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VANISHING THEOREMS OF THE BASIC P-HARMONIC FORMS ON COMPLETE FOLIATED RIEMANNIAN MANIFOLDS

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Abstract: In this paper, we study the basic p-harmonic forms on the complete foliated Riemannian manifolds. By using the method in [1], we show that if the basic mean curvature form is bounded and co-closed, and the transversal curvature operator is nonnegative and positive at least one point, then we obtain a vanishing theorem for L^p-integrably p-harmonic r-forms.

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1 Introduction

An interesting problem in Riemannian geometry is to study the relationship between the curvature and the topology of a Riemannian manifold. It is well-known that the space of harmonic r-forms is isomorphic to its r-th de Rham cohomology group for compact manifolds. When the manifold is non-compact, it is natural to study the square integrable harmonic forms. So it is an interesting problem in geometry and topology to find sufficient conditions on a complete manifold such that we can obtain vanishing theorems of harmonic forms.

Let $(M, g, \mathcal{F})$ be a $(n+m)$-dimensional complete foliated Riemannian manifold with a foliation $\mathcal{F}$ of codimention m and a bundle-like metric g with respect to $\mathcal{F}$. Let d_B be the restriction of the exterior differential operator d on the basic forms $\Omega^r_B(\mathcal{F})$. Then the formal dual operator of d_B is defined by

$$\delta_B = (-1)^{m(r+1)+1} \star_B (d_B - \kappa_B \wedge) \star_B,$$

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where $\star_B$ is the basic Hodge star operator and κ_B is the basic part of the mean curvature form κ. Then the basic Laplician operator Δ_B acting on the space of basic r-forms $\Omega_B^r(\mathcal{F})$ is given by

$$\Delta_B = d_B\delta_B + \delta_B d_B.$$

A basic r-form $\omega \in \Omega_B^r$ is said to be harmonic if $\Delta_B\omega = 0$. It is well-known that ω is harmonic if and only if $d_B\omega = 0$ and $\delta_B\omega = 0$ when M is compact. Hence, for any $p \geq 2$, we say a basic r-form $\omega \in \Omega_B^r(\mathcal{F})$ is a basic p-harmonic r-form if it satisfies the following properties:

$$\begin{cases} d_B\omega = 0, \\ \delta_B(|\omega|^{p-2}\omega) = 0. \end{cases}$$

It is easy to see that when $p = 2$ and M is compact, a basic p-harmonic r-form is exactly a basic harmonic r-form.

It is worth noting that the main tools to study the spaces of harmonic r-forms are the Bochner–Weizenböck type formulas. For basic harmonic forms, some vanishing theorems are obtained on compact foliated Riemannian manifolds. For example, M. Min-Oo et al. ([2]) proved that if the transversal curvature operator of $\mathcal{F}$ is positive definite, there is no non-trivial basic harmonic r-forms on a closed foliated Riemannian manifold. Recently, Jung–Liu ([3]) studied the basic r-forms on a complete foliated Riemannian manifold. Under the assumption that the mean curvature form is bounded and co-closed, they obtained some vanishing theorems on complete foliated Riemannian manifolds.

For p-harmonic 1-forms, Zhang ([4]) proved that if M is a complete non-compact Riemannian manifold with non-negative Ricci curvature, there is non-trivial L^q p-harmonic 1-form for $p > 1$ and $0 < q < \infty$. Later, inspired by Zhang's result, Chang–Guo–Sung ([5]) extended Zhang's result and obtained the compactness for any bounded set of p-harmonic 1-forms. In 2017, Dung ([1]) obtained some vanishing theorems for L^p p-harmonic r-forms on complete non-compact sub-manifolds of Euclidean space. For further details, the readers can refer to [6–8] and the references therein.

