cases} \tag{2.37}$$

where

$$\begin{aligned}
K_1 &= 1102743180\varepsilon\varepsilon_5\mu^{17}\mu' - 528660000\varepsilon_5c\mu^{15}\mu' + 199809720\mu^{15}\mu^{(3)}\\
&\quad + \cdots + 7406700\varepsilon\varepsilon_5\mu^3\mu'^5\mu''^2 - 2617300\varepsilon\varepsilon_5\mu^2\mu'^7\mu'' + 476000\varepsilon\varepsilon_5\mu\mu'^9,\\
K_2 &= 1102743180\varepsilon\varepsilon_5\mu^{17}\mu' - 528660000\varepsilon_5c\mu^{15}\mu' + 199809720\mu^{15}\mu^{(3)}\\
&\quad + \cdots + 7406700\varepsilon\varepsilon_5\mu^2\mu'^5\mu''^2 - 2617300\varepsilon\varepsilon_5\mu^2\mu'^7\mu'' + 476000\varepsilon\varepsilon_5\mu\mu'^9,\\
K_3 &= 6080540400\mu^{20} - 10535094900\varepsilon c\mu^{18} + 7668498420\varepsilon\varepsilon_5\mu^{17}\mu''\\
&\quad + \cdots - 11295900\varepsilon\varepsilon_5\mu'^6\mu''^2 + 3414600\varepsilon\varepsilon_5\mu\mu'^8\mu'' - 476000\varepsilon\varepsilon_5\mu'^{10},\\
K_4 &= 10231198560\mu^{19}\mu' - 13403764440\varepsilon c\mu^{17}\mu' + 17903160\varepsilon\varepsilon_5\mu^{17}\mu^{(3)}\\
&\quad + \ldots + 476000\varepsilon\varepsilon_5\mu\mu'^8\mu^{(3)} + 2462600\varepsilon\varepsilon_5\mu\mu'^7\mu''^2 - 476000\varepsilon\varepsilon_5\mu'^9\mu''.
\end{aligned}$$

and

$$\begin{aligned}
P_1 &= -96070985841600\varepsilon\varepsilon_5\mu^{26}\mu' + 85358626135200\varepsilon_5 c\mu^{24}\mu' \\
&\quad + \cdots + 6118007000\varepsilon\varepsilon_5\mu^3\mu'^{11}\mu'' - 661640000\varepsilon\varepsilon_5\mu^2\mu'^{13}, \\
P_2 &= 529736679648000\mu^{29} - 1134527927544000\varepsilon c\mu^{27} \\
&\quad + \cdots + 3062906000\varepsilon\varepsilon_5\mu^2\mu'^{12}\mu'' - 199920000\varepsilon\varepsilon_5\mu\mu'^{14}, \\
P_3 &= -96070985841600\varepsilon\varepsilon_5\mu^{26}\mu' + 85358626135200\varepsilon_5 c\mu^{24}\mu' \\
&\quad + \cdots + 6118007000\varepsilon\varepsilon_5\mu^3\mu'^{11}\mu'' - 661640000\varepsilon\varepsilon_5\mu^2\mu'^{13}, \\
P_4 &= 96070985841600\varepsilon\varepsilon_5\mu^{26}\mu'' + 1313334045084600\varepsilon\varepsilon_5\mu^{25}\mu'^2 \\
&\quad + \cdots - 4163348000\varepsilon\varepsilon_5\mu^2\mu'^{12}\mu'' + 461720000\varepsilon\varepsilon_5\mu\mu'^{14}, \\
P_5 &= 11507407958455200\mu^{28}\mu' - 486534246046200\varepsilon\lambda\mu^{26}\mu' \\
&\quad + \cdots - 2736132000\varepsilon\varepsilon_5\mu\mu'^{13}\mu'' + 199920000\varepsilon\varepsilon_5\mu'^{15}, \\
P_6 &= 891342018547200\mu^{28}\mu'' + 18704951237079360\mu^{27}\mu'^2 \\
&\quad + \cdots - 2997932000\varepsilon\varepsilon_5\mu\mu'^{12}\mu''^2 + 199920000\varepsilon\varepsilon_5\mu'^{14}\mu''.
\end{aligned}$$

From (2.37), we obtain the following algebraic polynomial equation

$$G_1(\mu,\mu',\mu'',\cdots,\mu^{(7)})T + G_2(\mu,\mu',\mu'',\cdots,\mu^{(8)}) = 0, \tag{2.38}$$

where $G_1(\mu,\mu',\mu'',\cdots,\mu^{(7)})$ and $G_2(\mu,\mu',\mu'',\cdots,\mu^{(8)})$ are polynomials of μ and its derivatives. By (2.4), the first equation of (2.7) and the symmetry of connection ∇, we conclude that

$$e_i(\mu) = e_i(\mu') = e_i(\mu'') = e_i(\mu^{(k)}) = 0,\ k \geq 3. \tag{2.39}$$

Acting on (2.38) by e_i, with $1 \leq i \leq 4$, combining (2.39), we know

$$G_1(\mu,\mu',\mu'',\cdots,\mu^{(7)})e_i(T) = 0. \tag{2.40}$$

Assume that $e_j(T) \neq 0$ for some $1 \leq j \leq 4$ on some open subset, then (2.40) implies $G_1 = 0$. It follows from (2.38) that $G_2 = 0$. We can eliminate $\mu',\mu'',\cdots,\mu^{(8)}$ from $G_1 = 0$ and $G_2 = 0$ step by step, and get a non-trival polynomial equation of μ. So, μ is a constant, a contradiction. Therefore, $e_j(T) = 0$ for any $1 \leq j \leq 4$.

Lemma 2.4 For $1 \leq i \leq 5$ and $1 \leq j \leq 4$, we have

$$e_j(\mu_i) = 0$$

on some open subset.

