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PSEUDO S-ASYMPTOTICALLY PERIODIC SOLUTIONS OF PARTIAL NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract: In this paper, we study the existence of pseudo S-asymptotically ω-periodic mild solutions of abstract partial neutral differential equations in Banach spaces. By using the principle of Banach contractive mapping, the existence and uniqueness of pseudo S-asymptotically ω-periodic mild solutions of abstract partial neutral differential equations are obtained. To illustrate the abstract result, a concrete example is given.

Keywords: pseudo S-asymptotically ω-periodic function; mild solution; partial neutral differential equations

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1 Introduction

Real systems usually exhibit internal variations or are submitted to external perturbations. In many situation we can assume that these variations are approximately periodic in a broad sense. In the literature have studied several concepts to represent the idea of approximately periodic function. Most of works deal with asymptotically periodic functions and almost periodic functions. In addition it has recently emerged the notion of S-asymptotically ω-periodic functions which has been shown to have interesting applications in several branches of differential equations. This has motivated considerable interest in the topic. Such concept is introduced in the literature by Henríquez et al in several studies [1,2], one can see previous studies [3-6] for more details. Recently, Pierri and Rolnik [7] introduced a new concept of a function called pseudo S-asymptotically ω-periodic function, which is general than S-asymptotically ω-periodic function and they have studied qualitative properties of this type of functions. In addition they discuss the existence of pseudo S-asymptotically ω-periodic mild solutions for abstract neutral functional equations. Some applications involving ordinary and partial differential equations with delay are presented. Since then,

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many applications in abstract differential equations are investigated. Existence and uniqueness of pseudo S-asymptotically ω-periodic solutions of fractional differential equations are investigated in Cuevas et al [8], Xia [9], Yang and Wang [10]. The authors have studied the pseudo S-asymptotically ω-periodicity of hyperbolic evolutions equations [11], the damped wave equation [12], and second-order abstract Cauchy problems [13], respectively. The function of pseudo S-asymptotically ω-periodic in Stepanov-like sense is introduced in Xia [14] and gives the applications in Volterra integro-differential equations.

However, the existence of pseudo S-asymptotically ω-periodic mild solutions for abstract partial neutral differential equations have still rarely been treated in the literature. Motivated by these works, the main purpose of this paper is to investigate the existence and uniqueness of pseudo S-asymptotically ω-periodic mild solutions to the following abstract partial neutral differential equations

$$\frac{d}{dt}D(t,x_t) = AD(t,x_t) + g(t,x_t),\ \ t \geq 0 \tag{1.1}$$

with

$$x_0 = \varphi \in \mathcal{C} \tag{1.2}$$

where $D(t,\psi) = \psi(0) - f(t,\psi)$ and $f, g : R \times \mathcal{C} \to X$ are appropriate functions. Throughout this paper, X is a Banach space endowed with a norm $\|\cdot\|$. A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $(T(t))_{t\geq 0}$ defined on X [15,16]. The function x_t, which is usually known as the segment of $x(\cdot)$ at t, is defined by $x_t : (-\infty, 0) \to X$, $x_t(\theta) = x(t+\theta)$. We assume that $x_t \in \mathcal{C}$, where $\mathcal{C} = C([-r,0],X)$ denotes the space of continuous funtion from $[-r,0]$ to X with the supremum norm, r is a positive real number.

The paper is organized as follows. In Section 2, we introduce the notion of pseudo S-asymptotically ω-periodic functions and study some of their basic properties. In Section 3, we prove the existence and uniqueness of a pseudo S-asymptotically ω-periodic mild solution to abstract partial neutral differential equations. In Section 4, an example is given to illustrate our main results.