Motivated by these results, in this paper, our aim is to study basic L^p p-harmonic r-forms on a complete foliated Riemannian manifold with nonnegative transversal curvature operator. As usual, we define the space of the basic p-harmonic r-forms on M by

$$\mathcal{H}_B^{r,p}(L^p(M)) = \{\omega \in \Omega_B^r(\mathcal{F}) | d_B\omega = 0, \delta_B(|\omega|^{p-2}\omega) = 0, \int_M |\omega|^p \mu_M < \infty\}.$$

In fact, we establish the following vanishing theorem.

Theorem 1.1 Let $(M, g, \mathcal{F})$ be a foliated Riemannian manifold with all leaves compact. Assume that the basic part of the mean curvature form is bounded and co-closed. If the transversal curvature operator of $\mathcal{F}$ is nonnegative and positive at least one point, any basic L^p p-harmonic r-form is trivial.

We should remark that our main theorem generalized Jun–Liu's result ([3, Main Theorem]) when $p = 2$.

The rest of this paper is organized as follows. In Section 2, we recall some preliminary knowledge about foliated Riemannian manifolds and some useful lemmas. In Section 3, we will give a detailed proof of our main Theorem 1.1.

2 Preliminary

Let $(M, g, \mathcal{F})$ be a $(n+m)$-dimensional complete foliated Riemannian manifold with a foliation $\mathcal{F}$ of co-dimension m and a bundle-like metric g with respect to $\mathcal{F}$. Let TM be the tangent bundle of M, $T\mathcal{F}$ its integrable subbundle given by $\mathcal{F}$, and $Q = TM/T\mathcal{F}$ the corresponding normal bundle of $\mathcal{F}$. Then we have an exact sequence of vector bundles

$$0 \to T\mathcal{F} \to TM \xrightarrow{\pi} Q \to 0,$$

where $\pi : TM \to Q$ is a projection. Then we have a bundle map $\sigma : Q \to T\mathcal{F}^{\perp} \subset TM$ satisfying $\pi \circ \sigma = \mathrm{Id}$.

Let g_Q be the holonomy invariant metric on Q induced by g. That is, for any vector fields $X \in \Gamma(T\mathcal{F})$, we have $\mathcal{L}_X g_Q = 0$, where $\mathcal{L}_X$ is the transverse Lie derivative. Let $\nabla = \nabla^Q$ be the transverse Levi-Civita connection on the normal bundle Q ([9, 10]). In fact, for any $s \in \Gamma(Q)$ and $Y_s = \sigma(s) \in \Gamma(T\mathcal{F}^{\perp})$, the connection ∇ is defined by

$$\nabla_X s = \begin{cases} \pi([X, Y_s]), & \text{for} \quad X \in \Gamma(T\mathcal{F}), \\ \pi(\nabla^M_X Y_s), & \text{for} \quad X \in \Gamma(T\mathcal{F}^{\perp}), \end{cases}$$

where ∇^M is the Levi-Civita connection with respect to the Riemannian metric g on M. Then the transverse curvature tensor R^Q of ∇ is given by

$$R^Q(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s,$$

for any $X, Y \in \Gamma(TM)$ and $s \in \Gamma(Q)$. Let Ric^Q be the transversal Ricci operator of $\mathcal{F}$ with respect to the transversal Levi-Civita connection ∇.

A differential form $\omega \in \Omega^r(M)$ is called basic, if for any $X \in \Gamma(T\mathcal{F})$, we have

$$\iota_X \omega = 0, \quad \iota_X d\omega = 0,$$

where ι_X is the interior product with respect to X. In a local chart $(x^1, \cdots, x^n; y^1, \cdots, y^m)$ of $\mathcal{F}$, a basic r-form ω can be written as

$$\omega = \sum_{i_1 < \cdots < i_r} \omega_{i_1 \cdots i_r}(y^1, \cdots, y^m) dy^{i_1} \wedge \cdots \wedge dy^{i_r},$$

where the coefficient functions $\omega_{i_1 \cdots i_r}$ are independent of x. We denote $\Omega^r_B(\mathcal{F})$ by the set of all basic r-forms on M. Then we have $\Omega^r(M) = \Omega^r_B(\mathcal{F}) \oplus \Omega^r_B(\mathcal{F})^{\perp}$.