Proof For the case that M_r^5 has at most three distinct principal curvatures, the conclusion has been obtained in [13]. We suppose that M_r^5 has five or four distinct principal curvatures. According to Lemma 2.3, we find that $e_i(T) = e_i(T') = e_i(T'') = e_i(T^{(k)}) = 0$ for $k \geq 3$ and $1 \leq i \leq 4$. It follows from (2.14) that $e_i(F_k) = 0$ for $1 \leq i, k \leq 4$, that is

$$\sum_{l=1}^{4}(\Gamma_{l5}^{l})^{k-1}e_i(\Gamma_{l5}^{l}) = 0. \tag{2.41}$$

When M_r^5 has five distinct principal curvatures, i.e. μ_1, μ_2, μ_3 and μ_4 are distinct, we know from (2.12) that $\Gamma_{15}^1, \Gamma_{25}^2, \Gamma_{35}^3$ and Γ_{45}^4 are distinct on some open subset. Then, the coefficient determinant of the system (2.41)

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \Gamma_{15}^1 & \Gamma_{25}^2 & \Gamma_{35}^3 & \Gamma_{45}^4 \\ (\Gamma_{15}^1)^2 & (\Gamma_{25}^2)^2 & (\Gamma_{35}^3)^2 & (\Gamma_{45}^4)^2 \\ (\Gamma_{15}^1)^3 & (\Gamma_{25}^2)^3 & (\Gamma_{35}^3)^3 & (\Gamma_{45}^4)^3 \end{vmatrix} = \prod_{1\le i<j\le 4} (\Gamma_{j5}^j - \Gamma_{i5}^i) \neq 0.$$

Therefore, (2.41) admits only zero solutions, i.e.,

$$e_i(\Gamma_{15}^1) = e_i(\Gamma_{25}^2) = e_i(\Gamma_{35}^3) = e_i(\Gamma_{45}^4) = 0, \quad 1 \le i \le 4. \tag{2.42}$$

Furthermore, we have

$$e_j e_5(\Gamma_{i5}^i) = 0. \tag{2.43}$$

Differentiating (2.12) along e_j, $1 \le j \le 4$, combining (2.42) and (2.43), we obtain $e_j(\mu_i) = 0$ for $1 \le i, j \le 4$, which together with (2.4) leads to the result.

For the case that M_r^5 has four distinct principal curvatures, without loss of generality, we suppose that μ_1, μ_2 and μ_3 are distinct and $\mu_4 = \mu_3$. It follows from the second equation of (2.11) and (2.12) that $\Gamma_{45}^4 = \Gamma_{35}^3$ and $\Gamma_{15}^1, \Gamma_{25}^2, \Gamma_{35}^3$ are distinct on some open subset. The system (2.41) gives that

$$\begin{cases} e_i(\Gamma_{15}^1) + e_i(\Gamma_{25}^2) + 2e_i(\Gamma_{35}^3) = 0, \\ \Gamma_{15}^1 e_i(\Gamma_{15}^1) + \Gamma_{25}^2 e_i(\Gamma_{25}^2) + 2\Gamma_{35}^3 e_i(\Gamma_{35}^3) = 0, \\ (\Gamma_{15}^1)^2 e_i(\Gamma_{15}^1) + (\Gamma_{25}^2)^2 e_i(\Gamma_{25}^2) + 2(\Gamma_{35}^3)^2 e_i(\Gamma_{35}^3) = 0, \end{cases}$$

which have nonzero coefficient determinant. So, $e_i(\Gamma_{j5}^j) = 0$, $1 \le i, j \le 4$. Furthermore, $e_j(\mu_i) = 0$ for $1 \le i \le 5$ and $1 \le j \le 4$.

3 Proof of Main Theorems

Theorem 3.1 Let M_r^5 be a λ-biharmonic hypersurface of $N_p^6(c)$ with diagonalizable shape operator, then it has constant mean curvature.

Proof We employ the method of contradiction to prove this Theorem. Assume that H is not a constant, now we use the equations and lemmas in Section 2 to derive contradictions. For the case that the number of distinct principal curvatures is not more than three, the contradiction has been derived in [13]. We only need to consider the case that M_r^5 has four or five distinct principal curvatures.

Applying Lemma 2.4, we obtain from the second equation of (2.7) that

$$\Gamma_{ij}^i = 0 \quad \text{for} \quad 1 \le i, j \le 4. \tag{3.1}$$

By using Gauss equation for $\langle R(e_i, e_j)e_i, e_j\rangle$, and combining (2.5), (2.9) and (3.1), we derive that

$$\varepsilon_5 \varepsilon_i \varepsilon_j \Gamma_{i5}^i \Gamma_{j5}^j - 2\sum_{k\neq i,j} \varepsilon_k \Gamma_{ij}^k \Gamma_{ji}^k = -\varepsilon\varepsilon_i\varepsilon_j \mu_i\mu_j - \varepsilon_i\varepsilon_j c, \tag{3.2}$$

for distinct i, j, and $1 \leq i, j \leq 4$.

Case 1: The terms of $\{\Gamma_{23}^1,\ \Gamma_{24}^1,\ \Gamma_{34}^1,\ \Gamma_{34}^2\}$ are all zero.

In this case, (3.2) is reduced to

$$\varepsilon_5 \Gamma_{i5}^i \Gamma_{j5}^j = -\varepsilon \mu_i \mu_j - c, \tag{3.3}$$

for distinct i, j and $1 \leq i, j \leq 4$, which implies that

$$\varepsilon_5 \Gamma_{k5}^k (\Gamma_{i5}^i - \Gamma_{j5}^j) = -\varepsilon \mu_k (\mu_i - \mu_j), \tag{3.4}$$

i.e.

$$\Gamma_{k5}^k = \varphi \mu_k, \quad 1 \leq k \leq 4, \tag{3.5}$$

where $\varphi = -\varepsilon \varepsilon_5 \frac{\mu_i - \mu_j}{\Gamma_{i5}^i - \Gamma_{j5}^j}$ for $1 \leq i, j \leq 4$, $i, j \neq k$ and $\mu_i \neq \mu_j$. Notice that φ does not depend on the indices i, j, or k, and satisfies that

$$\varepsilon_5 \varphi^2 + \varepsilon = 0.$$

When $c \neq 0$, it follows from (3.3) and (3.5) that

$$0 = (\varepsilon_5 \varphi^2 + \varepsilon) \mu_i \mu_j = -c \neq 0, \quad \text{for } i \neq j \text{ and } i, j = 1, 2, 3, 4,$$

a contradiction.

