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SIMONS-TYPE INTEGRAL AND HEIGHT FUNCTION IN SPHERES

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Abstract: This article delves Chern's conjecture for hypersurfaces with constant second fundamental form squared length S in the spherical space. At present, determining whether the third gap point of S is $2n$ remains unsolved yet. First, we investigate the height functions and their properties of the position vector and normal vector in natural coordinate vectors, and then prove the existence of a Simons-type integral formula on the hypersurface that simultaneously includes the first, second, and third gap point terms of S. These results can provide new avenues of thought and methods for solving Chern's conjecture.

Keywords: Chern's conjecture; Height function; Simons-Type integral

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1 Introduction

In the late 1960s, Chern proposed the following notable conjecture[1, 2]:

Conjecture 1.1 Let M^n be a closed immersed minimal hypersurface of the unit sphere $\mathbb{S}^{n+1}$ with constant scalar curvature R_M. Then for any dimension n, the set of all possible values of R_M is discrete.

With further study of the above conjecture, mathematicians not only realized the importance of Conjecture 1.1, but also proposed a strengthened version:

Conjecture 1.2 (Chern's Conjecture) Let M^n be a closed immersed minimal hypersurface of the unit sphere $\mathbb{S}^{n+1}$ with constant scalar curvature, then M^n is isoparametric.

Up to now, the conjecture remains far from being completely resolved. S. T. Yau has reintroduced it as the 105th problem in reference[3]. For the latest research on this conjecture, see the works of Scherfner-Weiss[4], Scherfner-Weiss-Yau[5], Ge-Tang[6], and others. In 1968, J. Simons proved the following theorem in reference[7]:

Theorem 1.3 (Simon's Inequality) Let M^n be a closed immersed minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$ with squared norm S of the second fundamental form. Then the inequality

$$\int_{M^n} S(S-n)dV_M \geq 0 \tag{1.1}$$

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holds eternally on M^n. In particular, if $0 \leq S \leq n$, then $S \equiv 0$ or $S \equiv n$ on M^n.

In Theorem 1.3, it is evident that the minimal hypersurface satisfying $S \equiv 0$ is the equatorial sphere. The classification of $S \equiv n$ is proven by Chern-do Carmo-Kobayshi[8, 9] and Lawson[10]: the Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})(1 \leq k \leq n-1)$ is the only closed minimal hypersurface in S^{n+1} with $S \equiv n$. By classifying isoparametric hypersurfaces with constant principal curvatures, it can be seen that the above two hypersurfaces are isoparametric hypersurfaces with multiplicity of principal curvatures being 1 and 2, respectively. For closed immersed minimal hypersurfaces in $\mathbb{S}^{n+1}$, it is easy to observe the following relationship from the Gauss equation:

$$R_M = n(n-1) - S. \tag{1.2}$$

Therefore, Simons inequality is regarded as the first breakthrough for Conjecture 1.1.

In 1983, Peng and Terng made significant progress on Chern's Conjecture 1.1, proving that if $S > n$, then $S > n + \frac{1}{12n}$. In 1993, Chang[11] completed the proof of Chern's Conjecture 1.2 for the case $n = 3$: if $S > 3$, then $S \geq 6$. In recent years, Yang-Cheng[12] and Suh-Yang[13] have further improved the constant term from $\frac{1}{12n}$ to $\frac{3n}{7}$, and Cheng-Wei[14] proved that if $S > n$, then $S > 1.8252n - 0.712898$ under the additional condition that trh^3 is constant. However, the question for higher dimension, "If $S \equiv$ constant $> n$, then $S \geq 2n$?" remains open.

In Section 3, this article systematically studies the height functions φ^A and ψ^A (defined below) and their important properties.