2 Preliminaries

In this section we give some definitions and study some of their basic properties which will be used in the sequel. For concepts of the theory of strongly continuous semigroups we refer the reader to [15, 16]. Some additional notations are the following. Let $R^+ = [0,+\infty)$, $C_b(R^+,X)$ (respectively, $C_b([-r,\infty),X)$) denotes the space formed by the bounded continuous functions from R^+ (respectively, $[-r,\infty)$) into X, endowed with the norm $\|\cdot\|_{C_b(R^+,X)}$ (respectively,$\|\cdot\|_{C_b([-r,\infty),X)}$). $C_b([0.\infty), R^+)$ denotes the space formed by the bounded continuous functions from $[0.\infty)$ into the positive real number set R^+, endowed with the norm $\|\cdot\|_{C_b([0,\infty),R^+)}$.We denote by $B(X)$ the Banach space of bounded linear operators from X into X and $\|\cdot\|_\infty$ the norm of the uniform convergence in any of these spaces.

We now recall some notations and properties related to pseudo S-asymptotically ω-periodic functions, more details see [7,8].

Definition 2.1[7]. A function $f \in C_b(R^+, X)$ is called pseudo S-asymptotically periodic if there exists $\omega > 0$ such that

$$\lim_{h\to\infty} \frac{1}{h}\int_0^h \|f(t+\omega) - f(t)\|_X dt = 0.$$

In this case, we say that f is pseudo S-asymptotically ω-periodic.

The collection of all such functions will be denoted by $PSAP_\omega(R^+, X)$. $PSAP_\omega(R^+, X)$ is a Banach space when it is equipped with the norm of uniform convergence.

Definition 2.2 A continuous function $f : R^+ \times X \to Y$ is said to be uniformly pseudo S-asymptotically ω-periodic if for every bounded subset $K \subseteq X$, the set $\{f(t, x) : t \geq 0, x \in K\}$ is bounded and

$$\lim_{h\to\infty} \frac{1}{h}\int_0^h \|f(t+\omega, x) - f(t, x)\|_Y dt = 0,$$

uniformly for $x \in K$.

Denote by $PSAP_\omega(R^+ \times X, Y)$ the set of all such functions.

Definition 2.3 A function $f : R^+ \times X \to Y$ is said to be asymptotically uniformly continuous on bounded sets of X if for every $\epsilon > 0$ and all bounded set $K \subseteq X$, there are constants $T \geq 0$ and $\delta > 0$ such that $\|f(t, x) - f(t, y)\|_Y \leq \epsilon$ for all $t \geq T$ and $x, y \in K$ with $\|x - y\|_X < \delta$.

Definition 2.4 A function $f : R^+ \times X \to Y$ is bounded on bounded sets of X if for every bounded subset $K \subseteq X$, the set $\{f(t, x) : t \geq 0, x \in K\}$ is bounded.

Lemma 2.5[8] Assume that $f : R^+ \times X \to Y$ is bounded on bounded sets of X and asymptotically uniformly continuous on bounded sets of X. Then for every $u, v \in C_b(R^+, X)$, $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t \|u(s) - v(s)\|_X dt = 0$ implies

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \|f(s, u(s)) - f(s, v(s))\|_Y dt = 0.$$

Lemma 2.6[14] Let $f \in PSAP_\omega(R^+, X)$, then $f(\cdot+\tau) \in PSAP_\omega(R^+, X)$ for all $\tau > 0$.

Lemma 2.7 Let $x \in C_b([-r, \infty), X)$ and $x|_{[0,\infty)} \in PSAP_\omega(R^+, X)$, then the function $t \to x_t$ belongs to $PSAP_\omega(R^+, \mathcal{C})$.

Proof Note that

$$\begin{aligned}
\frac{1}{h}\int_0^h \|x_{t+\omega} - x_t\|_{\mathcal{C}} dt &= \frac{1}{h}\int_0^h \sup_{-r\leq\theta\leq 0} \|x(t+\omega+\theta) - x(t+\theta)\| dt \\
&\leq \frac{1}{h}\int_{-r}^h \|x(t+\omega) - x(t)\| dt \\
&= \frac{1}{h}\int_0^h \|x(t+\omega) - x(t)\| dt + \frac{1}{h}\int_{-r}^0 \|x(t+\omega) - x(t)\| dt
\end{aligned}$$

$$\leq \frac{1}{h}\int_0^h \|x(t+\omega)-x(t)\|dt + \frac{2r}{h}\|x\|_{C_b([-r,\infty),X)}$$

By $x|_{[0,\infty)} \in PSAP_\omega(R^+,X)$, $r>0$, it is easy to see that

$$\lim_{h\to\infty}\frac{1}{h}\int_0^h \|x_{t+\omega}-x_t\|_{\mathcal{C}}dt = 0.$$

Hence $x_t \in PSAP_\omega(R^+,\mathcal{C})$. The proof is complete.