When $c = 0$, we obtain $\varepsilon \varepsilon_5 = -1$ and $\varphi^2 = 1$. Differentiating both sides of (2.3) with respect to e_5, combining the second equation of (2.11) and (3.5), we have

$$3\mu' = 12\varphi\mu^2 + 6\mu\varphi \sum_{i=1}^{3} \mu_i + 2\varphi \sum_{i=1}^{3} \mu_i^2 + 2\varphi \sum_{1 \leq i < j \leq 3} \mu_i \mu_j. \tag{3.6}$$

By differentiating (3.6) along e_5 and applying the second equation of (2.11), we obtain

$$3\mu'' = 96\mu^3 + 12\mu_1\mu_2\mu_3 + 78\mu^2 \sum_{i=1}^{3} \mu_i + 26\mu \sum_{i=1}^{3} \mu_i^2 + 6 \sum_{c_1} \mu_i^2 \mu_j + 44\mu \sum_{1 \leq i < j \leq 3} \mu_i \mu_j, \tag{3.7}$$

where c_1 means i, j are distinct and $1 \leq i, j \leq 3$. Using (2.3) and (3.6), the first equation of (2.11) turns into

$$\mu'' = 2\mu^3 + \varepsilon_5 \lambda \mu. \tag{3.8}$$

Combining (3.7) and (3.8) gives

$$90\mu^3 + 12\mu_1\mu_2\mu_3 + 78\mu^2 \sum_{i=1}^{3} \mu_i + 26\mu \sum_{i=1}^{3} \mu_i^2 + 6 \sum_{c_1} \mu_i^2 \mu_j + 22\mu \sum_{c_1} \mu_i \mu_j - 3\varepsilon_5 \lambda \mu = 0. \tag{3.9}$$

Since $\varphi^2 = 1$, we know $\Gamma_{i5}^i = \pm\mu_i$. If $\Gamma_{i5}^i = \mu_i$, then differentiating (3.9) two times along e_5, applying (3.6), the second equation of (2.11) and (3.5), we have

$$\begin{aligned}
&3240\mu^4 + 3726\mu^3 \sum_{i=1}^{3} \mu_i + 1710\mu^2 \sum_{i=1}^{3} \mu_i^2 + 1602\mu^2 \sum_{c_1} \mu_i\mu_j + 312\mu \sum_{i=1}^{3} \mu_i^3 + 52 \sum_{i=1}^{3} \mu_i^4 \\
&+ 104 \sum_{c_1} \mu_i^3\mu_j + 966\mu \sum_{c_1} \mu_i^2\mu_j - 36\varepsilon_5\lambda\mu^2 - 18\varepsilon_5\lambda\mu \sum_{i=1}^{3} \mu_i - 6\varepsilon_5\lambda \sum_{i=1}^{3} \mu_i^2 \\
&+ 78 \sum_{c_1} \mu_i^2\mu_j^2 + 140 \sum_{c_2} \mu_i^2\mu_j\mu_k - 3\varepsilon_5\lambda \sum_{c_1} \mu_i\mu_j + 1836\mu\mu_1\mu_2\mu_3 = 0,
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
&51840\mu^5 + 74358\mu^4 \sum_{i=1}^{3} \mu_i + 44370\mu^3 \sum_{i=1}^{3} \mu_i^2 + 42696\mu^3 \sum_{c_1} \mu_i\mu_j + 13056\mu^2 \sum_{i=1}^{3} \mu_i^3 \\
&+ 38118\mu^2 \sum_{c_1} \mu_i\mu_j^2 + 73656\mu^2\mu_1\mu_2\mu_3 + 2176\mu \sum_{i=1}^{3} \mu_i^4 + 7592\mu \sum_{c_1} \mu_i\mu_j^3 \\
&+ 5694\mu \sum_{c_1} \mu_i^2\mu_j^2 + 10958\mu \sum_{c_2} \mu_i^2\mu_j\mu_k + 540 \sum_{c_1} \mu_i\mu_j^4 + 1080 \sum_{c_2} \mu_i\mu_j\mu_k^3 \\
&+ 1080 \sum_{c_1} \mu_i^2\mu_j^3 + 1620 \sum_{c_2} \mu_i\mu_j^2\mu_k^2 - 234\varepsilon_5\lambda\mu^2 \sum_{i=1}^{3} \mu_i - 78\varepsilon_5\lambda\mu \sum_{i=1}^{3} \mu_i^2 \\
&- 288\varepsilon_5\lambda\mu^3 - 132\varepsilon_5\lambda\mu \sum_{c_1} \mu_i\mu_j - 18\varepsilon_5\lambda \sum_{c_1} \mu_i^2\mu_j - 36\varepsilon_5\lambda\mu_1\mu_2\mu_3 = 0,
\end{aligned} \tag{3.11}$$

where c_2 means i, j, k are distinct and $1 \leq i, j, k \leq 3$. When M_r^5 has four distinct principal curvatures, we suppose $\mu_1 = \mu_2$. By using (3.9)–(3.11), we may eliminate μ_1, μ_2, μ_3 and get a 165th-degree polynomial equation of μ with constant coefficients. Thus μ is a constant, a contradiction.

When M_r^5 has five distinct principal curvatures, i.e. $\mu_1, \mu_2, \mu_3, \mu_4$ are distinct, we differentiate (3.11) along e_5 and obtain that

$$\begin{aligned}
&3110400\mu^6 + 347458\mu^5 \sum_{i=1}^{3} \mu_i + 3943458\mu^4 \sum_{i=1}^{3} \mu_i^2 + 3837024\mu^4 \sum_{c_1} \mu_i\mu_j - 10368\varepsilon_5\lambda\mu^4 \\
&+ 1558152\mu^3 \sum_{i=1}^{3} \mu_i^3 + 4527162\mu^3 \sum_{c_1} \mu_i\mu_j^2 + 8826408\mu^3\mu_1\mu_2\mu_3 - 11502\varepsilon_5\lambda\mu^3 \sum_{i=1}^{3} \mu_i \\
&+ \cdots + 4352 \sum_{i=1}^{3} \mu_i^6 + 13056 \sum_{c_1} \mu_i\mu_j^5 + 30972 \sum_{c_1} \mu_i^2\mu_j^4 + 20092 \sum_{c_1} (\mu_i\mu_j)^3 \\
&- 156\varepsilon_5\lambda \sum_{i=1}^{3} \mu_i^4 - 132\varepsilon_5\lambda \sum_{c_1} \mu_i\mu_j^3 - 234\varepsilon_5\lambda \sum_{c_1} (\mu_i\mu_j)^2 = 0.
\end{aligned} \tag{3.12}$$

By using (3.9)–(3.12), we may eliminate μ_1, μ_2 and μ_3, and finally derive a 96th-degree polynomial equation of μ with constant coefficients, which yields that μ is a constant, a contradiction. If $\Gamma_{i5}^i = -\mu_i$, we can similarly deduce a contradiction.

Case 2: At least two terms of $\{\Gamma_{23}^1,\ \Gamma_{24}^1,\ \Gamma_{34}^1,\ \Gamma_{34}^2\}$ are nonzero.

