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POSITIVE GROUND STATE SOLUTIONS FOR A QUASILINEAR SCHRÖDINGER EQUATION

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Abstract: This paper is concerned with the positive ground state solutions for a quasilinear Schrödinger equation with a Hardy-type term. We obtain positive ground state solutions for the given quasilinear Schrödinger equation by using a change of variables and variational method.

Keywords: Quasilinear equation; schrödinger equation; positive ground state solutions; variational methods

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1 Introduction and main results

We consider the solitary wave solutions for quasilinear Schrödinger equations of the form

$$iz_t = -\Delta z + W(x)z - k(x,z) - \Delta l(|z|^2)l'(|z|^2)z, \tag{1.1}$$

where $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ is a complex function, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential function, $k : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ and $l : \mathbb{R} \to \mathbb{R}$ are suitable functions. Quasilinear equation (1.1) has been derived as models of several physical phenomena, for which we can refer to [1–8] and references therein. For example, for the special case $l(s) = s$, quasilinear equation(1.1) which has been called the superfluid film equation in fluid mechanics by Kurihara [9], can model the time evolution of the condensate wave function in super-fluid film ([9, 10]). For the special case $l(s) = (1+s)^{1/2}$, quasilinear equation(1.1) can model the self-channeling of a high-power ultra short laser in matter. Propagation of a high irradiance laser in a plasma produces an optical refractive index which is nonlinear related to the intensity of the light and gives rise to an interesting new nonlinear wave equation. (see [11–14]).

Set $z(t,x) = \exp(-iEt)u(x)$, where E is a real number and $u(x)$ is a real function, equation (1.1) can be simplified to a quasilinear elliptic equation (see [15])

$$-\Delta u + V(x)u - \Delta l(u^2)l'(u^2)u = k(x,u), \quad x \in \mathbb{R}^N. \tag{1.2}$$

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Thus, in order to find the solitary wave solution of (1.1), only the positive solution of the quasilinear elliptic equation (1.2) is required. If $l(s) = s$, the superfluid film equation in plasma physics can be obtained as follows:

$$-\Delta u + V(x)u - \Delta(u^2)u = k(x,u), \quad x \in \mathbb{R}^N. \tag{1.3}$$

Recent studies mainly focus on equation (1.3) with $k(x,u) = |u|^{q-2}u$ at infinity for $4 \leq q < 22^*,\ N \geq 3$, where $2^* = 2N/(N-2)$ is the Sobolev critical exponent. In the spirit of [20], such a nonlinear term is called subcritical growth. In [7] and [16], the existence of a positive ground state solution of the equation (1.3) has been proved. By using a constraint minimization argument, a solution of the equation with unknown Lagrangian multiplier λ in front of the nonlinear term has been given. In [17], the quasilinear equation is transformed into a semilinear equation by a variable transformation. Using an Orlitz space frame as the workspace, a positive solution of equation (1.3) is obtained according to the mountain-Pass lemma(e.g.,[18]). This method was later used for subcritical growth in [19]. Along this line of thought, one could also look for a sign-changing solution. For instance, in [20], Liu et al. used Nehari's method to deal with more general quasilinear equations and obtained positive and sign-changing solutions. Recently, the first two authors and Wang obtained infinitely many nodal radial solutions of the equation (1.3) through a construction argument in [21].

As shown in [20], the number 22^* is similar to the critical exponent of equation (1.3). As a matter of fact, in [20], using the variational identity given by Pucci and Serrin [22], it was proved that equation (1.3) has no positive solutions in $H^1(\mathbb{R}^N)$ with $u^2|\nabla u|^2 \in L^1(\mathbb{R}^N)$ if $k(u) = |u|^{p-2}u$, $p \geq 22^*$ and $\nabla V(x) \cdot x \geq 0$ in $\mathbb{R}^N$. As in [17] Liu et al. pointed out, the critical case for equation (1.3) is as good as a play. In this critical case, Moanemi dealt with the related singularly perturbed equation in reference [23], and obtained a positive radial solution in the case of radial symmetry. Later, in [24] a positive solution was proved to exist according to the mountain pass Lemma. Recently, Liu et al. used a perturbation method to obtain a positive solution of the general quasilinear elliptic equation like (1.3) in [25]. The existence of the nodal solution of the equation (1.3) with critical growth is studied by the variational method in [26].

We note that all of the above results are for the special case $l(s) = s$. A very natural question is whether there is a general way to study the equation (1.1) for the general function $l(s)$.

