Vol. 45 (2025)
No. 2

数 学 杂 志
J. of Math. (PRC)

REMARKS ON ASYMPTOTIC EXPANSIONS OF COMPOSITIONS WITH THE ELEMENTARY FUNCTIONS

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Abstract: In this paper, we study asymptotic power series of the composition $f(x) = h(g(x))$, where $g(x) = \sum\limits_{n=0}^{\infty} b_n x^{-n}$, $b_n \in \mathbb{R}$, and h is a given elementary function. The asymptotic expansions have been obtained for the composition with an exponential or logarithmic function. Using the recursive method, we present the asymptotic expansions for the composition with seven trigonometric functions, respectively. As an application, the asymptotic expansions of roots of some equations are given. Computational results show that our recursive formula is more efficient than the method of Lagrange's inverse theorem.

Keywords: asymptotic expansion; asymptotic power series; trigonometric function; composite function

2010 MR Subject Classification: 34E05; 26A06.

Document code: A **Article ID:** 0255-7797(2025)02-0131-09

1 Introduction

In asymptotic analysis, one kind of asymptotic expansions is the asymptotic power series, which is closely related to the formal power series. Both Copson [1] and Erdélyi [2] gave definitions of asymptotic expansions and asymptotic power series. The asymptotic expansion of the form

$$f(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots \qquad \text{as } x \to \infty$$

is called asymptotic power series.

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[*] **Received date:** 2024-09-01 **Accepted date:** 2024-09-25

Foundation item: Supported by The Innovation Fund of Postgraduate, Sichuan University of Science & Engineering (Y2024336), and NSF of Sichuan Province (2023NSFSC0065).

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It is interesting to consider the asymptotic power series of the composition $f(x) = h(g(x))$ where $g(x) = \sum\limits_{n=0}^{\infty} b_n x^{-n}$, $b_n \in \mathbb{R}$, and h is some elementary function. In fact, there are some properties on the composition of such power series.

Lemma 1.1 [[2], p. 20] If the function h has expansion into power series

$$h(x) = \sum_{n=0}^{\infty} c_n x^n, \qquad \text{as } x \to 0$$

and $g(x) = \sum_{n=0}^{\infty} b_n x^{-n}$ with leading coefficient $b_0 = 0$, then $h(g(x))$ has asymptotic expansion whose coefficients can be obtained by formal substitution and rearrangement of terms.

Lemma 1.2 [[2], p. 21] If the function $f(x) = \sum_{n=0}^{\infty} a_n x^{-n}$ is differentiable and if f' possesses an asymptotic power series expansion, then

$$f'(x) = -\sum_{n=1}^{\infty} n a_n x^{-n-1}, \qquad \text{as } x \to \infty.$$

In 2013, Chen et al. [3] gave respectively these recursive formula for the asymptotic power series compounded with the power function, the logarithm function and exponential function.

Lemma 1.3 [[3], Lemma 3] Let $g(x) = \sum_{n=1}^{\infty} b_n x^{-n}$. Then $f(x) = \exp(g(x))$ has asymptotic power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^{-n},$$

where $a_0 = 1$ and

$$a_n = \frac{1}{n} \sum_{k=1}^{n} k b_k a_{n-k}, \qquad n \geq 1.$$

Lemma 1.4 [[3], Lemma 4] Let $g(x) = \sum_{n=0}^{\infty} b_n x^{-n}$, $b_0 \neq 0$. Then $f(x) = \ln(g(x))$ has asymptotic power series

$$f(x) = \sum_{n=1}^{\infty} a_n x^{-n},$$

where

$$a_n = \frac{b_n}{b_0} - \frac{1}{n b_0} \sum_{k=1}^{n-1} k a_k b_{n-k}, \qquad n \geq 1.$$

Lemma 1.5 [[3], Lemma 5] Let $g(x) = \sum_{n=0}^{\infty} b_n x^{-n}$, $b_0 \neq 0$. Then $f(x) = [g(x)]^r$, for all real r, has asymptotic power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^{-n},$$

where

$$a_0 = b_0^r, \quad a_n = \frac{1}{nb_0}\sum_{k=1}^{n}[k(1+r)-n]b_k a_{n-k}.$$

These recursive formulas have a very wide range of applications to asymptotic expansions, for example, the new asymptotic expansions for Gamma function [4], the asymptotic expansion of the integral mean [5], the asymptotic expansion coefficients of Glaisher - Kinkelin type constants [6], Landau constants, Euler-Mascheroni constant, Stirling series [7] and explicit formulas for the Bell numbers and logarithmic polynomials [8].