Definition 1.4 (Height Function) Let M^n be a hypersurface isometrically immersed in $\mathbb{S}^{n+1}(\subset \mathbb{R}^{n+2})$, a be a fixed unit vector in $\mathbb{R}^{n+2}$, and $\{x|\, e_1, \cdots, e_n, e_{n+1}\}$ be the Darboux frame of the hypersurface M^n. Define smooth functions φ and ψ: $M^n \to \mathbb{R}$, satisfying

$$\varphi\left(p\right) = \left\langle x\left(p\right), a\right\rangle, \psi\left(p\right) = \left\langle e_{n+1}(p), a\right\rangle,\ \forall p \in M.$$

Then φ and ψ are regarded as the height functions of the position vector x and the normal vector e_{n+1} in the direction of the vector a. In particular, when a is the natural basis coordinate vector $E_A = (0, \cdots, 0, \underset{A}{1}, 0, \cdots 0)$, $\varphi^A = \langle x, E_A\rangle$ and $\psi^A = \langle e_{n+1}, E_A\rangle$ are the coordinate functions of x and e_{n+1} in $\mathbb{R}^{n+2}$ at the A-th position $(A = 1, 2, \cdots, n+2)$.

In Section 4, utilizing the important properties of the height functions φ^A and ψ^A, the motion equations of hypersurfaces in $\mathbb{S}^{n+1}$, the integrability conditions (Gauss-Codazzi equations), and the implications of Green's theorem, we calculate the Laplacians of the second and third covariant differential forms of φ^A, and obtain two integral equations for the first, second, and third gap point terms that simultaneously include the second fundamental form squared norm S.

Theorem 1.5 Let M^n be a closed immersed minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$ with constant scalar curvature. Then the integrals

$$\int_{M^n} S(n-S) + \|\nabla h\|^2 dV_M = 0, \tag{1.3}$$

$$\int_{M^n} S(S-n)(2n-S)+\left\|\nabla^2 h\right\|^2-3\left[Strh^4-\left(trh^3\right)^2-S(S-n)-S^2\right]dV_M=0, \quad (1.4)$$

holds true on M^n, where S is the squared norm of the second fundamental form h of the hypersurface, and trh^3 and trh^4 are represented in a local orthogonal coordinate system respectively as $\sum_{i,j,k} h_{ij}h_{jk}h_{ki}$ and $\sum_{i,j,k,l} h_{ij}h_{jk}h_{kl}h_{li}$.

Through the non-negativity of the square of the gradient norm of the second fundamental form, it is easy to see that formula (1.3) can be derived similarly to the Simons inequality: If $0 \le S \le n$, then $S \equiv 0$ or $S \equiv n$; for formula (1.4), if the non-negativity of integral

$$\int_{M^n} \left\|\nabla^2 h\right\|^2-3\left[Strh^4-\left(trh^3\right)^2-S(S-n)-S^2\right]dV_M$$

can be proven, it can be concluded that: If $S > n$, then $S \ge 2n$. Therefore, formulas (1.3) and (1.4) are collectively referred to as Simons-type integrals.

2 Background Material

Let M^n be a closed immersed minimal hypersurface with constant scalar curvature that is isometrically immersed in the unit sphere $\mathbb{S}^{n+1}$ $(\subset \mathbb{R}^{n+2})$, h denotes the second fundamental form of the hypersurface with respect to the unit normal vector field e_{n+1}. In $\mathbb{S}^{n+1}$, choose a set of orthogonal coordinate systems $e_1,\cdots,e_n,e_{n+1}$, such that their restriction on M^n, $e_1,\cdots,e_n$ are tangent to M^n and e_{n+1} is orthogonal to M^n, let $\omega_1,\cdots,\omega_n,\omega_{n+1}$ be the dual frame fields. In this article, we use the following range of indices:

$$1 \le i,j,k,\cdots \le n+1,\ 1 \le A,B,C,\cdots \le n+2.$$

Thus, the motion equations of the hypersurface M^n concerning the Darboux frame $\{x|\, e_1,\cdots,e_n,e_{n+1}\}$ can be written as

$$\begin{cases} dx=\omega_i e_i, \\ de_i=\omega_{ij}e_j+\omega_{in+1}e_{n+1}-\omega_i x, \\ de_{n+1}=\omega_{n+1i}e_i, \end{cases} \quad (2.1)$$