Lemma 2.8 [8] Assume that $f: R^+\times X \to Y$ is a function asymptotically bounded on bounded sets of X, asymptotically uniformly continuous on bounded sets of X and uniformly pseudo S-asymptotically ω-periodic on bounded sets of X. If $x: R^+ \to X$ is a pseudo S-asymptotically ω-periodic function, then $f(t,x(t)) \in PSAP_\omega(R^+,X)$.

Lemma 2.9 Let $f: R^+\times\mathcal{C}\to X$ be uniformly pseudo S-asymptotically ω-periodic on bounded sets of X that satisfies

$$\|f(t,\psi_2)-f(t,\psi_1)\| \leq L_f(t)\|\psi_2-\psi_1\|_{\mathcal{C}},$$

for all $t\geq 0$ and $\psi_1,\psi_2\in\mathcal{C}$, where $L_f\in C_b([0,\infty),R^+)$. If $x: R^+\to X$ is a pseudo S-asymptotically ω-periodic function, then $f(t,x_t)\in PSAP_\omega(R^+,X)$.

Proof Let $L=\|x\|_{C_b([-r,\infty),X)}$. By Lemma 2.7, we obtain the function $t\to x_t$ belongs to $PSAP_\omega(R^+,\mathcal{C})$, then for $\forall\ \varepsilon>0$, there exists $L_\varepsilon>0$, such that for each $h>L_\varepsilon$, we have

$$\|L_f\|_{C_b([0,\infty),R^+)}\frac{1}{h}\int_0^h \|x_{t+\omega}-x_t\|_{\mathcal{C}}dt<\varepsilon.$$

moreover, by $f\in PSAP_\omega(R^+,X)$, we have

$$\frac{1}{h}\int_0^h \sup_{\|x\|_{\mathcal{C}}\leq L}\|ft+\omega,x-ft,x\|dt<\varepsilon.$$

Thus for all $h>L_\varepsilon$, we can obtain

$$\begin{aligned}
&\frac{1}{h}\int_0^h \|f(t+\omega,x_{t+\omega})-f(t,x_t)\|dt\\
\leq&\frac{1}{h}\int_0^h \|f(t+\omega,x_{t+\omega})-f(t,x_{t+\omega})\|dt+\frac{1}{h}\int_0^h \|f(t,x_{t+\omega})-f(t,x_t)\|dt\\
\leq&\frac{1}{h}\int_0^h \sup_{\|x\|_{\mathcal{C}}\leq L}\|f(t+\omega,x)-f(t,x)\|dt+\|L_f\|_{C_b([0,\infty),R^+)}\frac{1}{h}\int_0^h \|x_{t+\omega}-x_t\|_{\mathcal{C}}dt\\
<&2\varepsilon.
\end{aligned}$$

It is easy to see that

$$\lim_{h\to\infty}\frac{1}{h}\int_0^h \|f(t+\omega,x_{t+\omega})-f(t,x_t)\|dt=0.$$

Hence $f(t,x_t)\in PSAP_\omega(R^+,X)$. The proof is complete.

3 Main Results

In this section, we establish the existence of pseudo S-asymptotically ω-periodic mild solutions to (1.1)-(1.2). Throughout this section, we suppose that the following assumptions hold:

$(H1)$ $(T(t))_{t\geq 0}$ is a uniformly exponentially stable semigroup that satisfies $\|T(t)\|\leq Me^{-\beta t}$, $t\geq 0$, if there exist constant $M\geq 1$ and $\beta>0$ is a fixed real number.

