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A NEW PROOF OF THE WEAK $(1,\frac{n}{n-\alpha})$ INEQUALITY FOR THE FRACTIONAL MAXIMAL OPERATORS

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Abstract: In this paper, we provide an alternative proof of the weak type $(1,\frac{n}{n-\alpha})$ inequality for the fractional maximal operators. By using the discretization technique, we can get the main result, which shows that the weak type $(1,\frac{n}{n-\alpha})$ bound of M_α is at worst $2^{n-\alpha}$. The weak type $(1,\frac{n}{n-\alpha})$ bound of M_α can be estimated more directly and easily in this method, which is different from the usual ways.

Keywords: Hardy-Littlewood maximal operator; fractional maximal operators; Dirac deltas; discrete method

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1 Introduction and the Main Result

For any $x \in \mathbb{R}^n$, $f \in L^1_{loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator over the Euclidean ball of radius r and centered at x is defined by

$$Mf(x) = \sup_{\epsilon>0} \frac{1}{|B(x,\epsilon)|} \int_{B(x,\epsilon)} |f(y)|dy,$$

where $|B(x,\epsilon)|$ is the Lebesgue measure of the ball. It is well known that the Hardy-Littlewood maximal operator is of weak type (1,1). That is, for any $\lambda > 0$, there exists a positive constant C_n only depends on n such that

$$|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \le \frac{C_n}{\lambda} \|f\|_{L_1}. \tag{1.1}$$

For any function f on $\mathbb{R}^n$ and $\epsilon > 0$, we denote $f_\epsilon(x) = \frac{1}{\epsilon^n} f(\frac{x}{\epsilon})$, $k(\cdot) = \frac{1}{v_n}\chi_{B(0,1)}(\cdot)$. Then, Hardy-Littlewood maximal operator can be written as

$$Mf(x) = \sup_{\epsilon>0} \frac{1}{v_n \epsilon^n} \int_{\mathbb{R}^n} |f(x-y)| \chi_{B(0,1)}(\frac{y}{\epsilon})dy = \sup_{\epsilon>0} (|f| * k_\epsilon)(x),$$

where v_n is the volume of the unit ball. The usual way to prove (1.1) is based on Vitali type covering lemma, for details see [1]. Until 1981, Guzmán introduced the discretization technique in [2, Theorem 4.1.1].

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Lemma 1.1 ([2]) Let $\{k_j\}_{j=1}^{\infty} \subseteq L^1(\Omega)$ be an ordinary sequence of functions, $\{K_j\}_{j=1}$ the sequence of convolution operators associated to it, and K^* the corresponding maximal operator. Then K^* is of weak type (1,1) if and only if K^* is of weak type (1,1) over finite sums of Dirac deltas.

Hence, by Lemma 1.1, we only need to prove (1.1) holds for combinations of finite delta functions, then the weak type (1,1) inequality of M holds for every $f \in L^1(\mathbb{R}^n)$. Based on this fact, Carlesson [3] proved that the Hardy-Littlewood maximal operator is of weak type (1,1) in another way. Other applications of discretization technique can be found in [4], [5], [6], [7] and [8].

For $f \in L^1_{loc}(\mathbb{R}^n)$ and $0 < \alpha < n$, the fractional maximal operator is defined as

$$M_\alpha f(x) = \sup_{\epsilon > 0} \frac{1}{|B(x,\epsilon)|^{1-\frac{\alpha}{n}}} \int_{B(x,\epsilon)} |f(y)| dy.$$

And for any $\lambda > 0$,

$$|\{x \in \mathbb{R}^n : M_\alpha f(x) > \lambda\}| \le \big(\frac{C_n \|f\|_1}{\lambda}\big)^{\frac{n}{n-\alpha}}, \tag{1.2}$$

where C_n is a constant only depending on n. It means that M_α is bounded from L^1 to $L^{\frac{n}{n-\alpha},\infty}$. The usual way to prove (1.2) is by using Riesz potential I_α to dominate M_α in some sense, that is

$$M_\alpha f(x) \le \gamma(\alpha) \cdot I_\alpha(|f|)(x),$$

where $\gamma(\alpha) = \pi^{\frac{n}{2}} 2^\alpha \Gamma(\frac{\alpha}{2}) / \Gamma(\frac{n-2}{2})$ and $I_\alpha(f)(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$. For an in-depth understand of fractional integral operators, one can refer to [9] and [10].