The second fundamental form of M^n is

$$h=-\langle dx, de_{n+1}\rangle=\omega_i\omega_{in+1}=h_{ij}\omega_i\omega_j, \quad (2.2)$$

where $\langle\cdot,\cdot\rangle$ is the inner product of M^n, x is the position vector of point p on M^n in $\mathbb{R}^{n+2}$ satisfying $\|x\|^2=1$, $\{\omega_{ij}\}$ is the connection form of M^n with respect to $\{\omega_i\}$, satisfying the following structural equation:

$$d\omega_i=\omega_j\wedge\omega_{ji},\ \omega_{ij}+\omega_{ji}=0, \quad (2.3)$$

$$\omega_{ij}=\omega_{ik}\wedge\omega_{kj}-\frac{1}{2}R_{ijkl}\omega_k\wedge\omega_l, \quad (2.4)$$

where R_{ijkl} is the component of the Riemann curvature tensor on M^n. The covariant derivative ∇h with components $h_{ij,k}$ is defined as

$$h_{ij,k}\omega_k = dh_{ij} + h_{kj}\omega_{ki} + h_{ik}\omega_{kj}. \tag{2.5}$$

Furthermore, the Gauss-Codazzi equations and Ricci formulas are given by

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk} + \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk},\ h_{ij,k} = h_{ik,j}, \tag{2.6}$$

$$h_{ij,kl} - h_{ij,lk} = \sum_m h_{im}R_{mjkl} + \sum_m h_{mj}R_{mikl}. \tag{2.7}$$

Finally, let's review the divergence theorem in Riemann geometry and its important applications:

Lemma 2.1 (Divergence Theorem) Let (M, g) be a compact oriented n dimensional Riemannian manifold with boundary, and $\overrightarrow{n}$ be the unit normal vector pointing inward to ∂M in M^n, then for any $X \in \mathfrak{X}(M)$, the following integral formula holds:

$$\int_M divX dV_M = -\int_{\partial M} g(\overrightarrow{n}, X)dV_{\partial M},$$

where ∂M has a direction induced by M, and dV_M is a volume element of ∂M.

Corollary 2.2 (Green's Formula) Assume the same as Lemma 2.1, then for any $f \in C^\infty(M)$, the following integral formula holds:

$$\int_M \Delta f dV_M = -\int_{\partial M} \overrightarrow{n}\,(f)dV_{\partial M}.$$

In particular,

$$\int_M \Delta f dV_M = 0$$

holds true when $\partial M = \emptyset$.

3 Important Properties of Height Function in Spheres

In this section, we introduce three important properties of height functions on hypersurfaces M^n in spherical space. First, define the smooth function φ^A to have first and second order covariant derivatives $\nabla\varphi^A$, $\nabla^2\varphi^A$ of the components φ^A_i and $\varphi^A_{i,j}$, respectively, as

$$\varphi^A_i\omega_i = d\varphi^A,\ \varphi^A_{i,j}\omega_j = d\varphi^A_i + \varphi^A_j\omega_{ji}. \tag{3.1}$$

Thus, there exists the following property:

Property 3.1 Let M^n be a minimal hypersurface in $\mathbb{S}^{n+1}\,(\subset \mathbb{R}^{n+2})$, φ^A and ψ^A be the height functions of the position vector x and the normal vector e_{n+1} defined on M^n in the direction of the natural basis coordinate vectors E_A. Then φ^A satisfies

$$\varphi^A_i = \langle e_i, E_A\rangle,\ \varphi^A_{i,j} = \psi^A h_{ij} - \varphi^A\delta_{ij},\ \Delta\varphi^A = -n\varphi^A. \tag{3.2}$$

Proof According to Definition 1.4, formula (3.1), (2.1), (2.2), and the compatibility of the connection ∇, it is calculated that

$$\varphi_i^A\omega_i = d\langle x, E_A\rangle = \langle dx, E_A\rangle = \langle e_i, E_A\rangle\,\omega_i, \tag{3.3}$$

and

$$\begin{aligned}\varphi_{i,j}^A\omega_j &= d\langle e_i, E_A\rangle + \langle e_j, E_A\rangle\,\omega_{ji}\\ &= \langle \omega_{ij}e_j + h_{ij}\omega_j e_{n+1} - \omega_i x + e_j\omega_{ji}, E_A\rangle\\ &= \left(\psi^A h_{ij} - \varphi^A\delta_{ij}\right)\omega_j.\end{aligned} \tag{3.4}$$