In [27], to handle the general case, Shen and Wang introduce the following new variable substitution

$$g^2(u) = 1 + \frac{(l(u^2)')^2}{2}.$$

Using this transformation, we can reduce (1.2) to quasilinear elliptic equation

$$-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = k(x,u), \quad x \in \mathbb{R}^N. \tag{1.4}$$

Setting $g^2(u) = 1 + 2u^2$, i.e., $l(s) = s$, we can get equation (1.3). Setting $g^2(u) =$

$1+\frac{u^2}{2(1+u^2)}$, i.e., $l(s)=(1+s)^{\frac{1}{2}}$, we can get the quasilinear Schrödinger equation

$$-\Delta u+V(x)u-[\Delta(1+u^2)^{\frac{1}{2}}]\frac{u}{2(1+u^2)^{\frac{1}{2}}}=k(x,u),\quad x\in\mathbb{R}^N \tag{1.5}$$

which models the self-channeling of a high-power ultrashort laser in matter.

According to this line of thought, in [27], Shen and Wang obtained a positive solitary wave solution of (1.2) with a general function $l(s)$ under some assumptions on g, V, and k which is a function of subcritical growth. In [21], Deng et al. obtained node solutions of equation (1.4) also with subcritical growth function $k(x,u)$.

What is the critical exponent of the quasilinear Schrödinger equation (1.2) (or (1.4)) with general function $l(s)$ (or $g(s)$)? What about the existence of positive solutions for such an equation with the critical exponent? These questions have recently been addressed by the authors in [28]. To be more precise, the critical exponents $\alpha 2^*$ of equation (1.4) with the general function $g(s)$ is obtained if $g(s)$ satisfies $\lim_{t\to+\infty}\frac{g(t)}{t^{\alpha-1}}=\beta>0$ for some $\alpha\geq 1$ (see [28]).

In recent years, we've found that there seems to be little progress on the existence of positive ground state solutions for equation (1.4) with a Hardy-type term.

In the present paper, we assume $V(x)\equiv 1$ and $k(x,t)=h(t)+\frac{g(t)|G(t)|^{2^*(a)-2}G(t)}{|x|^a}$. Whereupon, the quasilinear Schrödinger equation (1.4) can be rewritten as

$$-div(g^2(u)\nabla u)+g(u)g'(u)|\nabla u|^2+u=h(u)+\frac{g(u)|G(u)|^{2^*(a)-2}G(u)}{|x|^a},\quad x\in\mathbb{R}^N, \tag{1.6}$$

where $N\geq 3$, $2^*(a)=\frac{2(N-a)}{N-2}$, $0\leq a<2$, $G(t)=\int_0^t g(\tau)d\tau$ and $h:\mathbb{R}\to\mathbb{R}$ is a continuous function.

In order to establish the existence of positive radial ground state solutions for equation (1.6), we need to make some assumptions about $g(t)$ and $h(t)$.

(g_1) $g\in C^1(\mathbb{R})$ is a positive even function and $g'(t)\geq 0$ for $\forall t\geq 0$, $g(0)=1$;

(h_1) $h(t)\geq 0$ is differentiable for all $t\in[0,+\infty)$. Moreover, we extend $h(t)\equiv 0$ for all $t\in(-\infty,0)$;

(h_2) $\lim_{t\to+\infty}\frac{h(t)}{g(t)|G(t)|^{2^*(a)-1}}=0$ and $\lim_{t\to 0^+}\frac{h(t)}{g(t)G(t)}=0$;

(h_3) There exists $\delta\in(0,2^*(a)-2)$ such that for $\forall t>0$, there holds $(1+\delta)h(t)\leq G(t)(\frac{h(t)}{g(t)})'$;

Denoting $H(u)=\int_0^u h(\tau)d\tau$, we find that the natural variational functional

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^N}g^2(u)|\nabla u|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}|u|^2dx-\int_{\mathbb{R}^N}H(u)dx-\frac{1}{2^*(a)}\int_{\mathbb{R}^N}\frac{|G(u)|^{2^*(a)}}{|x|^a}dx$$

corresponding to (1.6) may be not well defined in $H^1(\mathbb{R}^N)$. In order to overcome this difficulty, we need to make a variable substitution which was constructed by Shen and Wang in [27], as

$$v = G(u) = \int_0^u g(t)dt.$$

And then we can obtain

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |G^{-1}(v)|^2 dx - \int_{\mathbb{R}^N} H(G^{-1}(v))dx - \frac{1}{2^*(a)}\int_{\mathbb{R}^N} \frac{|v|^{2^*(a)}}{|x|^a}dx.$$

Since $g(t)$ satisfies the assumption (g_1), we can get $|G^{-1}(v)| \le \frac{1}{g(0)}|v| = |v|$. It thus appears that, the functional $J(v)$ is well defined in $H^1(\mathbb{R}^N)$ and $J \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ if the function $h(t)$ satisfies the assumption (h_2).