Besides the above three elementary functions, it seems that no asymptotic expansion formula is given for the asymptotic power series compounded with trigonometric functions. The purpose of this paper is to present these recursive formula for the asymptotic power series compounded with seven trigonometric functions.

As an application, we use these results to give the asymptotic expansion of the roots of some equations, and our recursive method is more efficient than Lagrange's inverse theorem.

2 Main Results

First, we gives the asymptotic power series of $f(x) = \tan(g(x))$.

Theorem 2.1 Suppose that $g(x) = \sum\limits_{n=0}^{\infty} b_n x^{-n}$. Then $f(x) = \tan(g(x))$ has asymptotic power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^{-n},$$

where $a_0 = \tan(b_0)$, and

$$\begin{aligned} a_n &= \frac{1}{n}\sum_{k=1}^{n} kb_k c_{n-k}, \quad n \ge 1, \\ c_0 &= 1 + a_0^2, c_n = \sum_{k=0}^{n} a_k a_{n-k}, \quad n \ge 1; \end{aligned}$$

Proof Taking the derivative of $f(x) = \tan(g(x))$ on both sides, and using the identity $\sec^2(g(x)) = 1 + \tan^2(g(x))$, we have

$$f'(x) = \left(1 + f^2(x)\right) g'(x).$$

Let $1 + f^2(x) = \sum_{n=0}^{\infty} c_n x^{-n}$. Then

$$c_0 = 1 + a_0^2, \quad c_n = \sum_{k=0}^{n} a_k a_{n-k}, \quad n \ge 1.$$

From $f'(x) = \sum_{n=0}^{\infty}(-n)a_n x^{-n-1}$, $g'(x) = \sum_{n=0}^{\infty}(-n)b_n x^{-n-1}$, we see that

$$\left(\sum_{n=1}^{\infty}(-n)b_n x^{-n-1}\right)\left(\sum_{n=0}^{\infty} c_n x^{-n}\right) = \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-k)b_k c_{n-k} x^{-n-1} = \sum_{n=1}^{\infty}(-n)a_n x^{-n-1}.$$

Consequently,

$$a_0 = \tan b_0, \quad a_n = \frac{1}{n}\sum_{k=1}^{n} kb_k c_{n-k}, \quad n \geq 1.$$

This completes the proof of theorem 2.1.

Remark 2.1 With the similar argument, we can obtain the asymptotic power series of the composition with the cotangent function, arc-tangent function and arc-cotangent function.

Suppose that $g(x) = \sum_{n=0}^{\infty} b_n x^{-n}$. Then the following results hold:

(i) $f(x) = \cot(g(x))$ has asymptotic power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^{-n},$$

where $a_0 = \cot(b_0)$, and

$$\begin{aligned} a_n &= -\frac{1}{n}\sum_{k=1}^{n} kb_k c_{n-k}, \quad n \geq 1, \\ c_0 &= 1 + a_0^2, c_n = \sum_{k=0}^{n} a_k a_{n-k}, \quad n \geq 1; \end{aligned}$$

(ii) $f(x) = \arctan(g(x))$ has asymptotic power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^{-n},$$

where $a_0 = \arctan(b_0)$, and

$$a_1 = \frac{b_1}{1 + b_0^2}, \quad a_n = \frac{b_n}{1 + b_0^2} - \frac{1}{n(1 + b_0^2)}\sum_{k=1}^{n-1} ka_k c_{n-k}, \quad n \geq 2,$$

$$c_0 = 1 + b_0^2, \quad c_n = \sum_{k=0}^{n} b_k b_{n-k}, \quad n \geq 1;$$

(iii) $f(x) = \operatorname{arccot}(g(x))$ has asymptotic power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^{-n},$$