Furthermore, since M^n is minimal hypersurface ($H = \frac{1}{n}\sum_i h_{ii} \equiv 0$), we have $\Delta\varphi^A = -n\varphi^A$, which completes the proof of Property 3.1 .

Define the smooth function ψ^A with components ψ_i^A and first and second order covariant derivatives $\nabla\psi^A$ and $\nabla^2\psi^A$ as

$$\psi_i^A\omega_i = d\psi^A,\ \psi_{i,j}^A\omega_j = d\psi_i^A + \psi_j^A\omega_{ji}. \tag{3.5}$$

Thus, there exists the following property:

Property 3.2 Let M^n be a minimal hypersurface in $\mathbb{S}^{n+1}\,(\subset \mathbb{R}^{n+2})$, φ^A and ψ^A be the height functions of the position vector x and the normal vector e_{n+1} in the direction of the natural basis coordinate vectors E_A defined on M^n, and φ_i^A be the component of the covariant derivative $\nabla\varphi^A$. Then ψ^A satisfies

$$\psi_i^A = -\varphi_j^A h_{ij},\ \psi_{i,j}^A = -\varphi_k^A h_{ij,k} + \varphi^A h_{ij} - \psi^A h_{ik}h_{kj},\ \Delta\psi^A = -S\psi^A. \tag{3.6}$$

Proof According to Definition 1.4, formulas (2.1), (2.2), (2.5), (2.6), (3.2), (3.5) and the compatibility of the connection ∇, it is calculated that

$$\psi_i^A\omega_i = d\langle e_{n+1}, E_A\rangle = \langle de_{n+1}, E_A\rangle = \langle -h_{ij}\omega_i e_j, E_A\rangle = -\varphi_j^A h_{ij}\omega_i, \tag{3.7}$$

and

$$\begin{aligned}\psi_{i,j}^A\omega_j &= -d\left(\varphi_j^A\right)h_{ij} - \varphi_j^A d\left(h_{ij}\right) - \varphi_k^A h_{jk}\omega_{ji}\\ &= \varphi_k^A h_{ij}\omega_{kj} - \varphi_{j,k}^A h_{ij}\omega_k + \varphi_j^A h_{kj}\omega_{ki} + \varphi_j^A h_{ik}\omega_{kj} - \varphi_j^A h_{ij,k}\omega_k - \varphi_k^A h_{jk}\omega_{ji}\\ &= \left(-\varphi_k^A h_{ij,k} + \varphi^A h_{ij} - \psi^A h_{ik}h_{kj}\right)\omega_j.\end{aligned} \tag{3.8}$$

Furthermore, since M^n is a minimal hypersurface, we have $\Delta\psi^A = -S\psi^A$, which completes the proof of Property 3.2.

Through Property 3.1 and Property 3.2, it is not difficult to obtain the following property:

Property 3.3 Let M^n be a minimal hypersurface in $\mathbb{S}^{n+1}\,(\subset \mathbb{R}^{n+2})$, and φ^A, ψ^A are the height functions of the position vector x and the normal vector e_{n+1} in the direction of the natural basis coordinate vectors E_A , while φ_i^A, ψ_i^A are the components of the covariant derivatives $\nabla\varphi^A$ and $\nabla\psi^A$, respectively, then φ_i^A, ψ_i^A satisfy

$$\begin{gathered}\varphi^A\varphi^A=\psi^A\psi^A=1,\ \varphi_i^A\varphi_j^A=\delta_{ij},\ \varphi_i^A\psi_j^A=-h_{ij},\ \psi_i^A\psi_j^A=h_{ik}h_{kj},\\ \varphi^A\psi^A=\varphi^A\varphi_i^A=\varphi^A\psi_i^A=\psi^A\varphi_i^A=\psi^A\psi_i^A=0.\end{gathered}\tag{3.9}$$

Proof Using Definition 1.4, Property 3.1, and Property 3.2, as well as the fact that x, $e_i\,(1\le i\le n)$, and e_{n+1} are mutually orthogonal unit vectors, we can complete the proof of Property 3.3.