If u is a nontrivial solution of equation (1.6), then it should satisfy

$$\int_{\mathbb{R}^N} [g^2(u)\nabla u \nabla \varphi + g(u)'|\nabla u|^2\varphi + u\varphi - h(u)\varphi + \frac{g(u)|G(u)|^{2^*(a)-2}G(u)}{|x|^a}\varphi]dx = 0 \tag{1.7}$$

for $\forall \varphi \in C_0^\infty(\mathbb{R}^N)$.

Let $\varphi = \frac{1}{g(u)}\psi$, we immediately know (see[27]) that equation (1.7) is equivalent to

$$\langle J'(v), \psi\rangle = \int_{\mathbb{R}^N} [\nabla v \nabla \psi + \frac{G^{-1}(v)}{g(G^{-1}(v))}\psi - \frac{h(G^{-1}(v))}{g(G^{-1}(v))}\psi - \frac{|v|^{2^*(a)-2}v\psi}{|x|^a}]dx = 0 \tag{1.8}$$

for $\forall \psi \in C_0^\infty(\mathbb{R}^N)$.

Consequently, to find the nontrivial solutions of (1.6), it is sufficient to investigate the existence of nontrivial solutions to the following equation

$$-\Delta v + \frac{G^{-1}(v)}{g(G^{-1}(v))} - \frac{h(G^{-1}(v))}{g(G^{-1}(v))} - \frac{|v|^{2^*(a)-2}v}{|x|^a} = 0. \tag{1.9}$$

It is easy to prove that equation (1.6) is equivalent to equation (1.9) and the nontrivial critical point of $J(v)$ is the nontrivial solution of equation (1.9).

The main results of this paper can be stated by the following theorem:

Theorem 1.1 Assume that (g_1) and $(h_1)-(h_3)$ hold, then Equation (1.9) has at least one positive radial ground state solution solution if $N \ge 4$.

Remark 1 Since we want to study the existence of positive solutions to equation (1.9), we rewrite the corresponding variational function $J(v)$ into the following form:

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |G^{-1}(v)|^2 dx - \int_{\mathbb{R}^N} H(G^{-1}(v))dx - \frac{1}{2^*(a)}\int_{\mathbb{R}^N} \frac{(v^+)^{2^*(a)}}{|x|^a}dx,$$

where $v^+(x) = \max\{v(x), 0\}$.

We assert that all nontrivial critical points of the functional J are positive solutions of equation (1.9). As a matter of fact, let $v \in H^1(\mathbb{R}^N)$ be a nontrivial critical point of the functional J, then v must be a nontrivial solution of the equation

$$-\Delta v + v = v - \frac{G^{-1}(v)}{g(G^{-1}(v))} + \frac{h(G^{-1}(v))}{g(G^{-1}(v))} + \frac{(v^+)^{2^*(a)-1}}{|x|^a}. \tag{1.10}$$

According to standard regularity argument, we know that $v \in C^2(\mathbb{R}^N)$. Furthermore, by the assumption(h_2) and Lemma 2.1(2) in section 2, we can obtain that the right side of equation (1.10) is nonnegative since $G(u)g(u) > G^2(u)/u > u$. So, by the strong maximum principle, we know that v is positive.

This paper is organized as follows: In section 2, we will prove some useful lemmas. To be precise, firstly, we give some properties for $G, G^{-1}$and H. After that, based on these properties, we show that the functional $J(v)$ satisfies Mountain Pass geometry and that the corresponding $(PS)_c$ sequence is bounded. In Section 3, based on Section 2, we prove the main theorem 1.1 of this paper.

In what follows, we denote as usual the functions $u^+(x) = \max\{u(x), 0\}$ and $u^- = \max\{-u(x), 0\}$ by u^+ and u^- if $u \in H^1(\mathbb{R}^N)$. The usual Lebesgue space is denoted by $L^q(\mathbb{R}^N)$ with norms $\| u \|_q = \left(\int_{R^N} |u(x)|^q dx\right)^{\frac{1}{q}}, 1 \le q < \infty$, and the space of radial symmetric functions $\{u \in H^1(\mathbb{R}^N) : u(x) = u(|x|)\}$ is denoted by $H^1_r(\mathbb{R}^N)$.

2 Some Preliminary Lemmas

In this section, firstly, we give some properties for the fuctions h, g and H, G which are defined in the introduction. After that, we prove that the functional $J(v)$ satisfies Mountain Pass geometry and the $(PS)_c$ sequence corresponding to $J(v)$ is bounded. Finally, we establish a compactness theorem for variational functional $J(v)$.