Suppose Γ_{23}^1 and Γ_{24}^1 are nonzero, then μ_1, μ_2, μ_3 are distinct and μ_1, μ_2, μ_4 are also distinct by (2.8). It follows from (2.8) and (2.10) that

$$\frac{\Gamma_{25}^2 - \Gamma_{15}^1}{\mu_2 - \mu_1} = \frac{\Gamma_{35}^3 - \Gamma_{15}^1}{\mu_3 - \mu_1} = \frac{\Gamma_{35}^3 - \Gamma_{25}^2}{\mu_3 - \mu_2}, \tag{3.13}$$

and

$$\frac{\Gamma_{25}^2 - \Gamma_{15}^1}{\mu_2 - \mu_1} = \frac{\Gamma_{45}^4 - \Gamma_{15}^1}{\mu_4 - \mu_1} = \frac{\Gamma_{45}^4 - \Gamma_{25}^2}{\mu_4 - \mu_2}.$$

Since $e_i(\Gamma_{j5}^j) = e_i(\mu_j) = 0,\ 1 \le i, j \le 4$, we conclude from the above two equations that there exists two smooth functions ξ and η, with $e_i(\xi) = e_i(\eta) = 0,\ 1 \le i \le 4$, such that

$$\Gamma_{i5}^i = \xi\mu_i + \eta. \tag{3.14}$$

Since $\mu_1 \neq \mu_2$ and $\Gamma_{15}^1 \neq \Gamma_{25}^2$, we know $\xi \neq 0$. Differentiating both sides of (3.14) with respect to e_5, and combining the second equation of (2.11) and (2.12), we obtain that

$$(\xi' + \xi^2\mu + \varepsilon\varepsilon_5\mu + \xi\eta)\mu_i + \eta' + \xi\eta\mu + \eta^2 + \varepsilon_5 c = 0,\ i = 1, 2, 3, 4,$$

which gives that

$$\begin{cases} \xi' = -\xi^2\mu - \varepsilon\varepsilon_5\mu - \xi\eta, \\ \eta' = -\xi\eta\mu - \eta^2 - \varepsilon_5 c. \end{cases} \tag{3.15}$$

Applying (2.2), (2.3), the second equation of (2.11) and (3.14), we deduce that

$$\sum_{i=1}^{4} \Gamma_{i5}^i = -3\mu\xi + 4\eta, \tag{3.16}$$

and

$$-3\mu' = -\,\xi(S + 2\mu^2) + 7\mu\eta. \tag{3.17}$$

Using (3.16) and (3.17), the first equation of (2.11) can be written as

$$\mu'' = (\mu\xi - \frac{4}{3}\eta)\{\xi(2\mu^2 + S) - 7\mu\eta\} + \varepsilon_5\mu(\varepsilon S - 5c + \lambda). \tag{3.18}$$

Acting on (3.17) by e_5, and using (3.15) and (3.17), we derive that

$$-3\mu'' = -\xi S' + \xi\eta(9\mu^2 + \frac{10}{3}S) - \frac{1}{3}\xi^2\mu(S + 2\mu^2) + \mu(\varepsilon\varepsilon_5 S + 2\varepsilon\varepsilon_5\mu^2 - 7\varepsilon_5 c - \frac{70}{3}\eta^2),$$

which together with (3.18) yields

$$3\xi S' - 8\xi^2\mu(2\mu^2 + S) + 2\xi\eta(30\mu^2 + S) - \varepsilon_5\mu(14\varepsilon_5\eta^2 + 12\varepsilon S - 66c + 6\varepsilon\mu^2 + 9\lambda) = 0. \tag{3.19}$$

Taking the sum over the index i, j for $1 \leq i < j \leq 4$ in (3.2), and combining (3.14) and (2.9), we have

$$(\varepsilon_5\xi^2 + \varepsilon) \sum_{1\leq i<j\leq 4} \mu_i\mu_j + 3\xi\eta\varepsilon_5 \sum_{i=1}^{4} \mu_i + 6\varepsilon_5\eta^2 + 6c = 0. \tag{3.20}$$

Since

$$\sum_{1\leq i<j\leq 4} \mu_i\mu_j = 5\mu^2 - \frac{S}{2},$$

(3.20) turns into

$$(\varepsilon_5\xi^2 + \varepsilon)(5\mu^2 - \frac{1}{2}S) - 9\varepsilon_5\xi\eta\mu + 6\varepsilon_5\eta^2 + 6c = 0. \tag{3.21}$$

By differentiating (3.21) along e_5, combining (3.15) and (3.17), we have

$$\begin{aligned} &(\varepsilon_5\xi^2 + \varepsilon)(20\xi\mu^3 - 26\xi\mu S + 3S' + 86\eta\mu^2) + 6\varepsilon_5\xi^2\eta(2S + 7\mu^2) \\ &+ 18\varepsilon_5\eta^2(4\eta - 9\varepsilon_5\xi\mu) + 18c(4\eta - 3\xi\mu) = 0. \end{aligned} \tag{3.22}$$

If $\varepsilon_5\xi^2 + \varepsilon = 0$, then it follows from (3.15) that $\eta = 0$. And then, (3.21) gives $c = 0$. Follow the process of Case 1, we can deduce a contradiction. If $\varepsilon_5\xi^2 + \varepsilon \neq 0$ on some open subset, we can eliminate S' from (3.19) and (3.22), and obtain that

$$\begin{aligned} &(\varepsilon_5\xi^2 + \varepsilon)[68\mu^2\xi\eta + 10\xi\eta S + 6\mu^3(6\xi^2 + \varepsilon\varepsilon_5) + 6\mu S(2\varepsilon_5\varepsilon - 3\xi^2) - 120\varepsilon_5 c\mu + 9\varepsilon_5\lambda\mu \\ &- 148\eta^2\mu] - 6\xi\eta(7\varepsilon\mu^2 + 2\varepsilon S - 12c) + 18\eta^2(9\varepsilon\mu + 4\xi\eta\varepsilon_5) + 54c\mu\varepsilon\varepsilon_5 = 0. \end{aligned} \tag{3.23}$$