4 Simons-type Integral

Before addressing the two Simons type integrals, it is necessary to use the divergence theorem to establish the following lemma:

Lemma 4.1 Let M^n be a closed immersed minimal hypersurface in the unit sphere with constant scalar curvature R_M, h_{ij} be the components of the second fundamental form of M^n. Then the following integral formulas

$$\int_M(2h_{ij}h_{kl}h_{ik,m}h_{jl,m}+h_{ij}h_{jk}h_{lm}h_{ik,lm})dV_M=0,\tag{4.1}$$

$$\int_M\left[2h_{ij}h_{kl}h_{lm,i}h_{lm,k}+h_{ij}h_{kl}h_{il,m}h_{jk,m}+(n-S)trh^4\right]dV_M=0,\tag{4.2}$$

$$\int_M\left[h_{ij}h_{jk}h_{lm,i}h_{lm,k}+h_{ij}h_{jk}h_{lm}h_{ik,lm}+\left(trh^3\right)^2-Strh^4+S^2\right]dV_M=0,\tag{4.3}$$

$$\int_M\left[2h_{ij}h_{kl,i}h_{kl,j}+(n-S)trh^3\right]dV_M=0,\tag{4.4}$$

$$\int_M(h_{ij}h_{kl,i}h_{kl,j}+h_{ij}h_{kl}h_{ij,kl})dV_M=0,\tag{4.5}$$

$$\int_M\left[\|\nabla h\|^2+(n-S)S\right]dV_M=0\tag{4.6}$$

hold true on M^n.

Proof Define six 1-forms on M^n: $\Phi^\lambda=\Phi^\lambda_m\omega_m$ ($\lambda=1,2,\cdots,6$), where the components Φ^λ_m satisfy:

$$\begin{gathered}\Phi^1_m=h_{ij}h_{jk}h_{lm}h_{ik,l},\ \Phi^2_m=h_{ij}h_{jk}h_{kl}h_{il,m},\ \Phi^3_m=h_{ij}h_{mk}h_{kl}h_{li,j},\\ \Phi^4_m=h_{ij}h_{jk}h_{ik,m},\ \Phi^5_m=h_{ij}h_{km}h_{ij,k},\ \Phi^6_m=h_{ij}h_{ij,m}.\end{gathered}\tag{4.7}$$

Using formulas (2.6), (2.7), $R_M\equiv$ constant, and the minimal condition, we obtain

$$div\left(\Phi^1\right)=\Phi^1_{m,m}=2h_{ij}h_{kl}h_{ik,m}h_{jl,m}+h_{ij}h_{jk}h_{lm}h_{ik,lm},\tag{4.8}$$

$$div\left(\Phi^2\right)=\Phi^2_{m,m}=2h_{ij}h_{kl}h_{lm,i}h_{lm,k}+h_{ij}h_{kl}h_{il,m}h_{jk,m}+(n-S)trh^4,\tag{4.9}$$

$$div\left(\Phi^3\right)=\Phi^3_{m,m}=h_{ij}h_{jk}h_{lm,i}h_{lm,k}+h_{ij}h_{jk}h_{lm}h_{ik,lm}+\left(trh^3\right)^2-Strh^4+S^2, \tag{4.10}$$

$$div\left(\Phi^4\right)=\Phi^4_{m,m}=2h_{ij}h_{kl,i}h_{kl,j}+(n-S)trh^3, \tag{4.11}$$

$$div\left(\Phi^5\right)=\Phi^5_{m,m}=h_{ij}h_{kl,i}h_{kl,j}+h_{ij}h_{kl}h_{ij,kl}, \tag{4.12}$$

$$div\left(\Phi^6\right)=\Phi^6_{m,m}=\|\nabla h\|^2+(n-S)S. \tag{4.13}$$

Since M^n is closed, it follows that M^n is compact without boundary. By further utilizing the divergence theorem, the proof of Lemma 4.1 is completed.