Lemma 2.1 (See Lemma 2.1 in [29]) Assume that (g_1) and $(h_1)-(h_3)$ hold, then the functions $h(t), g(t)$ and $H(t) = \int_0^t h(\tau)d\tau, G(t) = \int_0^t g(\tau)d\tau$ enjoy the following properties:
(1) $G(t)$ and $G^{-1}(s)$ are odd functions.
(2) For all $t \ge 0, s \ge 0$, there hold $G(t) \le g(t)t, G^{-1}(s) \le \frac{s}{g(0)} = s$.
(3) For all $s \ge 0$, the function $\frac{G^{-1}(s)}{s}$ is nonincreasing and $\lim_{s\to 0} \frac{G^{-1}(s)}{s} = 1$. Moreover, if g is bounded, then $\lim_{s\to\infty} \frac{G^{-1}(s)}{s} = \frac{1}{g(\infty)}$; If g is unbounded, then $\lim_{s\to\infty} \frac{G^{-1}(s)}{s} = 0$.
(4) There exists a constant $\mu \in (2, 2^*(a))$ such that for $\forall t > 0$, there holds $h(t)G(t) \ge \mu g(t)H(t)$.
(5) $H(t) \ge \frac{H(M)}{G(M)}(G(t))^\mu$ for $t \ge M$.

(6) Denote $Q_1(s)=\dfrac{G^{-1}(s)}{g(G^{-1}(s))}$, $Q_2(s)=\dfrac{h(G^{-1}(s))}{g(G^{-1}(s))}$, then for all $s\in\mathbb{R}$,there hold

$$s^2Q_1'(s)\le Q_1(s)s,\quad s^2Q_2'(s)\ge(1+\delta)Q_2(s)s.$$

Denote

$$f(s)=s-\frac{G^{-1}(s)}{g(G^{-1}(s))}+\frac{h(G^{-1}(s))}{g(G^{-1}(s))},\tag{2.1}$$

$$F(s)=\int_0^s f(\tau)d\tau=\frac{1}{2}[s^2-(G^{-1}(s))^2]+H(G^{-1}(s)).\tag{2.2}$$

Lemma 2.2 (See Lemma 2.2 in [29]) Assume that (g_1) and $(h_1)-(h_3)$ hold, then the function $f(s)$, $F(s)$ enjoy the following properties:
(1) $f(s)\ge 0$ for all $s\ge 0$.

(2) $\lim\limits_{s\to0^+}\dfrac{f(s)}{s}=\lim\limits_{s\to0^+}\dfrac{F(s)}{s^2}=0$.

(3) $\lim\limits_{s\to+\infty}\dfrac{f(s)}{s^{2^*-1}}=\lim\limits_{s\to+\infty}\dfrac{F(s)}{s^{2^*}}=0$.

(4) $f(s)s\ge 2F(s)$ for $s\ge0$.

Based on the lemmas above, we can prove that the functional $J(v)$ satisfies the Mountain-Pass geometry.

Lemma 2.3 Under our assumptions, the functional $J(v)$ enjoy the following properties:
(1) there are $\alpha\in\mathbb{R}^+$, $\rho\in\mathbb{R}^+$, such that $J(v)\ge\alpha$ for all $\|v\|=\rho$;
(2) there is $\omega\in H^1(\mathbb{R}^N)$, such that $\|\omega\|>\rho$ and $J(\omega)<0$.

Proof By Lemma 2.2 (2),(3) and Sobolev-Hardy inequality, we can obtain, for any $\varepsilon>0$, there is a constant $C_\varepsilon>0$ such that

$$\begin{aligned}J(v)&=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}|G^{-1}(v)|^2dx-\int_{\mathbb{R}^N}H(G^{-1}(v))dx-\frac{1}{2^*(a)}\int_{\mathbb{R}^N}\frac{(v^+)^{2^*(a)}}{|x|^a}dx\\&=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v|^2+v^2)dx-\int_{\mathbb{R}^N}F(v)dx-\frac{1}{2^*(a)}\int_{\mathbb{R}^N}\frac{(v^+)^{2^*(a)}}{|x|^a}dx\\&\ge\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v|^2+v^2)dx-\int_{\mathbb{R}^N}(\varepsilon v^2+C_\varepsilon|v|^{2^*})dx-\frac{1}{2^*(a)}\int_{\mathbb{R}^N}\frac{|v|^{2^*(a)}}{|x|^a}dx\\&\ge(C-\varepsilon)\int_{\mathbb{R}^N}(|\nabla v|^2+v^2)dx-C_\varepsilon\int_{\mathbb{R}^N}|v|^{2^*}dx-C_1\Big(\int_{\mathbb{R}^N}(|\nabla v|^2\Big)^{\frac{2^*(a)}{2}}.\end{aligned}\tag{2.3}$$

Therefore, if we choose $\varepsilon>0$ small enough and $\rho>0$, then there holds

$$J(v)\ge C\|v\|^2-C_\varepsilon\|v\|^{2^*}-C_1\|v\|^{2^*(a)},$$

which implies the point (1) .