where $a_0 = \operatorname{arccot}(b_0)$, and

$$a_1 = -\frac{b_1}{1 + b_0^2}, \quad a_n = -\frac{b_n}{1 + b_0^2} - \frac{1}{n(1 + b_0^2)}\sum_{k=1}^{n-1} ka_k c_{n-k}, \quad n \geq 2,$$

$$c_0 = 1 + b_0^2, \quad c_n = \sum_{k=0}^{n} b_k b_{n-k}, \quad n \geq 1;$$

Next, we give the asymptotic power series of $f(x) = \sin(g(x))$. In the following Theorem, without loss of generality, we assume that $\cos(g(x)) > 0$. In fact, the case $\cos(g(x)) \leq 0$ is similar to the case $\cos(g(x)) > 0$.

Theorem 2.2 Suppose that $g(x) = \sum_{n=0}^{\infty} b_n x^{-n}$ and $\cos(g(x)) > 0$. Then $f(x) = \sin(g(x))$ has asymptotic power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^{-n},$$

where $a_0 = \sin b_0$, and

$$\begin{aligned} a_n &= \frac{1}{n} \sum_{k=1}^{n} k b_k d_{n-k}, \quad n \geq 1, \\ d_0 &= \sqrt{1 - a_0^2}, d_n = \frac{1}{nc_0} \sum_{k=1}^{n} (\frac{3k}{2} - n) c_k d_{n-k}, \quad n \geq 1, \\ c_0 &= 1 - a_0^2, c_n = -\sum_{k=0}^{n} a_k a_{n-k}, \quad n \geq 1; \end{aligned}$$

Proof From $\cos^2(g(x)) + \sin^2(g(x)) = 1$ and $\cos(g(x)) > 0$, differentiating both sides of $f(x) = \sin(g(x))$ yields that

$$g'(x)\sqrt{1 - f^2(x)} = f'(x). \tag{2.1}$$

Put $1 - (f(x))^2 = \sum_{n=0}^{\infty} c_n x^{-n}$. Then we have

$$c_0 = 1 - a_0^2, \quad c_n = -\sum_{k=0}^{n} a_k a_{n-k}, \quad n \geq 1,$$

since

$$(f(x))^2 = \left(\sum_{n=0}^{\infty} a_n x^{-n}\right)\left(\sum_{n=0}^{\infty} a_n x^{-n}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k a_{n-k} x^{-n}.$$

For the sake of convenience, put $\sqrt{1 - f^2(x)} = \sum_{n=0}^{\infty} d_n x^{-n}$. By Lemma 1.5, one see that

$$d_0 = \sqrt{1 - a_0^2}, \quad d_n = \frac{1}{nc_0} \sum_{k=1}^{n} (\frac{3k}{2} - n) c_k d_{n-k}, \quad n \geq 1.$$

Together with

$$g'(x) = \sum_{n=1}^{\infty} (-n) b_n x^{-n-1}, \quad f'(x) = \sum_{n=1}^{\infty} a_n (-n) x^{-n-1},$$

it follows from Eq. (2.1) that

$$\sum_{n=1}^{\infty} (-n) b_n x^{-n-1} \sum_{n=0}^{\infty} d_n x^{-n} = \sum_{n=1}^{\infty} (-n) a_n x^{-n-1}.$$

Comparing the coefficients of x^{-n-1}, one can see that

$$\sum_{k=1}^{n} kb_k d_{n-k} = na_n.$$

Therefore,

$$a_0 = \sin b_0, \quad a_n = \frac{1}{n}\sum_{k=1}^{n} kb_k d_{n-k}, \quad n \geq 1.$$

This completes the proof of Theorem 2.2.

Remark 2.2 With the similar argument, we can obtain the asymptotic power series of the composition with the arc-sine function and arc-cosine function.