Let $\star_B : \Omega^r_B(\mathcal{F}) \to \Omega^{m-r}_B(\mathcal{F})$ be the basic Hodge star operator defined by

$$\star_B \omega = (-1)^{n(m-r)} \star (\omega \wedge \chi_{\mathcal{F}}), \forall \omega \in \Omega^r_B(\mathcal{F}),$$

where $\star$ is the Hodge star operator with respect to g and $\chi_{\mathcal{F}}$ is the characteristic form of $\mathcal{F}$. Then we can easily check that, for any basic r-forms $\alpha, \beta \in \Omega_B^r(\mathcal{F})$,

$$\alpha \wedge \star_B \beta = \beta \wedge \star_B \alpha, \quad \star_B^2 \alpha = (-1)^{r(m-r)} \alpha.$$

Let ν be the transversal volume form such that $\star\nu = \chi_{\mathcal{F}}$, The point-wise inner product $\langle \cdot, \cdot \rangle$ on $\Omega_B^r(\mathcal{F})$ is given by

$$\langle \alpha, \beta \rangle \nu = \alpha \wedge \star_B \beta,$$

where $\alpha,\ \beta \in \Omega_B^r(\mathcal{F})$. So the global inner product can be defined by

$$(\alpha, \beta)_B = \int_M \langle \alpha, \beta \rangle \mu_M = \int_M \alpha \wedge \star_B \beta \wedge \chi_{\mathcal{F}},$$

where $\mu_M = \nu \wedge \chi_{\mathcal{F}}$ is the volume form.

Let d_B be the restriction of d to the basic forms, i.e., $d_B = d|_{\Omega_B^*(\mathcal{F})}$. Then the operator $\delta_B : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r-1}(\mathcal{F})$ is defined by

$$\delta_B \omega = (-1)^{m(r+1)+1} \star_B (d_B - \kappa_B \wedge) \star_B \omega,$$

where κ is the mean curvature form of $\mathcal{F}$ and κ_B is the basic part of κ ([11]). It is well that δ_B is the formal adjoint of d_B with respect to the global inner product $(\cdot, \cdot)_B$ ([12]). In general, δ_B is not a restriction of δ on $\Omega_B^r(\mathcal{F})$, i.e., $\delta_B \neq \delta|_{\Omega_B^r(\mathcal{F})}$, where δ is the formal adjoint of d. But for any basic 1-form ω, we have $\delta_B \omega = \delta \omega$. So the basic Laplacian Δ_B acting on $\Omega_B^*(\mathcal{F})$ is defined by $\Delta_B = d_B \delta_B + \delta_B d_B$. It is well known that $\Delta|_{\Omega_B^0(\mathcal{F})} = \Delta_B$, where δ is the Beltrami Laplacian of M ([13]).

In order to obtain the vanishing theorems for basic p-harmonic r-forms, we need the following lemmas.

Lemma 2.2 ([14, Lemma 3.2]) Let $\mathcal{F}$ be a Riemannian foliation. Then the operators d_B and δ_B on $\Omega_B^*(\mathcal{F})$ are given by

$$d_B = \sum_{a=1}^m \theta^a \wedge \nabla_{E_a}, \quad \delta_B = -\sum_{a=1}^m \iota_{E_a} \nabla_{E_a} + \iota_{\kappa_B^\sharp},$$

where $\{E_a\}_{a=1}^m$ is a local orthonormal basic frame in Q and $\{\theta^a\}_{a=1}^m$ its g_Q-dual 1-form.