Motivated by the work of Guzmán[2], Carlesson[3], we want to use the discrete method to prove the weak type $(1, \frac{n}{n-\alpha})$ inequality for M_α. The fractional maximal operator also can be written as the convolution form,

$$\begin{aligned} M_\alpha f(x) &= \sup_{\epsilon > 0} \frac{1}{|B(0,\epsilon)|^{1-\frac{\alpha}{n}}} \int_{B(0,\epsilon)} |f(x-y)| dy \\ &= \sup_{\epsilon > 0} \frac{1}{v_n^{\frac{n-\alpha}{n}} \epsilon^{n-\alpha}} \int_{\mathbb{R}^n} |f(x-y)| \chi_{B(0,\epsilon)}(y) dy \\ &= \sup_{\epsilon > 0} \frac{\epsilon^\alpha}{v_n^{\frac{n-\alpha}{n}} \epsilon^n} \int_{\mathbb{R}^n} |f(x-y)| \chi_{B(0,1)}(\frac{y}{\epsilon}) dy \\ &= \sup_{\epsilon > 0} (|f| * k_\epsilon)(x), \end{aligned}$$

where $k_\varepsilon(\cdot) = \frac{\epsilon^\alpha}{v_n^{\frac{n-\alpha}{n}}} \chi_{B(0,1)}(\cdot)$.

To prove our main result, one key lemma is needed.

Lemma 1.2 Let $\{k_j\}_{j=1}^{\infty} \subseteq L^1(\Omega)$ be an ordinary sequence of functions, $\{K_j\}_{j=1}$ the sequence of convolution operators associated to it, and K^* the corresponding maximal operator. Then K^* is of weak type $(1, \frac{n}{n-\alpha})$ if and only if K^* is of weak type $(1, \frac{n}{n-\alpha})$ over finite sums of Dirac deltas.

Proof By modifying the proof process of Lemma 1.1 step by step, we can prove Lemma 1.2, we omit the proof here.

Let $\mu = \sum\limits_{i=1}^{N} \delta_{t_i}$ and $\|\mu\|_1 = N$, where δ_{t_i} is the Dirac delta function at t_i. For any $\lambda > 0$, (1.2) is equivalent to

$$|\{x \in \mathbb{R}^n : M_\alpha \mu(x) > \lambda\}| \le \big(\frac{C_n N}{\lambda}\big)^{\frac{n}{n-\alpha}}.$$

The main result of the paper is the following.

Theorem 1.3 For any $\lambda > 0$, $|\{x \in \mathbb{R}^n : M_\alpha \mu(x) > \lambda\}| \le \big(\frac{2^{n-\alpha}N}{\lambda}\big)^{\frac{n}{n-\alpha}}$.

Remark 1 Theorem 1.3 shows that the weak type $(1,\frac{n}{n-\alpha})$ bound of M_α is at worst $2^{n-\alpha}$. By using the discrete method, the weak type $(1,\frac{n}{n-\alpha})$ bound of M_α can be estimated more directly and easily, which is different from other ways, see for example [9].

Remark 2 We get the Hardy-Littlewood maximal operator M is of weak type (1,1) and the weak (1,1) bound is at worst 2^n if $\alpha \to 0$.

2 Proof of Theorem 1.3

Proof Denote $E_\lambda = \{x \in \mathbb{R}^n : M_\alpha \mu(x) > \lambda\}$. The proof is by induction on N.

When $N = 1$, $\mu = \delta_{t_1}$,

$$M_\alpha \mu(x) = \sup_{\epsilon > 0} \frac{1}{|B(x,\epsilon)|^{1-\frac{\alpha}{n}}} \int_{B(x,\epsilon)} \delta_{t_1}(y)dy = \frac{1}{(v_n^{1/n}|x-t_1|)^{n-\alpha}},$$

then $E_\lambda = B\big(t_1, \frac{1}{v_n^{\frac{1}{n}} \lambda^{\frac{1}{n-\alpha}}}\big)$. Hence, $|E_\lambda| = \big(\frac{1}{\lambda}\big)^{\frac{n}{n-\alpha}} \le 2^n \big(\frac{1}{\lambda}\big)^{\frac{n}{n-\alpha}}$.

When $N > 1$, $\forall \lambda > 0$, we can choose a ball $B(x, \epsilon_x)$ for any $x \in E_\lambda$ such that

$$|B(x,\epsilon_x)|^{1-\frac{\alpha}{n}} < \frac{1}{\lambda} \int_{B(x,\epsilon_x)} |\mu(y)|dy. \tag{2.3}$$

Let $\epsilon = \sup\limits_{x \in E_\lambda} \epsilon_x$, then $0 < \epsilon < \infty$. Morever, fix $0 < \gamma < 1$, we can choose $x_0 \in E_\lambda$ such that $\epsilon_{x_0} > \gamma\epsilon$.

Set

$$\mu'(x) = \sum_{t_i \in B(x_0,\epsilon_{x_0})} \delta_{t_i}, \quad \mu''(x) = \sum_{t_i \in B(x,\epsilon_x), t_i \notin B(x_0,\epsilon_{x_0})} \delta_{t_i}.$$

Then $\mu = \mu' + \mu''$.