Next, we prove point (2). For a given nontrivial function $\varphi \in C_0^\infty(\mathbb{R}^N, [0,1])$, by Lemma 2.2(1) we can obtain that

$$\begin{aligned} J(t\varphi) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla t\varphi|^2+(t\varphi)^2)dx - \int_{\mathbb{R}^N}F(G^{-1}(t\varphi))dx - \frac{1}{2^*(a)}\int_{\mathbb{R}^N}\frac{(t\varphi)^{2^*(a)}}{|x|^a}dx \\ &\leq \frac{t^2}{2}\int_{\mathbb{R}^N}(|\nabla\varphi|^2+(\varphi)^2)dx - \frac{t^{2^*(a)}}{2^*(a)}\int_{\mathbb{R}^N}\frac{(\varphi)^{2^*(a)}}{|x|^a}dx \\ &\to -\infty \quad as \quad t\to+\infty. \end{aligned} \tag{2.4}$$

Therefore, if we take $\omega = t\varphi$ with t large enough, then we can see that point (2) is naturally true.

By Lemma 2.3 and the mountain pass Lemma, we know that for the constant

$$c := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} J(\gamma(t)) > 0, \tag{2.5}$$

where

$$\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)), \gamma(0)=0, \gamma(1)\neq 0, J^\infty(\gamma(1))<0\},$$

there is a $(PS)_c$ sequence $\{v_n\}$ in $H^1(\mathbb{R}^N)$ at the level c, that is,

$$J(v_n)\to c \quad \text{and} \quad J'(v_n)\to 0, \quad \text{as } n\to+\infty. \tag{2.6}$$

Lemma 2.4 The $(PS)_c$ sequence $\{v_n\}$ in (2.6) is bounded in $H^1(\mathbb{R}^N)$.

Proof Since $\{v_n\}\subset H^1(R^N)$ is a $(PS)_c$ sequence, we know that

$$\begin{aligned} J(v_n) &= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v_n|^2dx + \frac{1}{2}\int_{\mathbb{R}^N}|G^{-1}(v_n)|^2dx \\ &\quad - \int_{\mathbb{R}^N}H(G^{-1}(v_n))dx - \frac{1}{2^*(a)}\int_{\mathbb{R}^N}\frac{(v_n^+)^{2^*(a)}}{|x|^a}dx \to c \end{aligned} \tag{2.7}$$

and for $\forall\psi\in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned} \langle J'(v_n),\psi\rangle &= \int_{\mathbb{R}^N}[\nabla v_n\nabla\psi + \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}\psi - \frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}\psi - \frac{|v_n^+|^{2^*(a)-2}v_n^+\psi}{|x|^a}]dx \\ &= o(1)\|\psi\| \end{aligned} \tag{2.8}$$

as $n\to\infty$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, by taking $\psi = v_n$ we can obtain that

$$\begin{aligned} \langle J'(v_n),v_n\rangle &= \int_{\mathbb{R}^N}[|\nabla v_n|^2 + \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n - \frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}v_n - \frac{(v_n^+)^{2^*(a)}}{|x|^a}]dx \\ &= o(1)\|v_n\| \end{aligned} \tag{2.9}$$

as $n \to \infty$. By (2.7),(2.9) and Lemma 2.1(2),(4), we can deduce that

$$\begin{aligned}
&\mu c + o(1) - \langle J'(v_n), v_n\rangle \\
&= \mu J(v_n) - \langle J'(v_n), v_n\rangle \\
&= \frac{\mu-2}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} G^{-1}(v_n)\Big[\frac{1}{2}\mu G^{-1}(v_n) - \frac{1}{g(G^{-1}(v_n))}v_n\Big]dx \\
&\quad - \int_{\mathbb{R}^N}\Big[\mu H(G^{-1}(v_n)) - \frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}v_n\Big]dx - \int_{\mathbb{R}^N}\Big[\frac{\mu}{2^*(a)} - 1\Big]\frac{(v_n^+)^{2^*(a)}}{|x|^a}dx \\
&\geq \frac{\mu-2}{2}\Big[\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} |G^{-1}(v_n)|^2\Big].
\end{aligned} \tag{2.10}$$