By (3.21), (3.17) and (3.23) become

$$(\varepsilon_5\xi^2 + \varepsilon)\mu' = 4\varepsilon_5\xi^3\mu^2 - \frac{25}{3}\varepsilon_5\xi^2\eta\mu + 4\varepsilon\xi\mu^2 + 4\varepsilon_5\xi\eta^2 + 4\xi c - \frac{7}{3}\varepsilon\eta\mu, \tag{3.24}$$

and

$$\begin{aligned} &- 48\varepsilon\varepsilon_5\xi\eta^3 - 192\varepsilon_5\xi^3\eta^3 + 544\xi^4\eta^2\mu + 170\varepsilon\varepsilon_5\xi^2\eta^2\mu - 158\eta^2\mu - 492\xi^5\eta\mu^2 + 210\xi\eta\mu^2 \\ &- 24(3 + 5\varepsilon_5)c\xi^3\eta - 282\varepsilon\varepsilon_5\xi^3\eta\mu^2 - 48\varepsilon\xi\eta + 144\xi^6\mu^3 + 162\varepsilon\varepsilon_5\xi^4\mu^3 + 3\varepsilon_5(112c - 3\lambda)\xi^4\mu \\ &+ 6\varepsilon(43c - 3\lambda)\xi^2\mu - 108\xi^2\mu^3 - 126\varepsilon\varepsilon_5\mu^3 - 3\varepsilon_5(26c + 3\lambda)\mu = 0. \end{aligned} \tag{3.25}$$

Differentiating (3.25) along e_5 two times, combining (3.15) and (3.24), we deduce that

$$\begin{aligned} &6072\varepsilon\xi^3\eta^4 + 9984\varepsilon_5\xi^5\eta^4 - 1302\varepsilon_5\xi\eta^4 + 2198\varepsilon\eta^3\mu - 3888\varepsilon_5\xi\mu^4 - 31744\varepsilon_5\xi^6\eta^3\mu + 6660c\xi^7\mu^2 \\ &+ 9156\varepsilon_5\xi^2\eta^3\mu - 20898\varepsilon\xi^4\eta^3\mu - 7650\varepsilon\varepsilon_5 c\xi\mu^2 + 35376\varepsilon_5\xi^7\eta^2\mu^2 + 29268\varepsilon\xi^5\eta^2\mu^2 + 4608c^2\xi^5 \\ &- 12(176c + 9\lambda)\varepsilon_5\xi\eta^2 + 12(1216c - 9\lambda)\varepsilon_5\xi^5\eta^2 - 16812\varepsilon_5\xi^3\eta^2\mu^2 + 24(412c - 9\lambda)\varepsilon_5\varepsilon\xi^3\eta^2 \\ &- 10704\varepsilon\xi\eta^2\mu^2 - 16344\varepsilon_5\xi^8\eta\mu^3 + 9(37\lambda - 2800c)\xi^6\eta\mu - 18666\varepsilon\xi^6\eta\mu^3 + 13716\varepsilon_5\xi^4\eta\mu^3 \\ &+ 27\varepsilon\varepsilon_5\xi^4\eta\mu(9\lambda - 782c) + 27(17\lambda + 212c)\xi^2\eta\mu + 2016\varepsilon_5\eta\mu^3 + 63(26c + \lambda + 120\varepsilon_5 c)\varepsilon\varepsilon_5\eta\mu \\ &+ 2592\varepsilon_5\xi^9\mu^4 + 3888\varepsilon\xi^7\mu^4 - 108\varepsilon_5 c\lambda\xi^5 - 3888\varepsilon_5\xi^5\mu^4 + 5670\varepsilon\varepsilon_5 c\xi^5\mu^2 - 8640c\xi^3\mu^2 \\ &+ 18054\varepsilon\xi^2\eta\mu^3 - 9072\varepsilon\xi^3\mu^4 + 72\varepsilon\xi^3 c(53c - 3\lambda) - 36\varepsilon_5 c\xi(22c + 3\lambda) = 0, \end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
&54432\xi^{12}\mu^5-476280\xi^{11}\eta\mu^4+1685376\xi^{10}\eta^2\mu^3+89424\varepsilon\varepsilon_5\xi^{10}\mu^5+193428\varepsilon_5c\xi^{10}\mu^3\\
&-2836176\xi^9\eta^3\mu^2-2997\varepsilon_5\lambda\xi^9\eta\mu^2-1046700c\xi^9\eta\mu^2-1020780\varepsilon_5c\xi^9\eta\mu^2-647838\varepsilon\varepsilon_5\xi^9\eta\mu^4\\
&+2001852\varepsilon\varepsilon_5\xi^8\eta^2\mu^3+2147328c\varepsilon_5\xi^8\eta^2\mu+\cdots+270426\xi\eta^3\mu^2+78732\varepsilon\varepsilon_5\xi\eta\mu^4\\
&-2187\varepsilon_5\lambda\xi\eta\mu^2+34704\varepsilon\varepsilon_5c^2\xi\eta+1728\varepsilon\xi\eta^3\lambda+1080\varepsilon\varepsilon_5c\lambda\xi\eta+324\lambda\eta^2\mu+46176\varepsilon\varepsilon_5\xi\eta^5\\
&+11664\mu^5+16902c\varepsilon\mu^3+135c\lambda\mu-306\varepsilon_5\lambda\eta^2\mu=0.
\end{aligned} \tag{3.27}$$

We can eliminate ξ and η from (3.25), (3.26) and (3.27), and obtain an algebraic polynomial equation of μ. Thus, μ is a constant, which leads to a contradiction.

Case 3: Only one term of $\{\Gamma_{23}^1,\ \Gamma_{24}^1,\ \Gamma_{34}^1,\ \Gamma_{34}^2\}$ is nonzero.

Suppose $\Gamma_{23}^1\neq 0$ and $\Gamma_{24}^1=\Gamma_{34}^1=\Gamma_{34}^2=0$, then (3.13) holds. And then, we have

$$\Gamma_{i5}^i=\xi\mu_i+\eta,\quad i=1,2,3. \tag{3.28}$$

Here the smooth functions ξ and η satisfy (3.15) and $\xi\neq 0$ on some open subset. Letting $i=4, j=2,3$ respectively in (3.2), we obtain

$$\begin{cases}\varepsilon_5\Gamma_{45}^4\Gamma_{25}^2=-\varepsilon\mu_4\mu_2-c,\\ \varepsilon_5\Gamma_{45}^4\Gamma_{35}^3=-\varepsilon\mu_4\mu_3-c.\end{cases} \tag{3.29}$$

