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AN ELEGANT RELATION BETWEEN SELBERG'S INEQUALITY AND BESSEL'S INEQUALITY

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Abstract: The purpose of this paper is to study some famous inequalities in Euclidean space. We are able to reveal an elegant relation between the famous Selberg inequality and Bessel inequality in Euclidean space.

Keywords: the Selberg inequality; the Bessel inequality; Euclidean space; Orthonormal family

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1 Introduction

Let V be a linear space over the real number field $\mathbb{R}$. An inner product is a real-valued function $(\cdot,\cdot): V\times V\to\mathbb{R}$ if it satisfies the following conditions:

(1) $(\alpha,\beta)=(\beta,\alpha)$ for all $\alpha,\beta\in V$;

(2) $(k\alpha,\beta)=k(\alpha,\beta)$ for all $\alpha,\beta\in V$ and $k\in\mathbb{R}$;

(3) $(\alpha+\beta,\gamma)=(\alpha,\gamma)+(\beta,\gamma)$ for all $\alpha,\beta,\gamma\in V$;

(4) $(\alpha,\alpha)\geq 0$ for all $\alpha\in V,$ and the equality holds if and only if $\alpha=0$.

The linear space V becomes an Euclidean space $(V;(\cdot,\cdot))$ when endowed with an inner product $(\cdot,\cdot)$. Note that for the same linear space, the inner product is not necessarily unique.

The norm or length $\|\alpha\|$ of $\alpha\in V$ is defined by $\|\alpha\|^2=(\alpha,\alpha)$. If $\alpha,\beta\in V$ satisfies $(\alpha,\beta)=0$, they are called orthogonal elements. If a family of elements $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_k$ in V satisfies

$$(\varepsilon_i,\varepsilon_j)=\delta_{ij}=\begin{cases}1, \text{ if } i=j;\\ 0, \text{ if } i\neq j,\end{cases}$$

they are called an orthonormal family. In particular, if $k=\dim V$, they constitute an orthonormal basis of V.

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As a generalization of the basic fact the hypotenuse of a triangle is greater than the right angled side, there is the famous Bessel inequality [1] [2] in a general Euclidean space, which states: suppose that $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k$ are an orthonormal family of the Euclean space V, then for any $\alpha \in V$ we have

$$\|\alpha\|^2 \geq |(\alpha, \varepsilon_1)|^2 + |(\alpha, \varepsilon_2)|^2 + \cdots + |(\alpha, \varepsilon_k)|^2 = \sum_{i=1}^{k} |(\alpha, \varepsilon_i)|^2.$$

Furthermore, if $\varphi_1, \varphi_2, \cdots, \varphi_k$ (not necessarily orthonormal) and α are arbitrary elements of the Euclean space V, then the famous Selberg inequality [3] indicates that

$$\begin{aligned}\|\alpha\|^2 &\geq \frac{|(\alpha, \varphi_1)|^2}{\sum\limits_{j=1}^{k} |(\varphi_1, \varphi_j)|} + \frac{|(\alpha, \varphi_2)|^2}{\sum\limits_{j=1}^{k} |(\varphi_2, \varphi_j)|} + \cdots + \frac{|(\alpha, \varphi_k)|^2}{\sum\limits_{j=1}^{k} |(\varphi_k, \varphi_j)|} \\ &= \sum_{i=1}^{k} \frac{|(\alpha, \varphi_i)|^2}{\sum\limits_{j=1}^{k} |(\varphi_i, \varphi_j)|}.\end{aligned}$$

These inequalities have many important applications in mathematics. On can easily find that if $\varphi_1, \varphi_2, \cdots, \varphi_k$ are an orthonormal family, the Selberg inequality degenerates into the Bessel inequality since $\sum_{j=1}^{k} |(\varphi_i, \varphi_j)| = \sum_{j=1}^{k} \delta_{ij} = 1$ for any fixed $1 \leq i \leq k$.

In this paper, we would like to strengthen the Selberg inequality and the Bessel inquality by combining the two famous inequalities. It appears that it is the first time to reveal the elegant relation between these two famous inequalities.