$(H2)$ There exist constants $L_f>0$ and $L_g>0$ such that

$$\|f(t,\psi_2)-f(t,\psi_1)\|\leq L_f\|\psi_2-\psi_1\|_{\mathcal{C}},$$

$$\|g(t,\psi_2)-g(t,\psi_1)\|\leq L_g\|\psi_2-\psi_1\|_{\mathcal{C}},$$

for any $\psi_1,\psi_2\in\mathcal{C}$ and all $t\geq 0$.

$(H3)$ There exist continuous functions $L_f(t)>0$ and $L_g(t)>0$ such that

$$\|f(t,\psi_2)-f(t,\psi_1)\|\leq L_f(t)\|\psi_2-\psi_1\|_{\mathcal{C}},$$

$$\|g(t,\psi_2)-g(t,\psi_1)\|\leq L_g(t)\|\psi_2-\psi_1\|_{\mathcal{C}},$$

for any $\psi_1,\psi_2\in\mathcal{C}$ and all $t\geq 0$.

Definition 3.1 A function $x\in C_b([-r,\infty),X)$ is called a mild solution to problems (1.1) and (1.2) if $x_0=\varphi\in\mathcal{C}$ and

$$x(t)=T(t)(\varphi(0)-f(0,\varphi))+f(t,x_t)+\int_0^t T(t-s)g(s,x_s)ds \tag{3.1}$$

for all $t\geq 0$.

Lemma 3.2 Assume that condition $(H1)$ hold. If $x\in PSAP_\omega(R^+,X)$, then the function

$$u(t)=\int_0^t T(t-s)x(s)ds,\quad t\in R^+$$

is pseudo S-asymptotically ω-periodic.

Proof We have the estimate

$$\|u(t)\|\leq\int_0^t\|T(t-s)x(s)\|ds\leq\frac{M}{\beta}\|x\|_{C_b(R^+,X)},$$

this shows that $u(t)\in C_b(R^+,X)$. Now we shall show that

$$\lim_{h\to\infty}\frac{1}{h}\int_0^h\|u(t+\omega)-u(t)\|dt=0.$$

Note that

$$\begin{aligned} u(t+\omega)-u(t) &= \int_0^{t+\omega} T(t+\omega-s)x(s)ds - \int_0^t T(t-s)x(s)ds \\ &= \int_0^{\omega} T(t+\omega-s)x(s)ds + \int_0^t T(t-s)(x(s+\omega)-x(s))ds \\ &:= I(t)+J(t), \end{aligned}$$

We will estimate the terms $I(t), J(t)$ separately.

Since

$$\begin{aligned} \frac{1}{h}\int_0^h \|I(t)\|dt &\le \frac{1}{h}\int_0^h\int_0^{\omega} \|T(t+\omega-s)x(s)\|dsdt \\ &\le \frac{1}{h}\int_0^h\int_0^{\omega} Me^{-\beta(t+\omega-s)}\|x(s)\|dsdt \\ &\le \frac{1}{h}\frac{M}{\beta}\|x\|_{C_b(R^+,X)}\int_0^h e^{-\beta t}dt \\ &\le \frac{1}{h}\frac{M}{\beta^2}\|x\|_{C_b(R^+,X)}. \end{aligned}$$

So

$$\lim_{h\to\infty}\frac{1}{h}\int_0^h \|I(t)\|dt = 0.$$

Next, we will prove that

$$\lim_{h\to\infty}\frac{1}{h}\int_0^h \|J(t)\|dt = 0.$$

For $h \ge \iota \ge 0$, we have that

$$\begin{aligned} \frac{1}{h}\int_0^h \|J(t)\|dt \le& \frac{1}{h}\int_0^{\iota}\int_0^t \|T(t-s)(x(s+\omega)-x(s))\|dsdt \\ &+ \frac{1}{h}\int_{\iota}^h\int_0^{\iota} \|T(t-s)(x(s+\omega)-x(s))\|dsdt \\ &+ \frac{1}{h}\int_{\iota}^h\int_{\iota}^t \|T(t-s)(x(s+\omega)-x(s))\|dsdt \\ :=& J_1(t)+J_2(t)+J_3(t). \end{aligned}$$

We estimate the terms $J_1(t), J_2(t), J_3(t)$ separately.