It follows that

$$\varepsilon_5\Gamma_{45}^4(\Gamma_{25}^2-\Gamma_{35}^3)=-\varepsilon\mu_4(\mu_2-\mu_3),$$

which together with (3.28) shows that

$$\varepsilon_5\xi\Gamma_{45}^4=-\varepsilon\mu_4. \tag{3.30}$$

Substituting (3.28) into (3.29), and combining (3.30) gives

$$\varepsilon\eta\mu_4=c\xi. \tag{3.31}$$

Taking the sum over the index i,j for $1\leq i<j\leq 3$ in (3.2), and combining (2.9) and (3.28), we have

$$(\varepsilon_5\xi^2+\varepsilon)(\mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3)+2\varepsilon_5\xi\eta(\mu_1+\mu_2+\mu_3)+3\varepsilon_5\eta^2+3c=0. \tag{3.32}$$

By using (2.2) and (2.3), (3.32) turns into

$$(\varepsilon_5\xi^2+\varepsilon)(10\mu^2+6\mu\mu_4+2\mu_4^2-S)+2\varepsilon_5\eta(3\eta-2\xi\mu_4-6\xi\mu)+6c=0. \tag{3.33}$$

If $\varepsilon_5\xi^2+\varepsilon=0$, then $\eta=0$ and $c=0$ by (3.15) and (3.31). We can derive a contradiction as Case 1. In the following, we treat the case that $\varepsilon_5\xi^2+\varepsilon\neq 0$ on some open subset. From (2.3), (2.11), (3.28) and (3.30), we deduce that

$$3\xi\mu'=2\xi^2\mu^2+\xi^2\mu\mu_4-\xi^2\mu_4^2+\xi^2S-6\xi\eta\mu-\xi\eta\mu_4+\varepsilon\varepsilon_5\mu\mu_4-\varepsilon\varepsilon_5\mu_4^2, \tag{3.34}$$

and

$$\begin{aligned}3\xi^2\mu'' =&6\xi^4\mu^3 + 5\xi^4\mu^2\mu_4 - 2\xi^4\mu\mu_4^2 - \xi^4\mu_4^3 + 3\xi^4\mu S + \xi^4\mu_4 S - 24\xi^3\eta\mu^2 - 12\xi^3\eta\mu\mu_4\\ &+ 2\xi^3\eta\mu_4^2 - 3\xi^3\eta S + 3\varepsilon\varepsilon_5\xi^2\mu S + 5\varepsilon\varepsilon_5\xi^2\mu^2\mu_4 + 18\xi^2\eta^2\mu - \varepsilon\varepsilon_5\xi^2\mu\mu_4^2 + 3\xi^2\eta^2\mu_4\\ &+ 3(\lambda - 5c)\varepsilon_5\xi^2\mu - 2\varepsilon\varepsilon_5\xi^2\mu_4^3 + \varepsilon\varepsilon_5\xi^2\mu_4 S - 9\varepsilon\varepsilon_5\xi\eta\mu\mu_4 + 2\varepsilon\varepsilon_5\xi\eta\mu_4^2 + \mu\mu_4^2 - \mu_4^3.\end{aligned} \tag{3.35}$$

Differentiating (3.34) along e_5, combining (3.15), (3.31) and (3.35), we get

$$\begin{aligned}&16\xi^4\mu^3 - 2\xi^4\mu_4^3 + 12\xi^4\mu^2\mu_4 + 2(4\mu + \mu_4)\xi^4 S - 6\xi^4\mu\mu_4^2 - 20\xi^3\eta\mu\mu_4 - 48\xi^3\eta\mu^2\\ &- 2\xi^3\eta\mu_4^2 + (3 + 9\varepsilon\varepsilon_5)\xi^2\mu^2\mu_4 - (9 + \varepsilon\varepsilon_5)\xi^2\mu\mu_4^2 + (9\lambda - 63c)\varepsilon_5\xi^2\mu - 3\varepsilon_5 c\xi^2\mu_4\\ &- 3\xi^3 S' + (6 - 4\varepsilon\varepsilon_5)\xi^2\mu_4^3 + 12\varepsilon\varepsilon_5\xi^2\mu S + 2\varepsilon\varepsilon_5\xi^2\mu_4 S + 6\varepsilon\varepsilon_5\xi^2\mu^3 + (3 + 4\varepsilon\varepsilon_5)\xi\eta\mu_4^2\\ &- 3(1 + 6\varepsilon\varepsilon_5)\xi\eta\mu\mu_4 + 3(\varepsilon\varepsilon_5 - 1)\mu^2\mu_4 + (5 - 9\varepsilon\varepsilon_5)\mu\mu_4^2 + 2(3\varepsilon\varepsilon_5 - 1)\mu_4^3 = 0.\end{aligned} \tag{3.36}$$

Acting on (3.33) by e_5 and combining (3.36), we derive that

$$\begin{aligned}&36\xi^6\mu^3 + 4\xi^6\mu_4^3 + 20\xi^6\mu\mu_4^2 + 16\xi^6\mu^2\mu_4 - 4\xi^6\mu_4 S - 18\xi^6\mu S + 4\xi^5\eta\mu_4^2 + 6\xi^5\eta S + 84\xi^5\eta\mu^2\\ &+ 60\xi^5\eta\mu\mu_4 + 6(\varepsilon\varepsilon_5 - 1)\xi^4\mu_4^3 - 36(\mu_4 + 3\mu)\xi^4\eta^2 + (21 + 5\varepsilon\varepsilon_5)\xi^4\mu^2\mu_4 - 15\varepsilon_5 c\xi^4\mu_4\\ &+ 9(\lambda - 11c)\varepsilon_5\xi^4\mu - (6\mu + 2\mu_4)\varepsilon\varepsilon_5\xi^4 S + (33\varepsilon\varepsilon_5 - 15)\xi^4\mu\mu_4^2 + 42\varepsilon\varepsilon_5\xi^4\mu^3 + 36\varepsilon\varepsilon_5\xi^3\eta\mu^2\\ &+ 3(6\varepsilon\varepsilon_5 - 5)\xi^3\eta\mu\mu_4 + (15 - 4\varepsilon\varepsilon_5)\xi^3\eta\mu_4^2 + 36\varepsilon_5 c\xi^3\eta + 36\xi^3\eta^3 + (24\varepsilon\varepsilon_5 - 14)\xi^2\mu^2\mu_4\\ &+ 6(3 - 4\varepsilon\varepsilon_5)\xi^2\mu\mu_4^2 + 2(6\mu + \mu_4)\xi^2 S + 9(\lambda - 7c)\varepsilon\xi^2\mu - 3\varepsilon c\xi^2\mu_4 + (4 + 3\varepsilon\varepsilon_5)\xi\eta\mu_4^2\\ &+ 6\xi^2\mu^3 + (6 - 2\varepsilon\varepsilon_5)\mu_4^3 + 3(1 - \varepsilon\varepsilon_5)\mu^2\mu_4 + (5\varepsilon\varepsilon_5 - 9)\mu\mu_4^2 - 3(6 + \varepsilon\varepsilon_5)\xi\eta\mu\mu_4 = 0.\end{aligned} \tag{3.37}$$