The third, fourth, and fifth order covariant derivatives $\nabla^3\varphi^A$, $\nabla^4\varphi^A$, and $\nabla^5\varphi^A$, and the second order covariant derivatives $\nabla^2 h$, which have components $\varphi^A_{i,jk}$, $\varphi^A_{i,jkl}$, $\varphi^A_{i,jklm}$, and $h_{ij,kl}$, are defined as follows:

$$\varphi^A_{i,jk}\omega_k=d\varphi^A_{i,j}+\varphi^A_{k,j}\omega_{ki}+\varphi^A_{i,k}\omega_{kj}, \tag{4.14}$$

$$\varphi^A_{i,jkl}\omega_l=d\varphi^A_{i,jk}+\varphi^A_{l,jk}\omega_{ki}+\varphi^A_{i,lk}\omega_{kj}+\varphi^A_{i,jl}\omega_{lk}, \tag{4.15}$$

$$\varphi^A_{i,jklm}\omega_m=d\varphi^A_{i,jkl}+\varphi^A_{m,jkl}\omega_{mi}+\varphi^A_{i,mkl}\omega_{mj}+\varphi^A_{i,jml}\omega_{mk}+\varphi^A_{i,jkm}\omega_{ml}, \tag{4.16}$$

$$h_{ij,kl}\omega_l=dh_{ij,k}+h_{lj,k}\omega_{li}+h_{il,k}\omega_{lj}+h_{ij,l}\omega_{lk}. \tag{4.17}$$

Finally, we present the proof of Theorem 1.5:

Proof First, for the proof of formula (1.3): Based on formulas (4.14) and (4.15), combined with formula (3.2), it can be calculated that

$$\varphi^A_{i,jk}=\psi^A_k h_{ij}+\psi^A h_{ij,k}-\varphi^A_k\delta_{ij}, \tag{4.18}$$

$$\varphi^A_{i,jkl}=\psi^A_{k,l}h_{ij}+\psi^A_k h_{ij,l}+\psi^A_l h_{ij,k}+\psi^A h_{ij,kl}-\varphi^A_{k,l}\delta_{ij}, \tag{4.19}$$

Using formulas (3.2) and (3.6) , we have

$$\varphi^A_{i,jkk}=-\psi^A\cdot Sh_{ij}+\psi^A_k\cdot 2h_{ij,k}+\varphi^A\cdot n\delta_{ij}+\psi^A h_{ij,kk}. \tag{4.20}$$

From $R_M\equiv$ constant, it can be inferred that $S\equiv$ constant. Combining this with formulas (2.6), (2.7), and the minimal condition, we calculate to obtain

$$h_{ij}h_{ij,kk}=h_{ij}\left[(h_{ki,jk}-h_{ki,kj})+h_{kk,ij}\right]=h_{ij}\left(h_{pi}R_{pj}+h_{kp}R_{pijk}\right)=S\left(n-S\right). \tag{4.21}$$

Therefore, from formulas (2.6), (3.9), (4.18), (4.20), (4.21) and the minimal condition, we have

$$\begin{aligned}\frac{1}{2}\Delta\sum_A\left\|\nabla^2\varphi^A\right\|^2&=\left(\varphi^A_{i,jk}\right)^2+\varphi^A_{i,j}\varphi^A_{i,jkk}\\&=\left(S^2+\|\nabla h\|^2+n^2\right)+\left(-S^2-n^2+h_{ij}h_{ij,kk}\right)\\&=\|\nabla h\|^2+S(n-S).\end{aligned} \tag{4.22}$$

Since M^n is closed, the formula (1.3) holds using a corollary of the Green formula.