Proposition 2.1 For any basic form $\phi \in \Omega_B^r(\mathcal{F})$, the operator δ_B can be also written as

$$\delta_B \phi = (-1)^{m(r+1)+1} \star_B d_B \star_B \phi + \iota_{\kappa_B^\sharp} \phi.$$

Furthermore, for any function $f \in C^\infty(M)$, we have

$$\delta_B(f\phi) = f\delta_B \phi + (-1)^{m(r+1)+1} \star_B (d_B f \wedge \star_B \phi).$$

Proof By direct calculation, we have

$$\begin{aligned}\delta_B(f\phi) =&(-1)^{m(r+1)+1}\star_B d_B \star_B (f\phi) + f\iota_{\kappa_B^\sharp}\phi\\ =&(-1)^{m(r+1)+1}\star_B d_B(f\star_B \phi) + f\iota_{\kappa_B^\sharp}\phi\\ =&(-1)^{m(r+1)+1}\star_B (d_B f\wedge \star_B\phi + f d_B\star_B \phi) + f\iota_{\kappa_B^\sharp}\phi\\ =&f\delta_B\phi + (-1)^{m(r+1)+1}\star_B (d_B f\wedge\star_B\phi).\end{aligned}$$

Lemma 2.3 On a Riemannian foliation $\mathcal{F}$, we have that for any $\phi\in\Omega_B^r(\mathcal{F})$,

$$\Delta_B\phi = \nabla_{\rm tr}^*\nabla_{\rm tr}\phi + F(\phi) + A_{\kappa_B^\sharp}\phi,$$

where $A_{\kappa_B^\sharp}\phi = L_{\kappa_B^\sharp}\phi - \nabla_{\kappa_B^\sharp}\phi$, $L_{\kappa_B^\sharp} = d_B\iota(\kappa_B^\sharp) + \iota(\kappa_B^\sharp)d_B$, and F is the transverse curvature operator locally given by

$$F(\phi) = \sum_{a,b=1}^{m}\theta^a\wedge\iota_{E_b}R^Q(E_b,E_a)\phi.$$

Furthermore, since $\frac{1}{2}\Delta_B|\phi|^2 = \langle\nabla_{\rm tr}^*\nabla_{\rm tr}\phi,\phi\rangle - |\nabla_{\rm tr}\phi|^2$, we obtain that for any $\phi\in\Omega_B^r(\mathcal{F})$,

$$\frac{1}{2}\Delta_B|\phi|^2 = \langle\Delta_B\phi,\phi\rangle - |\nabla_{\rm tr}\phi|^2 - \langle F(\phi),\phi\rangle - \langle A_{\kappa_B^\sharp}\phi,\phi\rangle.$$

Remark 1 The above operator $A_{\kappa_B^\sharp}:\Omega_B^r(\mathcal{F})\to\Omega_B^r(\mathcal{F})$ is $C^\infty(M)$ linear. That means for every $f\in C^\infty(M)$, it holds $A_{\kappa_B^\sharp}(f\phi) = fA_{\kappa_B^\sharp}(\phi)$.

Proof

$$A_{\kappa_B^\sharp}(f\phi) = d_B\iota(\kappa_B^\sharp)(f\phi) + \iota(\kappa_B^\sharp)d_B(f\phi) - \nabla_{\kappa_B^\sharp}(f\phi) =: I_1 + I_2 - I_3.$$

By direct calculation, we have

$$\begin{aligned}I_1 :=& d_B\iota(\kappa_B^\sharp)(f\phi) = d_B(f\iota(\kappa_B^\sharp)\phi) = d_Bf\wedge[\iota(\kappa_B^\sharp)\phi] + fd_B\iota(\kappa_B^\sharp)\phi,\\ I_2 :=&\iota(\kappa_B^\sharp)d_B(f\phi) = \iota(\kappa_B^\sharp)(d_Bf\wedge\phi + fd_B\phi)\\ =&\kappa_B^\sharp(f)\varphi - d_Bf\wedge[\iota(\kappa_B^\sharp)\phi] + f\iota(\kappa_B^\sharp)d_B\phi\\ I_3 :=&\nabla_{\kappa_B^\sharp}(f\phi) = \kappa_B^\sharp(f)\phi + f\nabla_{\kappa_B^\sharp}\phi.\end{aligned}$$