Theorem 1.1 Let V be an Euclidean space and $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ two inner products on V, which satisfy the following metric property, i.e.

$$\|\alpha\|_1 = \sqrt{(\alpha, \alpha)_1} \geq \|\alpha\|_2 = \sqrt{(\alpha, \alpha)_2}$$

for any $\alpha \in V$. Furthermore, suppose that $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k$ are an orthonormal family of $(V; (\cdot,\cdot)_2)$, and $\varphi_1, \varphi_2, \cdots, \varphi_\ell$ are arbitrary elements in $(V; (\cdot,\cdot)_1)$ satisfying

$$L(\varphi_1, \varphi_2, \cdots, \varphi_\ell) \subseteq L(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k).$$

Then we have

$$\|\alpha\|_1^2 - \sum_{i=1}^{\ell} \frac{|(\alpha, \varphi_i)_1|^2}{\sum\limits_{j=1}^{\ell} |(\varphi_i, \varphi_j)_1|} \geq \|\alpha\|_2^2 - \sum_{i=1}^{k} |(\alpha, \varepsilon_i)_2|^2 \geq 0.$$

Here $L(\varphi_1, \varphi_2, \cdots, \varphi_\ell)$ and $L(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k)$ are linear spaces spanned by $\varphi_1, \varphi_2, \cdots, \varphi_\ell$ and $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k$ respectively.

When $\varphi_1, \varphi_2, \cdots, \varphi_\ell$ are an orthonormal family of $(V; (\cdot,\cdot)_1)$, Theorem 1.1 gives the result of Dragomir [4], which considers an elegant monotonicity property of Bessel's inequality.

Corollary 1.1 Let V be an Euclidean space and $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ two inner products on V, which satisfy the following metric property $\|\alpha\|_1 \geq \|\alpha\|_2$ for any $\alpha \in V$. Suppose that $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k$ are an orthonormal family of $(V;(\cdot,\cdot)_2)$, and $\varphi_1, \varphi_2, \cdots, \varphi_\ell$ are an orthonormal family of $(V;(\cdot,\cdot)_1)$ satisfying

$$L(\varphi_1, \varphi_2, \cdots, \varphi_\ell) \subseteq L(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k).$$

Then we have

$$\|\alpha\|_1^2 - \sum_{i=1}^{\ell} |(\alpha, \varphi_i)_1|^2 \geq \|\alpha\|_2^2 - \sum_{i=1}^{k} |(\alpha, \varepsilon_i)_2|^2 \geq 0.$$

For the same norm in the Euclidean space $(V;(\cdot,\cdot))$, we still have similar results. As an example, we have

Corollary 1.2 Suppose that $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k$ are an orthonormal family of $(V;(\cdot,\cdot))$, and $\varphi_1, \varphi_2, \cdots, \varphi_\ell$ are arbitrary elements in $(V;(\cdot,\cdot))$ satisfying

$$L(\varphi_1, \varphi_2, \cdots, \varphi_\ell) \subseteq L(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k).$$

Then we have

$$\|\alpha\|^2 - \sum_{i=1}^{\ell} \frac{|(\alpha, \varphi_i)_1|^2}{\sum_{j=1}^{\ell} |(\varphi_i, \varphi_j)_1|} \geq \|\alpha\|^2 - \sum_{i=1}^{k} |(\alpha, \varepsilon_i)|^2 \geq 0.$$

Remark Even the case $\ell = 1$ in Corollary 1.2 is not trivial. In fact, suppose that $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k$ are an orthonormal family. Then for any $\alpha \in \mathbb{R}$, and $\varphi \in L(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k)$, we have

$$\frac{|(\alpha, \varphi)|^2}{|(\varphi, \varphi)|} = \left| \left(\alpha, \frac{\varphi}{\|\varphi\|} \right) \right|^2 \leq \sum_{i=1}^{k} |(\alpha, \varepsilon_i)|^2.$$

On the one hand, it formally implies the Cauchy-Schwartz inequality $|(\alpha, \varphi)| \leq \|\alpha\|\|\varphi\|$ by virtue of Bessel's equality. On the other hand, in geometry it shows that the length of the projection of α onto the subspace $L(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k)$ is larger than that of the projection of α onto any element φ in this subspace.

2 Proof of Main Results

Lemma 2.1 Let V be an Euclidean space endowed with an inner product $(\cdot,\cdot)$. Suppose that $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k$ are an orthonormal family of $(V;(\cdot,\cdot))$. Then for any sequence $\lambda_i \in \mathbb{R}$, $i = 1, 2, \cdots, k$ and any $\alpha \in V$, we have that

$$\left\| \alpha - \sum_{i=1}^{k} \lambda_i \varepsilon_i \right\|^2 \geq \|\alpha\|^2 - \sum_{i=1}^{k} |(\alpha, \varepsilon_i)|^2.$$

Proof For completeness, we give a detailed proof from scratch. Let

$$u = \sum_{i=1}^{k} (\alpha, \varepsilon_i)\varepsilon_i.$$

Obviously, we have

$$\left(\alpha - u, u - \sum_{i=1}^{k} \lambda_i \varepsilon_i\right) = 0. \tag{2.1}$$