(i) $f(x) = \arcsin(g(x))$ has asymptotic power series $f(x) = \sum_{n=0}^{\infty} a_n x^{-n}$, where

$$\begin{aligned} a_0 &= \arcsin b_0, \quad & a_1 &= \frac{b_1}{\sqrt{1-b_0^2}}, \quad a_n = \frac{b_n}{d_0} - \frac{1}{nd_0}\sum_{k=1}^{n-1} ka_k d_{n-k}, \quad n \geq 2, \\ d_0 &= \sqrt{1-b_0^2}, \quad & d_n &= \frac{1}{nc_0}\sum_{k=1}^{n}(\frac{3k}{2} - n)c_k d_{n-k}, \quad n \geq 1, \\ c_0 &= 1 - b_0^2, \quad & c_n &= -\sum_{k=0}^{n} b_k b_{n-k}, \quad n \geq 1; \end{aligned}$$

(ii) $f(x) = \arccos(g(x))$ has asymptotic power series $f(x) = \sum_{n=0}^{\infty} a_n x^{-n}$, where

$$\begin{aligned} a_0 &= \arccos b_0, \quad & a_1 &= \frac{b_1}{\sqrt{1-b_0^2}}, \quad a_n = -\frac{b_n}{d_0} - \frac{1}{nd_0}\sum_{k=1}^{n-1} ka_k d_{n-k}, \quad n \geq 2, \\ d_0 &= \sqrt{1-b_0^2}, \quad & d_n &= \frac{1}{nc_0}\sum_{k=1}^{n}(\frac{3k}{2} - n)c_k d_{n-k}, \quad n \geq 1, \\ c_0 &= 1 - b_0^2, \quad & c_n &= -\sum_{k=0}^{n} b_k b_{n-k}, \quad n \geq 1. \end{aligned}$$

3 Applications

In this section, we will use Theorem 2.1 to give the asymptotic behavior of roots of the equation $x = \cot x$.

Example 3.1 We can see that there exists a unique root x_n of the equation $x = \cot x$ such that $x_n \in (n\pi, (n+1)\pi)$ for every fixed $n \in \mathbb{Z}$. Then x_n has asymptotic power series

$$x_n = n\pi + \frac{1}{n\pi} + \frac{b_2}{(n\pi)^2} + \frac{b_3}{(n\pi)^3} + \cdots, \quad n \to \infty,$$

where

$$b_1 = 1; b_{2n} = 0;$$

$$b_3 = -\frac{4}{3}; b_5 = \frac{53}{15}; b_7 = -\frac{1226}{105}; b_9 = \frac{13597}{315}; b_{11} = -\frac{65782}{385};$$
$$b_{13} = \frac{478551932}{675675}; b_{15} = -\frac{6152345618}{2027025}; b_{17} = \frac{3217326704057}{241215975}.$$

Proof Let $\omega = n\pi$ and $z = \sum_{n=1}^{\infty} b_n\omega^{-n}$. It follows from Theorem 2.1 that

$$\tan(z) = \sum_{n=1}^{\infty} a_n\omega^{-n},$$

where

$$b_0 = 0, a_0 = 0, c_0 = 1, b_1 = 1,$$
$$a_n = \frac{1}{n}\sum_{k=1}^{n} kb_kc_{n-k}, \quad c_n = \sum_{k=0}^{n} a_ka_{n-k}, \quad n \geq 1.$$

From $(\omega + z)\tan(\omega + z) = (\omega + z)\tan(z) = 1$, we see that

$$\sum_{n=1}^{\infty} a_n\omega^{-n+1} + \left(\sum_{n=1}^{\infty} b_n\omega^{-n}\right)\left(\sum_{n=1}^{\infty} a_n\omega^{-n}\right) = \sum_{n=0}^{\infty} a_{n+1}\omega^{-n} + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_kb_{n-k}\omega^{-n} = 1,$$

where $b_0 = 1, a_1 = 1$. Then

$$\sum_{n=1}^{\infty} a_{n+1}\omega^{-n} + \sum_{n=1}^{\infty}\sum_{k=1}^{n} a_kb_{n-k}\omega^{-n} = 0.$$

Comparing the coefficients of ω^{-n}, we obtain

$$a_{n+1} = -\sum_{k=1}^{n} a_kb_{n-k}.$$

Thus,

$$\frac{1}{n+1}\sum_{k=1}^{n+1} kb_kc_{n+1-k} = b_{n+1}c_0 + \frac{1}{n+1}\sum_{k=1}^{n} kb_kc_{n+1-k} = -\sum_{k=1}^{n} a_kb_{n-k},$$