By Lemma 2.1(4),(5), we can obtain that

$$\frac{h(t)}{g(t)}G(t) \geq \mu H(t) \geq CG(t)^{\mu} \geq CG(t)^2 \tag{2.11}$$

for all $t \geq 1$. Therefore, for the case $x \in \{x : |G^{-1}(v_n)| > 1\}$, we have that

$$\begin{aligned}
\int_{\{x:|G^{-1}(v_n)|>1\}} |v_n|^2 dx &\leq C\int_{\{x:|G^{-1}(v_n)|>1\}} H(G^{-1}(v_n))dx \\
&\leq \int_{\mathbb{R}^N} H(G^{-1}(v_n))dx + \frac{C}{2^*(a)}\int_{\mathbb{R}^N}\frac{|v_n^+|^{2^*(a)}}{|x|^a}dx \\
&\leq C\Big[c + o(1) + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} |G^{-1}(v_n)|^2\Big] \\
&\leq C.
\end{aligned} \tag{2.12}$$

On the other hand, since $g(t)$ is a nondecreasing function, for the case $x \in \{x : |G^{-1}(v_n)| \leq 1\}$ we have that

$$\frac{1}{g^2(1)}\int_{\{x:|G^{-1}(v_n)|\leq 1\}} |v_n|^2 dx \leq C\int_{\{x:|G^{-1}(v_n)|\leq 1\}} |G^{-1}(v_n)|^2 dx. \tag{2.13}$$

Substiuting (2.12)-(2.13) into (2.10), we immediately know that the $(PS)_c$ sequence $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$.

The following lemma provides the interval in which the $(PS)_c$ condition holds for $J(v)$.

Lemma 2.5 $J(v)$ defined on $H_r^1(\mathbb{R}^N)$ satisfies $(PS)_c$ condition if the level value

$$c < \frac{2-a}{2(N-a)}A_a^{\frac{N-a}{2-a}}$$

where A_a is the best Sobolev-Hardy constant defined as follows:

$$A_a := \inf_{u \in D^{1,2}(\mathbb{R}^{\mathbb{N}})-\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\Big(\int_{\mathbb{R}^N} \frac{|u|^{2^*(a)}}{|x|^a}dx\Big)}.$$

Proof Let $\{v_n\} \subset H_r^1(\mathbb{R}^N)$ be a $(PS)_c$ sequence of $J(v)$. Similar to the proof of Lemma 2.4, we can easily prove that $\{v_n\}$ is bounded in $H_r^1(\mathbb{R}^N)$. Then it follows from Strauss Lemma [30] that there is a subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$) such that

$$\begin{aligned} &v_n \rightharpoonup v \quad \text{weakly in } H_r^1(\mathbb{R}^N), \\ &v_n \to v \quad \text{strongly in } L^p(\mathbb{R}^N), \quad 2 < p < 2^* \\ &v_n \to v \quad \text{a.e. in } \mathbb{R}^N, \end{aligned} \tag{2.14}$$

and v is a solution of (1.10). It follows from Strauss lemma [30] and (2.14) that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} F(v_n)dx = \int_{\mathbb{R}^N} F(v)dx, \quad \lim_{n\to\infty}\int_{\mathbb{R}^N} f(v_n)v_n dx = \int_{\mathbb{R}^N} f(v)vdx. \tag{2.15}$$

Therefore, we can obtain that

$$\int_{\mathbb{R}^N}\left[|\nabla v|^2 + v^2\right]dx - \int_{\mathbb{R}^N}\frac{(v^+)^{2^*(a)}}{|x|^a}dx - \int_{\mathbb{R}^N} f(v)vdx = 0 \tag{2.16}$$

It is clear that for all $A, B \in \mathbb{R}$, there holds

$$||A+B| - |A|| \le |B|.$$

Hence, we can obtain that

$$\left|\left|\frac{v_n^{2^*(a)}}{|x|^a}\right| - \left|\frac{v_n^{2^*(a)}}{|x|^a} - \frac{v^{2^*(a)}}{|x|^a}\right| - \left|\frac{v^{2^*(a)}}{|x|^a}\right|\right| \le 2\left|\frac{v^{2^*(a)}}{|x|^a}\right|.$$

By the Sobolev-Hardy inequality,

$$\int_{\mathbb{R}^N}\frac{|v|^{2^*(a)}}{|x|^a}dx \le C\left(\int_{\mathbb{R}^N}|\nabla v|^2 dx\right)^{\frac{2^*(a)}{2}} < \infty.$$

It follows from Lebesgue theorem that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left|\left|\frac{v_n^{2^*(a)}}{|x|^a}\right| - \left|\frac{v_n^{2^*(a)}}{|x|^a} - \frac{v^{2^*(a)}}{|x|^a}\right| - \left|\frac{v^{2^*(a)}}{|x|^a}\right|\right| = 0. \tag{2.17}$$