In fact, let

$$W = L(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k).$$

Then u is the projection of α onto W, and then $\alpha - u \perp W$. Hence (2.1) holds since

$$u - \sum_{i=1}^{k} \lambda_i \varepsilon_i \in W.$$

By the Pythagorean theorem, we have

$$\|\alpha - u\|^2 \leq \|\alpha - u\|^2 + \left\|u - \sum_{i=1}^{k} \lambda_i \varepsilon_i\right\|^2 = \left\|\alpha - \sum_{i=1}^{k} \lambda_i \varepsilon_i\right\|^2. \tag{2.2}$$

Notice that

$$(\alpha, u) = \left(\alpha, \sum_{i=1}^{k} (\alpha, \varepsilon_i)\varepsilon_i\right) = \sum_{i=1}^{k} (\alpha, \varepsilon_i)^2,$$

and that

$$(u, u) = \left(\sum_{i=1}^{k} (\alpha, \varepsilon_i)\varepsilon_i, \sum_{j=1}^{k} (\alpha, \varepsilon_j)\varepsilon_j\right) = \sum_{i=1}^{k} \sum_{j=1}^{k} (\alpha, \varepsilon_i)(\alpha, \varepsilon_j)\delta_{ij} = \sum_{i=1}^{k} (\alpha, \varepsilon_i)^2.$$

We have

$$\begin{aligned} \|\alpha - u\|^2 &= (\alpha - u, \alpha - u) = \|\alpha\|^2 - 2(\alpha, u) + (u, u) \\ &= \|\alpha\|^2 - \sum_{i=1}^{k} |(\alpha, \varepsilon_i)|^2. \end{aligned} \tag{2.3}$$

From (2.2) and (2.3), we establish this lemma.

Now we start to prove Theorem 1.1.

Proof For any sequence $c_i \in \mathbb{R}$, $i = 1, 2, \cdots, \ell$, we consider

$$\left\|\alpha - \sum_{i=1}^{\ell} c_i \varphi_i\right\|_1^2 \geq \left\|\alpha - \sum_{i=1}^{\ell} c_i \varphi_i\right\|_2^2 = \left\|\alpha - \sum_{i=1}^{\ell} c_i \sum_{j=1}^{k} (\varphi_i, \varepsilon_j)_2 \varepsilon_j\right\|_2^2,$$

where we used that $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k$ are an orthonormal family of $(V; (\cdot, \cdot)_2)$, and hence for any φ_i, $i = 1, 2, \cdots, k$, we have

$$\varphi_i = \sum_{j=1}^{k} (\varphi_i, \varepsilon_j)_2 \varepsilon_j.$$

After switching the summations, we find that

$$\left\|\alpha - \sum_{i=1}^{\ell} c_i \varphi_i\right\|_1^2 \geq \left\|\alpha - \sum_{j=1}^{k} \left(\sum_{i=1}^{\ell} c_i (\varphi_i, \varepsilon_j)_2\right) \varepsilon_j\right\|_2^2. \tag{2.4}$$

By Lemma 2.1 with $\lambda_j = \sum_{i=1}^{\ell} c_i(\varphi_i, \varepsilon_j)_2$, we have

$$\left\| \alpha - \sum_{j=1}^{k} \left(\sum_{i=1}^{\ell} c_i(\varphi_i, \varepsilon_j)_2 \right) \varepsilon_j \right\|_2^2 \geq \|\alpha\|_2^2 - \sum_{j=1}^{k} \left|(\alpha, \varepsilon_j)_2\right|^2 . \tag{2.5}$$

Now we consider the term

$$\left\| \alpha - \sum_{i=1}^{\ell} c_i \varphi_i \right\|_1^2$$

on the left-hand side of (2.4). By the definition of the norm, we observe that

$$\begin{aligned} \left\| \alpha - \sum_{i=1}^{\ell} c_i \varphi_i \right\|_1^2 &= \left(\alpha - \sum_{i=1}^{\ell} c_i \varphi_i, \alpha - \sum_{i=1}^{\ell} c_i \varphi_i \right)_1 \\ &= (\alpha, \alpha)_1 - 2 \sum_{i=1}^{\ell} c_i \left(\alpha, \varphi_i\right)_1 + \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} c_i c_j \left(\varphi_i, \varphi_j\right)_1 \\ &\leq (\alpha, \alpha)_1 - 2 \sum_{i=1}^{\ell} c_i \left(\alpha, \varphi_i\right)_1 + \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} |c_i||c_j| \left|(\varphi_i, \varphi_j)_1\right| . \end{aligned} \tag{2.6}$$