For the first term on the right hand side we have

$$\begin{aligned} J_1(t) &= \frac{1}{h}\int_0^{\iota}\int_0^t \|T(t-s)(x(s+\omega)-x(s))\|dsdt \\ &\le \frac{2M\|x\|_{C_b(R^+,X)}}{h}\int_0^{\iota}\int_0^t e^{-\beta s}dsdt \\ &\le \frac{2M\iota\|x\|_{C_b(R^+,X)}}{\beta}\frac{1}{h}, \end{aligned}$$

which implies that $J_1(t) \to 0$ as $h \to \infty$.

For the second term on the right hand side we obtain

$$\begin{aligned}
J_2(t) &= \frac{1}{h}\int_{\iota}^{h}\int_{0}^{\iota} \|T(t-s)(x(s+\omega)-x(s))\|dsdt \\
&\le \frac{2M\|x\|_{C_b(R^+,X)}}{h}\int_{\iota}^{h}\int_{t-\iota}^{t} e^{-\beta s}dsdt \\
&\le \frac{2M\iota\|x\|_{C_b(R^+,X)}}{\beta^2}\frac{1}{h},
\end{aligned}$$

therefore $J_2(t) \to 0$ as $h \to \infty$.

Finally,

$$\begin{aligned}
J_3(t) &= \frac{1}{h}\int_{\iota}^{h}\int_{\iota}^{t} \|T(t-s)(x(s+\omega)-x(s))\|dsdt \\
&\le \frac{M}{h}\int_{\iota}^{h}\int_{s}^{h} e^{-\beta(t-s)}\|x(s+\omega)-x(s)\|dtds \\
&\le \frac{M}{h}\int_{\iota}^{h}\int_{0}^{h-s} e^{-\beta t}dt\|x(s+\omega)-x(s)\|ds \\
&\le \frac{M}{\beta}\frac{1}{h}\int_{0}^{h} \|x(s+\omega)-x(s)\|ds,
\end{aligned}$$

which shows that $J_3(t) \to 0$ as $h \to \infty$.

This completes the proof that $u(t) \in PSAP_\omega(R^+, X)$.

Theorem 3.3 Assume that assumptions (H_1),(H_2) hold. Let $f, g : R^+ \times \mathcal{C} \to X$ be functions asymptotically bounded on bounded sets of X, asymptotically uniformly continuous on bounded sets of X and uniformly pseudo S-asymptotically ω-periodic on bounded sets of X. If $(L_f + \frac{ML_g}{\beta}) < 1$, then (1.1) and (1.2) have a unique pseudo S-asymptotically ω periodic mild solution.

Proof We consider the space $\mathcal{B} = \{x : [-r, \infty) \to X, x_0 = \varphi, x|_{[0,\infty)} \in PSAP_\omega(R^+, X)\}$ endowed with the metric defined by $d(u, v) = \|u - v\|_{C_b([-r,\infty),X)}$.

Let Γ be the map defined by

$$(\Gamma x)(t) = T(t)(\varphi(0) - f(0, \varphi)) + f(t, x_t) + \int_0^t T(t-s)g(s, x_s)ds.$$

We obtain the following estimate

$$\begin{aligned}
\|(\Gamma x)(t)\| &\le \|T(t)\|[\|\varphi(0)\| + \|f(0,\varphi)\|] + \|f(t,x_t)\| + \int_0^t Me^{-\beta(t-s)}\|g(s,x_s)\|dsdt \\
&\le Me^{-\beta t}[\|\varphi(0)\| + \|f(0,\varphi)\|] + \|f(t,x_t)\|_{C_b(R^+,X)} + \frac{M}{\beta}\|g(t,x_t)\|_{C_b(R^+,X)}.
\end{aligned}$$