Taking $v_n' = v_n - v$, by Brezis-Lieb's lemma [31] and (2.15)-(2.17), we can obtain that

$$J(v) + \frac{1}{2}\int_{\mathbb{R}^N}\left[|\nabla v_n'|^2 + |v_n'|^2\right] - \frac{1}{2^*(a)}\int_{\mathbb{R}^N}\frac{|v_n'|^{2^*(a)}}{|x|^a} = c + o(1), \tag{2.18}$$

and

$$\int_{\mathbb{R}^N}\left[|\nabla v_n'|^2 + |v_n'|^2\right] - \int_{\mathbb{R}^N}\frac{|v_n'|^{2^*(a)}}{|x|^a} = o(1).$$

Suppose that v_n does not converge to v in $H^1(\mathbb{R}^N)$, we may assume that $\int_{\mathbb{R}^N}\frac{|v_n'|^{2^*(a)}}{|x|^a} = l + o(1)$, where $l > 0$. Then

$$\int_{\mathbb{R}^N}(|\nabla v_n'|^2 + |v_n'|^2) = l + o(1).$$

By the Sobolev-Hardy inequality,

$$o(1)+A_a l^{\frac{2}{2^*(a)}}=A_a\Big(\int_{\mathbb{R}^N}\frac{|v_n'|^{2^*(a)}}{|x|^a}\Big)^{\frac{2}{2^*(a)}}\le\int_{\mathbb{R}^N}|\nabla v_n'|^2\le\int_{\mathbb{R}^N}(|\nabla v_n'|^2+|v_n'|^2)=l+o(1).$$

Hence,

$$l\ge(A_a)^{\frac{N-a}{2-a}}.$$

By (2.18)we deduce that

$$J(v)=c-\Big(\frac{1}{2}-\frac{1}{2^*(a)}\Big)l\le c-\frac{2-a}{2(N-a)}(A_a)^{\frac{N-a}{2-a}}<0. \tag{2.19}$$

On the other hand, by Lemma 2.1, we have that

$$\frac{1}{2}f(v)v-F(v)\ge\frac{1}{2}\Big(|G^{-1}(v)|^2-\frac{G^{-1}(v)v}{g(G^{-1}(v))}\Big)-H(G^{-1}(v))+\frac{1}{2}\frac{h(G^{-1}(v))v}{g(G^{-1}(v))}\ge 0. \tag{2.20}$$

Thus we can obtain that

$$J(v)=\int_{\mathbb{R}^N}\big(\frac{1}{2}-\frac{1}{2^*(a)}\big)\frac{|v|^{2^*(a)}}{|x|^a}dx+\int_{\mathbb{R}^N}\big(\frac{1}{2}f(v)v-F(v)\big)dx\ge 0.$$

This is in contradiction with (2.19). Therefore $l=0$. By the definition of v_n' we can conclude that J satisfies $(PS)_c$ condition. And then the proof is done.

By Lemmas 2.3-2.5 and the mountain pass Lemma, we can easily obtain the following lemma.

Lemma 2.6 Assume that there is $v_0\in H_r^1(\mathbb{R}^N), v_0\ne 0$ such that

$$\sup_{t\ge 0}J(tv_0)<\frac{2-a}{2(N-a)}A_a^{\frac{N-a}{2-a}}. \tag{2.21}$$

Then equation (1.9) has at least one positive weak solution.

3 Proof of Main Theorem

Now, we will show that the level value c is in the interval in which the $(PS)_c$ condition holds. For this purpose, we introduce a well-known fact that the function

$$u_\epsilon(x)=\frac{((N-a)(N-2)\epsilon)^{\frac{N-2}{2(2-a)}}}{(\epsilon+|x|^{2-a})^{\frac{N-2}{2-a}}}$$

solve the equation

$$-\Delta u=\frac{|u|^{2^*(a)-2}}{|x|^a}u \quad\text{in}\quad \mathbb{R}^N-\{0\}$$

and satisfy

$$\int_{\mathbb{R}^N}|\nabla u_\epsilon|^2dx=\int_{\mathbb{R}^N}\frac{|u_\epsilon|^{2^*(a)}}{|x|^a}dx=(A_a)^{\frac{N-a}{2-a}}.$$

Let $\varphi\in C_0^\infty(\mathbb{R}^N,[0,1])$ be a radial cut-off function such that $\varphi(|x|)=1$ for $|x|\le\rho_\epsilon$, $\varphi(|x|)\in(0,1)$ for $\rho_\epsilon<|x|<2\rho_\epsilon$, and $\varphi(|x|)=0$ for $|x|\ge 2\rho_\epsilon$, where $\rho_\epsilon=\epsilon^\tau,\ \tau\in(\frac{1}{4},\frac{1}{2})$. Taking $v_\epsilon(x)=\varphi(x)u_\epsilon(x)$, and then we can obtain the following estimations (see [32]):

Lemma 3.1 The function $v_\epsilon(x)$ satisfies the estimations as follows: as $\epsilon \to 0$,

(1) $\|\nabla v_\epsilon\|_2^2 = (A_a)^{\frac{N-a}{2-a}} + O(\epsilon^{N-2})$.