By using (3.31) and (3.33), (3.34) and (3.37) reduce to

$$\begin{aligned}3\eta^2(\xi^2 + \varepsilon\varepsilon_5)\mu' =&\xi^5 c^2 + 7\varepsilon c\xi^4\eta\mu + 12\xi^3\eta^2\mu^2 - 5\varepsilon c\xi^3\eta^2 - 18\xi^2\eta^3\mu + 8\varepsilon_5 c\xi^2\eta\mu\\ &+ 12\varepsilon\varepsilon_5\xi\eta^2\mu^2 + 6\xi\eta^4 + 5\varepsilon_5 c\xi\eta^2 - \xi c^2 - 6\varepsilon\varepsilon_5\eta^3\mu + \varepsilon c\eta\mu,\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}&4\varepsilon_5\xi^{10}c^3 + 40\varepsilon\varepsilon_5 c^2\xi^9\eta\mu + 2(\varepsilon + 3\varepsilon_5)c^3\xi^8 - 32\varepsilon\varepsilon_5 c^2\xi^8\eta^2 - 216\varepsilon_5 c\xi^7\eta^3\mu + 144\varepsilon\varepsilon_5\xi^7\eta^3\mu^3\\ &+ (19 - 15\varepsilon\varepsilon_5)c^2\xi^6\eta^2 + 132\varepsilon_5 c\xi^8\eta^2\mu^2 + (15\varepsilon\varepsilon_5 + 31)c^2\xi^7\eta\mu + 3(61\varepsilon - 7\varepsilon_5)c\xi^6\eta^2\mu^2\\ &+ 6(\varepsilon - \varepsilon_5)c^3\xi^6 - 360\varepsilon\varepsilon_5\xi^6\eta^4\mu^2 + 84\varepsilon_5 c\xi^6\eta^4 + (39 - 63\varepsilon\varepsilon_5)c^2\xi^5\eta\mu + 288\varepsilon\varepsilon_5\xi^5\eta^5\mu\\ &+ 15\varepsilon_5 c\xi^5\eta^3 + 9\varepsilon(5c - \lambda)\xi^5\eta^3\mu + 162\xi^5\eta^3\mu^3 - 2(3\varepsilon_5 + \varepsilon)c^3\xi^4 + 2(19\varepsilon\varepsilon_5 - 9)c^2\xi^4\eta^2\\ &- 9(5\varepsilon + 3\varepsilon_5)c\xi^4\eta^2\mu^2 - 252\xi^4\eta^4\mu^2 - 72\varepsilon\varepsilon_5\xi^4\eta^6 - 24\varepsilon c\xi^4\eta^4 + 144\xi^3\eta^5\mu - 108\varepsilon\varepsilon_5\xi^3\eta^3\mu^3\\ &+ (33\varepsilon\varepsilon_5 - 59)c^2\xi^3\eta\mu + 18(\varepsilon c + 15\varepsilon_5 c - \lambda)\xi^3\eta^3\mu - 36\xi^2\eta^6 - 48\varepsilon_5 c\xi^2\eta^4 + 108\varepsilon\varepsilon_5\xi^2\eta^4\mu^2\\ &- 3c(25\varepsilon + 9\varepsilon_5)\xi^2\eta^2\mu^2 + 2(\varepsilon_5 - 3\varepsilon)c^3\xi^2 - 72\varepsilon\varepsilon_5\xi\eta^5\mu + 9\varepsilon(c - \lambda)\xi\eta^3\mu + 3\varepsilon_5 c\xi\eta^3\\ &+ (9 - 5\varepsilon\varepsilon_5)c^2\xi\eta\mu - 126\xi\eta^3\mu^3 - (13 + 3\varepsilon\varepsilon_5)c^2\xi^2\eta^2 + 3c(\varepsilon_5 - \varepsilon)\eta^2\mu^2 = 0.\end{aligned} \tag{3.39}$$

Differentiating (3.39) two times along e_5 and using (3.15) and (3.38), we have

$$\begin{aligned}
&40\varepsilon\varepsilon_5c^4\xi^{14}+424\varepsilon_5c^3\xi^{13}\eta\mu+31c^4\xi^{12}+15\varepsilon\varepsilon_5c^4\xi^{12}-536\varepsilon_5c^3\xi^{12}\eta^2\\
&+1560\varepsilon\varepsilon_5c^2\xi^{12}\eta^2\mu^2+495\varepsilon c^3\xi^{11}\eta\mu-81\varepsilon_5c^3\xi^{11}\eta\mu+2232\varepsilon_5c\xi^{11}\eta^3\mu^3\\
&+\cdots+702\varepsilon\lambda\xi^3\eta^5\mu+18\varepsilon_5c\xi^2\eta^6+45\varepsilon_5c\eta^4\mu^2+96c^2\xi\eta^3\mu+1944\xi\eta^7\mu\\
&-45\varepsilon\varepsilon_5c\lambda\xi^2\eta^4-15c^2\eta^2\mu^2+9\varepsilon c^2\lambda\xi^2\eta^2+72c\lambda\xi\eta^3\mu-9c^4\xi^2+27\varepsilon\lambda\eta^4\mu^2\\
&+1224c^2\xi^2\eta^2\mu^2-6156\xi^2\eta^6\mu^2+108\varepsilon_5c\xi\eta^3\mu^3-81\varepsilon c\eta^4\mu^2=0,
\end{aligned}\tag{3.40}$$

and

$$\begin{aligned}
&424c^5\xi^{18}+4408\varepsilon c^4\xi^{17}\eta\mu+495\varepsilon\varepsilon_5c^5\xi^{16}-8312\varepsilon c^4\xi^{16}\eta^2-81c^5\xi^{16}\\
&+15816c^3\xi^{16}\eta^2\mu^2+5501\varepsilon_5c^4\xi^{15}\eta\mu-2055\varepsilon c^4\xi^{15}\eta\mu-74160c^3\xi^{15}\eta^3\mu\\
&-217944\varepsilon c^2\xi^{14}\eta^4\mu^2-1216c^5\xi^{14}-4692c^3\xi^{14}\eta^2\mu^2+10014\varepsilon\varepsilon_5c^3\xi^{14}\eta^2\mu^2\\
&+\cdots+630c^2\lambda\xi^2\eta^4+3024\varepsilon_5c\lambda\xi^2\eta^6-14742\varepsilon\varepsilon_5\lambda\xi^2\eta^6\mu^2-1227c^3\xi^2\eta^2\mu^2\\
&-2898\varepsilon\varepsilon_5c^3\xi^2\eta^4-684\varepsilon\varepsilon_5c^2\lambda\xi\eta^4\mu+48c^3\xi\eta^3\mu-126\varepsilon_5c^3\lambda\xi^2\eta^2+36\varepsilon_5c^4\xi\eta\mu\\
&-16362\varepsilon_5c^2\xi\eta^3\mu^3-1134\lambda\eta^6\mu^2-486\varepsilon_5c\lambda\eta^4\mu^2+8c^5\xi^2+468\varepsilon_5c^2\eta^4\mu^2=0.
\end{aligned}\tag{3.41}$$