Furthermore, it is obvious that

$$\frac{1}{h}\int_0^h \|T(t+\omega)(\varphi(0)+f(0,\varphi))-T(t)(\varphi(0)+f(0,\varphi))\|dt \le \frac{2M}{\beta}\frac{1}{h}[\|\varphi(0)\|+\|f(0,\varphi)\|].$$

Whence

$$\lim_{h\to\infty}\frac{1}{h}\int_0^h \|T(t+\omega)(\varphi(0)+f(0,\varphi))-T(t)(\varphi(0)+f(0,\varphi))\|dt = 0.$$

So $T(t)(\varphi(0)+f(0,\varphi)) \in PSAP_\omega(R^+,X)$. It follows from Lemma 2.7 that $x_t \in PSAP_\omega(R^+,\mathcal{C})$. By Lemma 2.8, $f(t,x_t), g(t,x_t) \in PSAP_\omega(R^+,X)$. From Lemma 3.2, we get $\int_0^t T(t-s)g(t,x_t)ds \in PSAP_\omega(R^+,X)$. So Γ is a map from $\mathcal{B}$ into $\mathcal{B}$.

Furthermore, for all $x,y\in\mathcal{B}$ and $t\ge 0$, we can deduce that

$$\begin{aligned}\|(\Gamma x)(t)-(\Gamma y)(t)\| &\le \|f(t,x_t)-f(t,y_t)\| + \int_0^t \|T(t-s)\|\|g(s,x_s)-g(s,y_s)\|ds\\ &\le L_f\|x_t-y_t\|_{\mathcal{C}} + ML_g\int_0^t e^{-\beta(t-s)}ds\|x_t-y_t\|_{\mathcal{C}}\\ &\le (L_f+\frac{ML_g}{\beta})\|x-y\|_{C_b([-r,\infty),X)},\end{aligned}$$

from the above estimates it follows that Γ is a contraction on $\mathcal{B}$. Therefore we can affirm that Γ has a unique fixed point $x\in PSAP_\omega(R^+,X)$, which is the mild solution of (1.1) and (1.2). The proof is complete.

Theorem 3.4 Assume that assumptions (H_1),(H_3) hold. Let $f,g: R^+\times\mathcal{C}\to X$ be functions asymptotically bounded on bounded sets of X, asymptotically uniformly continuous on bounded sets of X and uniformly pseudo S-asymptotically ω-periodic on bounded sets of X. If

$$\|L_f\|_{C_b([0,\infty),R^+)} + M\sup_{t\ge 0}W_g(t) < 1,$$

where $W_g(t)=\int_0^t L_g(s)e^{-\beta(t-s)}ds$, then (1.1) and (1.2) have a unique pseudo S-asymptotically ω periodic mild solution.

Proof We still define the space $\mathcal{B}$ and Γ as in Theorem 3.3. Since f,g are asymptotically bounded on bounded sets of X, then the functions $f(\cdot,0), g(\cdot,0)$ are bounded functions in R^+. We obtain the following estimate

$$\begin{aligned}\|(\Gamma x)(t)\| &\le \|T(t)\|[\|\varphi(0)\|+\|f(0,\varphi)\|] + [\|f(t,x_t)-f(t,0)\| + \|f(t,0)\|_{C_b(R^+,X)}]\\ &\quad + \int_0^t Me^{-\beta(t-s)}[\|g(s,x_s)-g(s,0)\|+\|g(s,0)\|_{C_b(R^+,X)}]ds\\ &\le Me^{-\beta t}[\|\varphi(0)\|+\|f(0,\varphi)\|] + \|L_f\|_{C_b([0,\infty),R^+)}\|x\|_{C_b([-r,\infty),X)} + \|f(t,0)\|_{C_b(R^+,X)}\\ &\quad + \frac{M}{\beta}\|L_g\|_{C_b([0,\infty),R^+)}\|x\|_{C_b([-r,\infty),X)} + \|g(t,0)\|_{C_b(R^+,X)},\end{aligned}$$

similarly as the proof of Theorem 3.3, Γ is well defined .