And then

$$I_1 + I_2 - I_3 = fd_B\iota(\kappa_B^\sharp)\phi + f\iota(\kappa_B^\sharp)d_B\phi - f\nabla_{\kappa_B^\sharp}\phi = fA_{\kappa_B^\sharp}(\phi).$$

Lemma 2.4 ([15, Lemma 3.4]) For any d_B-closed r-form ϕ and $f\in C^\infty(M)$, we have

$$|d_B(f\phi)| = |d_Bf\wedge\phi| \le |d_Bf|\cdot|\phi|. \tag{2.1}$$

Lemma 2.5 ([15, Lemma 3.5]) Let ϕ be a basic r-form and $\kappa_B^\sharp$ the dual vector field of κ_B.Then we have

$$|\iota_{\kappa_B^\sharp}\phi| \le |\kappa_B^\sharp||\phi| \le |\kappa_B||\phi|.$$

3 Proof of Theorem 1.1

Proof Let ω be any basic p-harmonic r-form on M, $1\leq r\leq m-1$. Then we have

$$\begin{cases} d_B\omega=0,\\ \delta_B(|\omega|^{p-2}\omega)=0. \end{cases}$$

Applying the Bochner formula to the form $|\omega|^{p-2}\omega$ and Lemma 1 we obtain

$$\begin{aligned}\frac{1}{2}\Delta_B|\omega|^{2p-2}=&\langle\Delta_B(|\omega|^{p-2}\omega),|\omega|^{p-2}\omega\rangle-|\nabla_{\rm tr}(|\omega|^{p-2}\omega)|^2\\ &-\langle F(|\omega|^{p-2}\omega),|\omega|^{p-2}\omega\rangle-|\omega|^{2p-4}\langle A_{\kappa_B^\sharp}(\omega),\omega\rangle.\end{aligned}$$

On the other hand, it holds

$$\frac{1}{2}\Delta_B|\omega|^{2p-2}=\frac{1}{2}\Delta_B(|\omega|^{p-1})^2=|\omega|^{p-1}\Delta_B|\omega|^{p-1}-|d_B|\omega|^{p-1}|^2.$$

By the first Kato's inequality, we have $|\nabla_{\rm tr}(|\omega|^{p-2}\omega)|\geq|d_B|\omega|^{p-1}|$. Then, we have

$$|\omega|\Delta_B|\omega|^{p-1}+|\omega|^{p-2}\langle F(\omega),\omega\rangle\leq\langle\Delta_B(|\omega|^{p-2}\omega),\omega\rangle-|\omega|^{p-2}\langle A_{\kappa_B^\sharp}(\omega),\omega\rangle. \tag{3.1}$$

Let $B_l=\{y\in M|\rho(y)\leq l\}$, where $\rho(y)$ is the distance between leaves through y and a fixed point. Then we can define a basic function φ_l on M satisfying the following properties:

$$\begin{cases} 0\leq\varphi_l(y)\leq 1, & \text{for any}\quad y\in M,\\ \operatorname{supp}\varphi_l\subset B_{2l}, & \\ \varphi_l(y)=1, & \text{for any}\quad y\in B_l, \end{cases} \tag{3.2}$$

and

$$\lim_{l\to+\infty}\varphi_l=1,\quad |d_B\varphi_l|\leq\frac{C_1}{l}, \tag{3.3}$$

where C_1 is a positive constant independent of l. ([16]) In the following, we will denote φ_l by φ without confusion.