Eliminating ξ and η from (3.39)–(3.41), we obtain a polynomial equation of μ. Therefore μ is a constant, which is a contradiction.

Applying Theorem 3.1, we have the following corollary for biharmonic hypersurfaces of $N_p^6(c)$.

Corollary 3.2 Let M_r^5 be a biharmonic hypersurface of $N_p^6(c)$ with diagonalizable shape operator and $c\varepsilon \leq 0$, where $\varepsilon = \langle \xi, \xi \rangle$ with ξ an unit normal vector field, then it must be minimal.

Proof According to Theorem 3.1, the mean curvature H is a constant. It follows from the first equation of (2.1) that

$$H(\mathrm{tr}A^2 - 5c\varepsilon) = 0. \tag{3.42}$$

Let $\mu_1, \mu_2, \cdots, \mu_5$ are principal curvatures of M_r^5. Assume that $H \neq 0$, then $\mu_k \neq 0$ for some $1 \leq k \leq 5$. Considering that the shape operator A is diagonalizable, we know $\mathrm{tr}A^2 = \sum_{i=1}^5 \mu_i^2 > 0$, which together with $c\varepsilon \leq 0$ tells us that $\mathrm{tr}A^2 - 5c\varepsilon > 0$. However, (3.42) implies that $\mathrm{tr}A^2 - 5c\varepsilon = 0$, a contradiction.

Remark From Corollary 3.1, the hypersurface M_r^5 of $N_p^6(c)$ is minimal when $c = 0$, or $c > 0$ and the normal vector field is time-like, or $c < 0$ and the normal vector field is space-like. Specially, when $r = p = 0$, the above results degenerate into the results in [15].

References

[1] Arvanitoyeorgos A, Defever F, Kaimakamis G. Hypersurfaces of $\mathbb{E}_s^4$ with proper mean curvature vector[J]. J. Math. Soc. Japan., 2007, 59(3): 797–809.

[2] Chen Bang-yen. Pseudo-Riemannian geometry δ-Invariants and applications[M]. Hackensack: World Scientific Publishing, 2011.

[3] Inoguchi J. Biminimal submanifolds in contact 3-manifolds[J]. Balkan J. Geom. Appl., 2007, 12(1): 56–67.

[4] Guan Zhida, Li Haizhong, Vrancken L. Four dimensional biharmonic hypersurfaces in nonzero space forms have constant mean curvature[J]. J. Geom. Phys., 2021, 160: 103984.

[5] Ou Yelin. Biharmonic hypersurfaces in Riemannian manifolds[J]. Pacific J. Math., 2010, 248(1): 217–232.

[6] Chen Bang-yen. Null 2-type surfaces in E^3 are circular cylinders[J]. Kodai Math. J., 1988, 11(2): 295–299.

[7] Ferrández A, Lucas P. On surfaces in the 3-dimensional Lorentz-Minkowski space[J]. Pacific J. Math., 1992, 152(1): 93–100.

[8] Defever F. Hypersurfaces of $\mathbb{E}^4$ satisfying $\Delta\vec{H} = \lambda\vec{H}$[J]. Michigan Math. J., 1997, 44(2): 355–363.

[9] Arvanitoyeorgos A, Kaimakamis G, Magid M. Lorentz hypersurfaces in E_1^4 satisfying $\Delta\vec{H} = \alpha\vec{H}$[J]. Illinois J. Math., 2009, 53(2): 581–590.

[10] Arvanitoyeorgos A, Kaimakamis G. Hypersurfaces of type M_2^3 in E_2^4 with proper mean curvature vector[J]. J. Geom. Phys., 2013, 63: 99–106.

[11] Fu Yu, Zhan Xin. Hypersurfaces satisfying $\Delta\vec{H} = \lambda\vec{H}$ in $\mathbb{E}^5$[J]. J. Math. Anal. Appl., 2021, 503(2): 125337.

[12] Yang Chao, Liu Jiancheng, Du Li. Four dimensional hypersurfaces with proper mean curvature vector field in pseudo-Riemannian space forms. arxiv: 2303. 02362v1.

[13] Du Li, Zhang Juan, Xie Xun. Hypersurfaces satisfying $\tau_2(\varphi) = \eta\tau(\varphi)$ in pseudo-Riemannian space forms[J]. Math. Phys. Anal. Geom., 2017, 20: 17.

[14] Du Li. On η-biharmonic hypersurfaces with constant scalar curvature in higher dimensional pseudo-Riemannian space forms[J]. J. Math. Anal. Appl., 2023, 518(1): 126670.

[15] Fu Yu, Hong Minchun, Zhan Xin. Biharmonic conjectures on hypersurfaces in a space form[J]. Trans. Amer. Math. Soc., 2023, 376(12): 8411–8445.

六维伪黎曼空间型中的 $\lambda-$双调和超曲面

段珍萍，杨　超，刘建成，陈佳蕊

(西北师范大学数学与统计学院, 甘肃 兰州 730070)

摘要: 本文研究了具有常截曲率 c 的六维伪黎曼空间型 $N_p^6(c)$ (指标 $0 \leq p \leq 6$) 中的 λ-双调和超曲面 M_r^5 (其中 $r = p - 1$ 或 p), 证明了当超曲面 M_r^5 的形状算子可对角化时, 其平均曲率必为常数. 应用该结论, 我们证得 $N_p^6(c)$ 中的一类双调和超曲面必定是极小的.

关键词: $\lambda-$双调和超曲面; 伪黎曼空间型; 常平均曲率; 形状算子; 极小

MR(2010)**主题分类号:** 53C50; 53C40 **中图分类号:** O186.12