For $x, y \in \mathcal{B}$ and $t \geq 0$, we can deduce that

$$\begin{aligned}\|(\Gamma x)(t) - (\Gamma y)(t)\| &\leq \|f(t, x_t) - f(t, y_t)\| + \int_0^t \|T(t-s)\|\|g(s, x_s) - g(s, y_s)\| ds \\ &\leq L_f(t)\|x_t - y_t\|_{\mathcal{C}} + M \int_0^t L_g(s) e^{-\beta(t-s)} ds \|x_t - y_t\|_{\mathcal{C}} \\ &\leq (\|L_f\|_{C_b([0,\infty), R^+)} + M \sup_{t\geq 0} W_g(t))\|x - y\|_{C_b([-r,\infty), X)},\end{aligned}$$

which proves that Γ is a contraction on $\mathcal{B}$. Therefore we can affirm that Γ has a unique fixed point $x \in PSAP_\omega(R^+, X)$, which is the mild solution of (1.1) and (1.2). The proof is complete.

Theorem 3.5 Assume that assumptions (H_1),(H_3) hold. Let $f, g : R^+ \times \mathcal{C} \to X$ be functions asymptotically bounded on bounded sets of X and uniformly pseudo S-asymptotically ω-periodic on bounded sets of X. If

$$\|L_f\|_{C_b([0,\infty), R^+)} + \frac{M}{\beta}\|L_g\|_{C_b([0,\infty), R^+)} < 1,$$

then (1.1) and (1.2) have a unique pseudo S-asymptotically ω periodic mild solution.

Proof We still define the space $\mathcal{B}$ and Γ as in Theorem 3.3. We can easily prove that

$$\begin{aligned}\|(\Gamma x)(t)\| \leq\ & M e^{-\beta t}[\|\varphi(0)\| + \|f(0, \varphi)\|] + \|L_f\|_{C_b([0,\infty), R^+)}\|x\|_{C_b([-r,\infty), X)} + \|f(t, 0)\|_{C_b(R^+, X)} \\ &+ \frac{M}{\beta}\|L_g\|_{C_b([0,\infty), R^+)}\|x\|_{C_b([-r,\infty), X)} + \|g(t, 0)\|_{C_b(R^+, X)},\end{aligned}$$

and $T(t)(\varphi(0) + f(0, \varphi)) \in PSAP_\omega(R^+, X)$. It follows from Lemma 2.7 that $x_t \in PSAP_\omega(R^+, \mathcal{C})$. By Lemma 2.9, $f(t, x_t), g(t, x_t) \in PSAP_\omega(R^+, X)$. From Lemma 3.2, we get $\int_0^t T(t-s) g(t, x_t) ds \in PSAP_\omega(R^+, X)$. So Γ is a map from $\mathcal{B}$ into $\mathcal{B}$. Γ is well defined.

Let $x, y \in \mathcal{B}$ and $t \geq 0$, we can deduce that

$$\begin{aligned}\|(\Gamma x)(t) - (\Gamma y)(t)\| &\leq \|f(t, x_t) - f(t, y_t)\| + \int_0^t \|T(t-s)\|\|g(s, x_s) - g(s, y_s)\| ds \\ &\leq L_f(t)\|x_t - y_t\|_{\mathcal{C}} + M \int_0^t L_g(s) e^{-\beta(t-s)} ds \|x_t - y_t\|_{\mathcal{C}} \\ &\leq (\|L_f\|_{C_b([0,\infty), R^+)} + M\|L_g\|_{C_b([0,\infty), R^+)} \int_0^t e^{-\beta(t-s)} ds)\|x - y\|_{C_b(R^+, X)} \\ &\leq (\|L_f\|_{C_b([0,\infty), R^+)} + \frac{M}{\beta}\|L_g\|_{C_b([0,\infty), R^+)})\|x - y\|_{C_b([-r,\infty), X)},\end{aligned}$$

because $\|L_f\|_{C_b([0,\infty), R^+)} + \frac{M}{\beta}\|L_g\|_{C_b([0,\infty), R^+)} < 1$, it follows that Γ is a contraction on $\mathcal{B}$. We know that Γ has a unique fixed point $x \in PSAP_\omega(R^+, X)$, which is the mild solution of (1.1) and (1.2). The proof is complete.