Then multiplying both sides of inequality (3.1) by φ^2 and integrating over M, we obtain

$$\begin{aligned}&\int_M\varphi^2|\omega|\Delta_B|\omega|^{p-1}\mu_M+\int_M|\omega|^{p-2}\langle F(\omega),\varphi^2\omega\rangle\mu_M\\ \leq&\int_M\langle\Delta_B(|\omega|^{p-2}\omega),\varphi^2\omega\rangle\mu_M-\int_M\varphi^2|\omega|^{p-2}\langle A_{\kappa_B^\sharp}(\omega),\omega\rangle\mu_M.\end{aligned} \tag{3.4}$$

In order to express the following estimate explicitly, we set

$$\begin{aligned}L_1&:=\int_M\varphi^2|\omega|\Delta_B|\omega|^{p-1}\mu_M,\\ R_1&:=\int_M\langle\Delta_B(|\omega|^{p-2}\omega),\varphi^2\omega\rangle\mu_M,\\ R_2&:=\int_M\varphi^2|\omega|^{p-2}\langle A_{\kappa_B^\sharp}(\omega),\omega\rangle\mu_M.\end{aligned}$$

Step 1: We compute the above three terms separately.

Firstly, we deal with L_1 :

$$\begin{aligned}L_1=&\int_M\langle d_B(\varphi^2|\omega|),d_B|\omega|^{p-1}\rangle\mu_M\\=&2(p-1)\int_M\varphi|\omega|^{p-1}\langle d_B\varphi,d_B|\omega|\rangle\mu_M+(p-1)\int_M\varphi^2|\omega|^{p-2}|d_B|\omega||^2\mu_M\\\geq&-2(p-1)\int_M\varphi|\omega|^{p-1}|d_B\varphi|\cdot\big|d_B|\omega|\big|\mu_M+(p-1)\int_M\varphi^2|\omega|^{p-2}|d_B|\omega||^2\mu_M.\end{aligned}$$

By Young's inequality, we have

$$\begin{aligned}&\varphi|\omega|^{p-1}|d_B\varphi|\cdot|d_B|\omega||=|\omega|^{p-2}(\varphi|d_B|\omega||)\cdot(|d_B\varphi||\omega|)\\\leq&|\omega|^{p-2}\left(\frac{\epsilon_1}{2}\varphi^2|d_B|\omega||^2+\frac{1}{2\epsilon_1}|d_B\varphi|^2|\omega|^2\right),\end{aligned}\tag{3.5}$$

where $\epsilon_1>0$ is a constant. Then we have

$$\begin{aligned}L_1\geq&-(p-1)\epsilon_1\int_M\varphi^2|\omega|^{p-2}|d_B|\omega||^2\mu_M-\frac{p-1}{\epsilon_1}\int_M|d_B\varphi|^2|\omega|^p\mu_M\\&+(p-1)\int_M\varphi^2|\omega|^{p-2}|d_B|\omega||^2\mu_M\\=&(p-1)(1-\epsilon_1)\int_M\varphi^2|\omega|^{p-2}|d_B|\omega||^2\mu_M-\frac{p-1}{\epsilon_1}\int_M|d_B\varphi|^2|\omega|^p\mu_M.\end{aligned}\tag{3.6}$$

Then we compute R_1 :

$$\begin{aligned}R_1=&\int_M\langle d_B(|\omega|^{p-2}\omega),d_B(\varphi^2\omega)\rangle\mu_M\\=&2(p-2)\int_M|\omega|^{p-3}\varphi\langle d_B|\omega|\wedge\omega,d_B\varphi\wedge\omega\rangle\mu_M\\\leq&2(p-2)\int_M\varphi|\omega|^{p-3}|d_B|\omega|\wedge\omega||d_B\varphi\wedge\omega|\mu_M\\\leq&2(p-2)\int_M\varphi|\omega|^{p-1}|d_B\varphi|\cdot\big|d_B|\omega|\big|\mu_M\\\leq&(p-2)\epsilon_1\int_M\varphi^2|\omega|^{p-2}|d_B|\omega||^2\mu_M+\frac{p-2}{\epsilon_1}\int_M|d_B\varphi|^2|\omega|^p\mu_M,\end{aligned}\tag{3.7}$$

where we used the Young's inequality (3.5) again and ϵ_1 is a positive constant.