Here we only use basic properties of inner product and the trivial inequality $x \leq |x|$ for all $x \in \mathbb{R}$. By the elementary inequality $|c_i||c_j| \leq \frac{1}{2}\left(|c_i|^2 + |c_j|^2\right)$, we further have

$$\begin{aligned} \left\| \alpha - \sum_{i=1}^{\ell} c_i \varphi_i \right\|_1^2 &\leq (\alpha, \alpha)_1 - 2 \sum_{i=1}^{\ell} c_i \left(\alpha, \varphi_i\right)_1 + \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} \frac{1}{2} \left(|c_i|^2 + |c_j|^2\right) \left|(\varphi_i, \varphi_j)_1\right| \\ &= \|\alpha\|_1^2 - 2 \sum_{i=1}^{\ell} c_i \left(\alpha, \varphi_i\right)_1 + \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} |c_i|^2 \left|(\varphi_i, \varphi_j)_1\right| . \end{aligned}$$

Take

$$c_i = (\alpha, \varphi_i)_1 \left(\sum_{j=1}^{\ell} \left|(\varphi_i, \varphi_j)_1\right| \right)^{-1} .$$

We find that

$$\begin{aligned} \left\| \alpha - \sum_{i=1}^{\ell} c_i \varphi_i \right\|_1^2 &\leq \|\alpha\|_1^2 - 2 \sum_{i=1}^{\ell} (\alpha, \varphi_i)_1^2 \left(\sum_{j=1}^{\ell} \left|(\varphi_i, \varphi_j)_1\right| \right)^{-1} \\ &\quad + \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} \left|(\varphi_i, \varphi_j)_1\right| (\alpha, \varphi_i)_1^2 \left(\sum_{j=1}^{\ell} \left|(\varphi_i, \varphi_j)_1\right| \right)^{-2} \\ &= \|\alpha\|_1^2 - \sum_{i=1}^{\ell} |(\alpha, \varphi_i)_1|^2 \left(\sum_{j=1}^{\ell} |(\varphi_i, \varphi_j)_1| \right)^{-1} . \end{aligned} \tag{2.7}$$

From (2.4), (2.5), and (2.7), we complete the proof of Theorem 1.1.

3 Further Discussion

Theorem 1.1 also holds true for the inner product space over the complex numbers $\mathbb{C}$. The only difference is that in (2.6) we obtain

$$\begin{aligned}\left\|\alpha-\sum_{i=1}^{\ell} c_i\varphi_i\right\|_1^2 &= \left(\alpha-\sum_{i=1}^{\ell} c_i\varphi_i, \alpha-\sum_{i=1}^{\ell} c_i\varphi_i\right)_1 \\ &= (\alpha,\alpha)_1 - 2\Re\sum_{i=1}^{\ell}\bar{c}_i\,(\alpha,\varphi_i)_1 + \sum_{i=1}^{\ell}\sum_{j=1}^{\ell} c_i\bar{c}_j\,(\varphi_i,\varphi_j)_1 \\ &\le (\alpha,\alpha)_1 - 2\Re\sum_{i=1}^{\ell}\bar{c}_i\,(\alpha,\varphi_i)_1 + \sum_{i=1}^{\ell}\sum_{j=1}^{\ell} |c_i||c_j|\left|(\varphi_i,\varphi_j)_1\right| \\ &\le (\alpha,\alpha)_1 - 2\Re\sum_{i=1}^{\ell}\bar{c}_i\,(\alpha,\varphi_i)_1 + \sum_{i=1}^{\ell}\sum_{j=1}^{\ell} |c_i|^2\left|(\varphi_i,\varphi_j)_1\right|.\end{aligned}$$

And eventually we also take

$$c_i = (\alpha,\varphi_i)_1\left(\sum_{j=1}^{\ell}\left|(\varphi_i,\varphi_j)_1\right|\right)^{-1}.$$

References

[1] Stein E M, Shakarchi R. Fourier analysis: An introduction [M]. Princeton: Princeton University Press, 2003.

[2] Nicholson W K. Linear algebra with applications (3rd ed.) [M]. Boston: PWS Publishing Company, 1995.

[3] Bombieri E. Le Grand Crible dans la Théorie Analytique des Nombres[M]. Paris: Société Mathématique de France, 1974.

[4] Dragomir S S. A Note on Bessel's inequality [J]. Austral. Math. Soc. Gaz., 2001, 28(50): 246–248.

关于Selberg不等式和Bessel不等式的一个关系

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摘 要: 本文研究了欧几里得空间著名的Selberg不等式和Bessel不等式的强化问题. 利用线性代数的方法, 揭示了Selberg不等式和Bessel不等式之间的一个未知关系.

关键词: Selberg不等式; Bessel不等式; 欧几里得空间; 标准正交向量组

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