Next, we prove formula (1.4): substituting formula (3.2) into formula (4.19), we can see

$$\begin{aligned}\varphi^A_{i,jkl} =& -\varphi^A_m h_{kl,m} h_{ij} + \varphi^A h_{ij} h_{kl} - \psi^A h_{km} h_{ml} h_{ij} + \psi^A_k h_{ij,l} \\ &+ \psi^A_l h_{ij,k} + \psi^A h_{ij,kl} - \psi^A h_{kl}\delta_{ij} + \varphi^A \delta_{ij}\delta_{kl}.\end{aligned} \tag{4.23}$$

According to formulas (1.4), (3.6), (4.16), $S \equiv$ constant and the minimal condition, it is noted that

$$\begin{aligned}h_{ij,k}h_{ij,kll} &= h_{ij,k}\left[(h_{ij,kl} - h_{ij,lk})_l + (h_{li,jkl} - h_{li,jlk}) + (h_{li,jl} - h_{li,lj})_k + h_{ll,ijk}\right] \\ &= h_{ij,k}\left[(h_{pj}R_{pikl} + h_{ip}R_{pjkl})_l + (h_{pi,j}R_{pk} + h_{lp,j}R_{pikl} + h_{li,p}R_{pjkl})\right. \\ &\qquad \left. +(h_{pi}R_{pj} + h_{lp}R_{pijl})_k\right] \\ &= 6h_{ij}h_{kl}h_{ik,m}h_{jl,m} - 3h_{ij}h_{jk}h_{lm,i}h_{lm,k} + (2n - S + 3)\left\|\nabla h\right\|^2.\end{aligned} \tag{4.24}$$

Furthermore, we calculated that

$$\begin{aligned}&\frac{1}{2}\Delta\sum_A \left\|\nabla^3\varphi^A\right\|^2 = \left(\varphi^A_{i,jkl}\right)^2 + \varphi^A_{i,jk}\varphi^A_{i,jkll} \\ =&(\left\|\nabla^2 h\right\|^2 + 3S\|\nabla h\|^2 + Strh^4 + S^2 + nS + n^2 - 2h_{ij}h_{km}h_{ml}h_{ij,kl} + 2h_{ij}h_{jk}h_{lm,i}h_{lm,j}) \\ &+ [2S^2(n-S) + 2h_{ij}h_{km}h_{ml}h_{ij,kl} - Strh^4 - S^2 - 2h_{ij}h_{jk}h_{ij,k}h_{ij,l} - S\|\nabla h\|^2 \\ &+h_{ij,k}h_{ij,kll} - nS - n^2] \\ =&\left\|\nabla^2 h\right\|^2 + (2n + S + 3)\left\|\nabla h\right\|^2 + 2S^2(n-S) + 6h_{ij}h_{kl}h_{ik,m}h_{jl,m} - 3h_{ij}h_{jk}h_{lm,i}h_{lm,k}.\end{aligned} \tag{4.25}$$

Finally, since M^n is closed, it can be inferred from the Green's formula that there holds an integral equation

$$\begin{aligned}\int_M &\left\|\nabla^2 h\right\|^2 + (2n + S + 3)\left\|\nabla h\right\|^2 + 2S^2(n-S) \\ &+6h_{ij}h_{kl}h_{ik,m}h_{jl,m} - 3h_{ij}h_{jk}h_{lm,i}h_{lm,k}dV_M = 0\end{aligned} \tag{4.26}$$

on M^n, Using the formulas (4.1), (4.3), and (4.6) to simplify the above expression, we obtain the formula (1.4), thus completing the proof of Theorem 1.5.

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球空间中高度函数与 Simons 型积分公式

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摘要： 本文研究了球空间中具有常第二基本形式模长平方 S 超曲面的陈猜想. 目前“S 的第三个间隙点取值是否为 $2n$”依然是一个公开性的问题. 首先我们研究了位置向量和法向量在自然坐标向量上的高度函数及其性质, 然后证明了在超曲面上存在同时具有 S 第一、二、三间隙点项的 Simons 型积分方程, 这些结果可以为解决陈猜想的研究提供全新的思考途径与求解方法.

关键词： 陈猜想; 高度函数; Simons 型积分

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