Substituting (3.6) and (3.7) into (3.4), we have

$$\begin{aligned}&A\int_M\varphi^2|\omega|^{p-2}|d_B|\omega||^2\mu_M+\int_M|\omega|^{p-2}\langle F(\omega),\varphi^2\omega\rangle\mu_M\\\leq&B\int_M|d_B\varphi|^2|\omega|^p\mu_M-\int_M\varphi^2|\omega|^{p-2}\langle A_{\kappa_B^\sharp}(\omega),\omega\rangle\mu_M,\end{aligned}\tag{3.8}$$

where $A:=A(\epsilon_1)=(p-1)-(2p-3)\epsilon_1$ and $B:=B(\epsilon_1)=\frac{2p-3}{\epsilon_1}$. Since $p-1>0$, we can choose proper $\epsilon_1>0$ such that $A>0$.

At last, we calculate R_2 :

$$\begin{aligned}R_2 :=&\int_M \varphi^2|\omega|^{p-2}\langle A_{\kappa_B^\sharp}(\omega),\omega\rangle\mu_M\\ =&\int_M \langle d_B(\iota_{\kappa_B^\sharp}\omega),(\varphi^2|\omega|^{p-2}\omega)\rangle\mu_M-\int_M \varphi^2|\omega|^{p-2}\langle\nabla_{\kappa_B^\sharp}\omega,\omega\rangle\mu_M\\ =:&R_{2.1}-R_{2.2}.\end{aligned}$$

From the Proposition 2.1, we have

$$\begin{aligned}R_{2.1}=&\int_M \langle d_B\iota_{\kappa_B^\sharp}\omega,(\varphi^2|\omega|^{p-2}\omega)\rangle\mu_M\\ =&\int_M \langle\iota_{\kappa_B^\sharp}\omega,\delta_B(\varphi^2|\omega|^{p-2}\omega)\rangle\mu_M\\ =&\int_M \langle\iota_{\kappa_B^\sharp}\omega,\varphi^2\delta_B(|\omega|^{p-2}\omega)\rangle\mu_M+\int_M \langle\iota_{\kappa_B^\sharp}\omega,(-1)^{m(r+1)+1}\star_B\left(2\varphi d_B\varphi\wedge\star_B(|\omega|^{p-2}\omega)\right)\rangle\mu_M\\ =&\int_M \langle\iota_{\kappa_B^\sharp}\omega,(-1)^{m(r+1)+1}\star_B\left(2\varphi d_B\varphi\wedge\star_B(|\omega|^{p-2}\omega)\right)\rangle\mu_M.\end{aligned}\tag{3.9}$$

So

$$\begin{aligned}|R_{2.1}|\le&2\int_M\left(\varphi|\iota_{\kappa_B^\sharp}\omega|\cdot|d_B\varphi\wedge\star_B\omega)|\right)|\omega|^{p-2}\mu_M\\ \le&2\int_M(\varphi|\kappa_B||\omega|\cdot|d_B\varphi||\omega|)|\omega|^{p-2}\mu_M\\ =&2\int_M\varphi|\kappa_B||d_B\varphi||\omega|^p\mu_M\\ \le&2\sup_M|\kappa_B|\int_M\varphi|d_B\varphi||\omega|^p\mu_M,\end{aligned}\tag{3.10}$$

where $\sup\limits_M|\kappa_B|$ is the supremum of the norm of κ_B.