4 Example

To complete this work, we apply the previous results to consider the following differential system

$$\frac{\partial}{\partial t}\left[u(t,\xi)+\int_{t-r}^{t}a(s-t)u(s,\xi)ds\right]$$
$$=\frac{\partial^2}{\partial \xi^2}\left[u(t,\xi)+\int_{t-r}^{t}a(s-t)u(s,\xi)ds\right]+\int_{t-r}^{t}b(s-t)u(s,\xi)ds+c(t)F(u(t,\xi))$$

with

$$u(t,0)=u(t,\pi)=0$$

$$u(\theta,\xi)=\varphi(\theta,\xi),-r\leq\theta\leq 0$$

for $t>0$ and $\xi\in[0,\pi]$, where r is a positive real number, $\varphi\in C([-r,0],X), c\in PSAP_\omega(R)$, $a,b\in C([-r,0],R)$.

In what follows we consider the space $X=L^2([0,\pi])$ and $A: D(A)\subseteq X\to X$ is the operator defined by $Ax=x''$ with domain $D(A)=\{x\in X|x''\in X, x(0)=x(\pi)=0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X. Furthermore, the spectrum of A is reduced to a point spectrum with eigenvalues of the form $-n^2$ for $n\in\mathbb{N}$, and corresponding normalized eigenfunctions given by $z_n(\xi)=(\frac{2}{\pi})^{\frac{1}{2}}\sin(n\xi)$. In addition, $T(t)x=\Sigma_{n=1}^{\infty}e^{-n^2t}\langle x,z_n\rangle z_n$ for $x\in X$ and $\|T(t)\|\leq e^{-t}$ for every $t\geq 0$.

Let $\mathcal{C}=C([-r,0],X)$. Define the functions $D,f,g:[0,\infty)\times\mathcal{C}\to X$ by

$$f(t,\psi)(\xi)=-\int_{-r}^{0}a(s)\psi(s,\xi)ds,$$

$$g(t,\psi)(\xi)=\int_{-r}^{0}b(s)\psi(s,\xi)ds+c(t)F(\psi(0,\xi)),$$
$$D(t,\psi)(\xi)=\psi(0)(\xi)-f(t,\psi)(\xi).$$

where $\psi(\theta)(\xi)=\psi(\theta,\xi)$. Therefore, the above system can be rewritten in the form of (1.1)-(1.2).

For every $t\geq 0$, $F:R\to R$ is globally Lipschitz continuous with Lipschitz constant $L_F>0$. It is easy to see that

$$\|F(t,\psi_1)-F(t,\psi_2)\|\leq(\|a\|_{L^2([-r,0),R)}r^{\frac{1}{2}})\|\psi_1-\psi_2\|_{\mathcal{C}},$$

$$\|G(t,\psi_1)-G(t,\psi_2)\|\leq(\|b\|_{L^2([-r,0),R)}r^{\frac{1}{2}}+|c(t)|L_F)\|\psi_1-\psi_2\|_{\mathcal{C}}$$

for $\psi_1,\psi_2\in\mathcal{C}$.

From Theorem 3.3, we have the following result.

Theorem 4.1 If $\|a\|_{L^2([-r,0),R)}r^{\frac{1}{2}}+\|b\|_{L^2([-r,0),R)}r^{\frac{1}{2}}+|c(t)|L_F<1$, then the above system have a unique pseudo S-asymptotically ω periodic mild solution.

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中立型偏微分方程的伪S-渐近周期解

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摘要: 本文研究了Banach空间中抽象中立型偏微分方程伪S-渐近ω-周期温和解的存在性. 利用Banach压缩映射原理, 得到了该类方程伪S-渐近ω-周期温和解的存在性和唯一性. 推广了中立型偏微分方程周期温和解的相关结论.

关键词: 伪S-渐近ω-周期函数; 温和解; 中立型偏微分方程

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