If $x \in E_\lambda \backslash B(x_0, \frac{2}{\gamma}\epsilon_{x_0})$, we have

$$|x - x_0| \ge \frac{2}{\gamma}\epsilon_{x_0} > 2\epsilon \ge \epsilon + \epsilon_{x_0}.$$

Thus, $B(x_0,\epsilon_{x_0})\cap B(x,\epsilon_x)=\varnothing$. By (2.3), we get

$$\begin{aligned}\lambda &< \frac{1}{|B(x,\epsilon_x)|^{1-\frac{\alpha}{n}}}\int_{B(x,\epsilon_x)}|\mu(y)|dy\\ &=\frac{1}{|B(x,\epsilon_x)|^{1-\frac{\alpha}{n}}}\sum_{t_i\in B(x,\epsilon_x)}1\\ &=\frac{1}{|B(x,\epsilon_x)|^{1-\frac{\alpha}{n}}}\sum_{t_i\in B(x,\epsilon_x),t_i\in B(x_0,\epsilon_{x_0})}1+\frac{1}{|B(x,\epsilon_x)|^{1-\frac{\alpha}{n}}}\sum_{t_i\in B(x,\epsilon_x),t_i\notin B(x_0,\epsilon_{x_0})}1\\ &=\frac{1}{|B(x,\epsilon_x)|^{1-\frac{\alpha}{n}}}\sum_{t_i\in B(x,\epsilon_x),t_i\notin B(x_0,\epsilon_{x_0})}1\\ &=\frac{1}{|B(x,\epsilon_x)|^{1-\frac{\alpha}{n}}}\int_{B(x,\epsilon_x)}|\mu''(y)|dy\\ &=M\mu''(x).\end{aligned}$$

That is, $M\mu''(x)>\lambda$.

Therefore, by the induction hypothesis, we have

$$\big|E_\lambda\backslash B(x_0,\frac{2}{\gamma}\epsilon_{x_0})\big|\le 2^n\big(\frac{\|\mu''\|_1}{\lambda}\big)^{\frac{n}{n-\alpha}}. \tag{2.4}$$

By using the following fact

$$\big|B(x_0,\frac{2}{\gamma}\epsilon_{x_0})\big|=\big(\frac{2}{\gamma}\big)^n|B(x_0,\epsilon_{x_0})|$$

and

$$|B(x_0,\epsilon_{x_0})|^{1-\frac{\alpha}{n}}\le\frac{1}{\lambda}\int_{B(x_0,\epsilon_{x_0})}|\mu(y)|dy=\frac{1}{\lambda}\sum_{t_i\in B(x_0,\epsilon_{x_0}}\delta_{t_i}(y)dy=\frac{1}{\lambda}\int_{\mathbb{R}^n}|\mu'(y)|dy=\frac{\|\mu'\|_1}{\lambda},$$

we get the estimate

$$\big|B(x_0,\frac{2}{\gamma}\epsilon_{x_0})\big|\le(\frac{2}{\gamma})^n\big(\frac{\|\mu'\|_1}{\lambda}\big)^{\frac{n}{n-\alpha}}. \tag{2.5}$$

Consequently, from (2.4) and (2.5), we get

$$\begin{aligned}|E_\lambda|&\le|E_\lambda\backslash B(x_0,\frac{2}{\gamma}\epsilon_{x_0})\big|+|B(x_0,\frac{2}{\gamma}\epsilon_{x_0})\big|\\ &\le 2^n\big(\frac{\|\mu''\|_1}{\lambda}\big)^{\frac{n}{n-\alpha}}+(\frac{2}{\gamma})^n\big(\frac{\|\mu'\|_1}{\lambda}\big)^{\frac{n}{n-\alpha}}\\ &\le(\frac{2}{\gamma})^n\big(\frac{\|\mu''\|_1}{\lambda}\big)^{\frac{n}{n-\alpha}}+(\frac{2}{\gamma})^n\big(\frac{\|\mu'\|_1}{\lambda}\big)^{\frac{n}{n-\alpha}}\\ &\le(\frac{2}{\gamma})^n\big(\frac{\|\mu\|_1}{\lambda}\big)^{\frac{n}{n-\alpha}}.\end{aligned}$$

Finally, let $\gamma\to 1$. Then we conclude

$$|E_\lambda|\le 2^n\big(\frac{N}{\lambda}\big)^{\frac{n}{n-\alpha}}.$$

We complete the proof of Theorem 1.3.

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分数次极大算子弱$(1,\frac{n}{n-\alpha})$不等式的一种新证明

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摘要: 本文给出了分数次极大算子弱型$(1,\frac{n}{n-\alpha})$不等式的另一种证明. 利用离散化方法，我们可以得到M_α的弱型$(1,\frac{n}{n-\alpha})$ 界最大是$2^{n-\alpha}$.这种方法与通常的做法不同, 可以更加简单直接地估计出M_α的弱型$(1,\frac{n}{n-\alpha})$界.

关键词: Hardy-Littlewood 极大算子; 分数次极大算子; Dirac delta 函数; 离散化方法

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