Next, we estimate

$$\begin{aligned}R_{2.2}:=&\int_M\langle\nabla_{\kappa_B^\sharp}\omega,\varphi^2|\omega|^{p-2}\omega\rangle\mu_M\\ =&\frac{1}{2}\int_M\varphi^2|\omega|^{p-2}\kappa_B^\sharp(|\omega|^2)\mu_M\\ =&\frac{1}{p}\int_M\varphi^2\kappa_B^\sharp(|\omega|^p)\mu_M\\ =&\frac{1}{p}\int_M\kappa_B^\sharp(\varphi^2|\omega|^p)\mu_M-\frac{1}{p}\int_M|\omega|^p\kappa_B^\sharp(\varphi^2)\mu_M\\ =&\frac{1}{p}\int_M\langle d_B(\varphi^2|\omega|^p),\kappa_B\rangle\mu_M-\frac{2}{p}\int_M\varphi|\omega|^p\kappa_B^\sharp(\varphi)\mu_M.\end{aligned}$$

Since κ_B is coclosed, we have

$$|R_{2.2}|=\left|-\frac{2}{p}\int_M\varphi|\omega|^p\kappa_B^\sharp(\varphi)\mu_M\right|\le\frac{2}{p}\int_M|\omega|^p(\varphi|\kappa_B|)|d_B\varphi|\mu_M\le\frac{2}{p}\sup_M|\kappa_B|\int_M\varphi|d_B\varphi||\omega|^p\mu_M.\tag{3.11}$$

Combining (3.8), (3.10), (3.11), we have

$$\begin{aligned} &A\int_M \varphi^2|\omega|^{p-2}|d_B|\omega||^2\mu_M+\int_M|\omega|^{2p-4}\langle F(\omega),\varphi^2\omega\rangle\mu_M\\ \leq &B\int_M|d_B\varphi|^2|\omega|^p\mu_M+\left(2+\frac{2}{p}\right)\sup_M|\kappa_B|\int_M\varphi|d_B\varphi||\omega|^p\mu_M. \end{aligned}\tag{3.12}$$

Step 2: Apply estimate (3.2) and (3.3) to inequality (3.12).

From (3.2) and (3.3), we have

$$\int_M|d_B\varphi|^2|\omega|^p\mu_M\leq\frac{C_1^2}{l^2}\int_M|\omega|^p\mu_M,$$

and

$$\int_M\varphi|d_B\varphi||\omega|^p\mu_M\leq\frac{C_1}{l}\int_M|\omega|^p\mu_M.$$

Therefore, we

$$A\int_M\varphi^2|\omega|^{p-2}|d_B|\omega||^2\mu_M+\int_M|\omega|^{p-2}\langle F(\omega),\varphi^2\omega\rangle\leq\left[\frac{BC_1^2}{l^2}+\frac{C_1}{l}\right]\int_M|\omega|^p\mu_M.$$

Letting $l\to+\infty$, it holds that

$$\lim_{l\to+\infty}\int_M\varphi^2|\omega|^{p-2}|d_B|\omega||^2\mu_M=\int_M|\omega|^{p-2}|d_B|\omega||^2\mu_M=0,$$

and

$$\lim_{l\to\infty}\int_M|\omega|^{p-2}\langle F(\omega),\varphi^2\omega\rangle=\int_M|\omega|^{p-2}\langle F(\omega),\omega\rangle\mu_M=0.$$

If the transverse curvature operator F is nonnegative and positive at some point, $|\omega|=0$. This completes the proof.

Remark 2 We should remark that, by applying the same method as above, we can obtain the vanishing theorem for basic $L^q(q\geq p)$ p-harmonic r-form when the basic part of mean curvature form of $\mathcal{F}$ vanishes.

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完备叶状黎曼流形上p-调和形式的消灭定理

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摘要: 在本论文中, 我们研究了完备叶状黎曼流形上的基本p-调和形式. 利用文献[1]中的方法, 我们得到了在基本平均曲率形式满足有界且余闭的, 以及横截曲率算子满足非负其至少在一点是正的假设下完备叶状黎曼流形上L^p-可积p-调和r- 形式的一个消灭定理.

关键词: 基本p-调和形式; 消灭定理; 叶状黎曼流形

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