Therefore,

$$b_{n+1} = -\frac{1}{n+1}\sum_{k=1}^{n} kb_kc_{n+1-k} - \sum_{k=1}^{n} a_kb_{n-k},$$

where

$$b_0 = 0, a_0 = 0, c_0 = 1, b_1 = 1, c_1 = 0, a_1 = 1,$$
$$a_n = \frac{1}{n}\sum_{k=1}^{n} kb_kc_{n-k}, \quad c_n = \sum_{k=0}^{n} a_ka_{n-k}, \quad n \geq 1.$$

Using the software MATLAB, it is easy to get the values of the previous terms:

$$b_1 = 1; b_{2n} = 0;$$

$$b_3 = -\frac{4}{3}; b_5 = \frac{53}{15}; b_7 = -\frac{1226}{105}; b_9 = \frac{13597}{315}; b_{11} = -\frac{65782}{385};$$
$$b_{13} = \frac{478551932}{675675}; b_{15} = -\frac{6152345618}{2027025}; b_{17} = \frac{3217326704057}{241215975}.$$

There is another method in [9, p. 24]. From the intersections of $y = x$ and $y = \cot x$, one can find that $x = \cot x$ has infinitely many roots. Further, we have

$$x_n \in (n\pi, (n+1)\pi), \quad n = \pm 1, \pm 2, \pm 3, \cdots .$$

Without loss of generality, assume $x_n > 0$ and $n\pi < x_n < (n+1)\pi$. Then

$$x_n = \cot x_n = \cot(x_n - n\pi).$$

Since $x_n \to \infty$, one has $x_n - n\pi \to 0$. Let $z = x_n - n\pi$ and $\omega = n\pi$. Thus $\omega + z = \cot z$. Consequently,

$$\omega^{-1} = \frac{\sin z}{\cos z - z\sin z} = \frac{z}{\frac{z(\cos z - z\sin z)}{\sin z}}.$$

Using the Lagrange's inverse theorem [9, p. 22], one has

$$\begin{aligned} z &= \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^{n-1}}{d\zeta^{n-1}} \left\{ \left(\frac{\zeta(\cos\zeta - \zeta\sin\zeta)}{\sin\zeta} \right)^n \right\}_{\zeta=0} \omega^{-n} \\ &= \omega^{-1} + b_2\omega^{-2} + b_3\omega^{-3} + \cdots . \end{aligned}$$

Namely,

$$x_n = n\pi + z = n\pi + \frac{1}{n\pi} + \frac{b_2}{(n\pi)^2} + \cdots = n\pi + \frac{1}{n\pi} + O\left(\frac{1}{n^2}\right).$$

Using the symbolic calculation software MAPLE, it is easy to get the values of the previous terms:

$$b_1 = 1; b_2 = b_4 = b_6 = b_8 = b_{10} = 0;$$
$$b_3 = -\frac{4}{3}; b_5 = \frac{53}{15}; b_7 = -\frac{1226}{105}; b_9 = \frac{13597}{315}; b_{11} = -\frac{65782}{385}.$$

Comparing these two methods, we can see that the recursive method is more efficient than Lagrange's inverse theorem. At the same time, we are sure that some higher derivatives can be solved by the recursive method.

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求解初等函数复合的渐近展开补注

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摘 要: 本文研究了复合函数$f(x) = h(g(x))$ 的渐近幂级数, 其中$g(x) = \sum\limits_{n=0}^{\infty} b_n x^{-n}$, $b_n \in \mathbb{R}$, h 为给定初等函数. 当h是指数或对数函数时，复合函数的渐近展开已有结果. 利用递推法, 本文分别获得了七个三角函数复合的渐近展开式. 作为应用, 还给出了方程根的渐近展开. 计算结果显示, 我们的递推公式比拉格朗日逆定理的方法更有效率.

关键词: 渐近展开; 渐近幂级数; 三角函数; 复合函数

MR(2010)**主题分类号**: 34E05; 26A06 **中图分类号**: O173.1; O241.5