(2) $\displaystyle\int_{\mathbb{R}^N} \frac{|v_\epsilon|^{2^*(a)}}{|x|^a} dx = (A_a)^{\frac{N-a}{2-a}} + O(\epsilon^{N-a})$.

(3) $$\|v_\epsilon\|_r^r = \begin{cases} O(\epsilon^{N-\frac{r(N-2)}{2}}) & for \quad r > \dfrac{2^*}{2}, \\ O(\epsilon^{\frac{r(N-2)}{2}} |\ln \epsilon|) & for \quad r = \dfrac{2^*}{2}, \\ O(\epsilon^{\frac{r(N-2)}{2}}) & for \quad r < \dfrac{2^*}{2}. \end{cases}$$

(4) $$\|v_\epsilon\|_2^2 = \begin{cases} O(\epsilon^2) & for \quad N \geq 5, \\ O(\epsilon^2 |\ln \epsilon|) & for \quad N = 4, \\ O(\epsilon^{N-2}) & for \quad N = 3. \end{cases}$$

(5) $\|v_\epsilon\|_1 = O(\epsilon^{\frac{N-2}{2}})$.

(6) $\|v_\epsilon\|_{2^*}^{2^*} = D + O(\epsilon^N)$.
where $D > 0$ is a constant.

Now we are going to show the following result.

Theorem 3.2 Assume that (g_1) and $(h_1) - (h_3)$ hold, then the equation (1.9) has at least one positive radial solution solution if $N \geq 4$.

Proof By Lemma 2.6, we just need to verify that condition (2.21) is naturally true.

Step 1. We claim that for $\epsilon > 0$ sufficiently small, there is a constant $t_\epsilon > 0$ such that

$$J(t_\epsilon v_\epsilon) = \max_{t \geq 0} J(tv_\epsilon)$$

and

$$0 < A_1 < t_\epsilon < A_2 < +\infty,$$

where A_1 and A_2 are two positive constants that do not depend on ϵ.

In fact, as a result of $J(0) = 0$ and $\lim_{t \to \infty} J(tv_\epsilon) = -\infty$, there is a $t_\epsilon > 0$ such that

$$J(t_\epsilon v_\epsilon) = \max_{t \geq 0} J(tv_\epsilon) \quad \text{and} \quad \frac{dJ(tv_\epsilon)}{dt}\Big|_{t=t_\epsilon} = 0.$$

Therefore we can obtain

$$\frac{\|v_\epsilon\|^2}{\int_{\mathbb{R}^N} \frac{|v_\epsilon|^{2^*(a)}}{|x|^a} dx} - t_\epsilon^{2^*(a)-2} - \frac{\int_{\mathbb{R}^N} f(t_\epsilon v_\epsilon) v_\epsilon dx}{t_\epsilon \int_{\mathbb{R}^N} \frac{|v_\epsilon|^{2^*(a)}}{|x|^a} dx} = 0. \tag{3.1}$$

By Lemma 2.2 and Lemma 3.1, for any $\delta > 0$ there exist positive constants k_1, k_2, k_3 such that

$$\frac{\int_{\mathbb{R}^N} f(t_\epsilon v_\epsilon)v_\epsilon dx}{t_\epsilon \int_{\mathbb{R}^N} \frac{|v_\epsilon|^{2^*(a)}}{|x|^a}dx} \leq k_1 \frac{\int_{\mathbb{R}^N} f(t_\epsilon v_\epsilon)v_\epsilon dx}{t_\epsilon} \leq k_1 \frac{1}{t_\epsilon}\int_{\mathbb{R}^N} (\delta t_\epsilon^{2^*-1} v_\epsilon^{2^*-1} + k_2 t_\epsilon v_\epsilon)v_\epsilon dx$$
$$= k_1 \int_{\mathbb{R}^N} (\delta t_\epsilon^{2^*-2} v_\epsilon^{2^*} + k_2 v_\epsilon^2)dx \to \delta k_3 t_\epsilon^{2^*-2} \quad (\epsilon \to 0).$$

It follows from (3.1) that

$$1 - t_\epsilon^{2^*(a)-2} - \delta k_3 t_\epsilon^{2^*-2} \leq 0 \quad (\epsilon \to 0).$$

Therefore there exists a constant $A_1 > 0$ such that $t_\epsilon > A_1$ if ϵ is sufficiently small.

On the other hand, by (3.1) we can obtain

$$t_\epsilon^{2^*(a)-2} = \frac{\|v_\epsilon\|^2}{\int_{\mathbb{R}^N} \frac{|v_\epsilon|^{2^*(a)}}{|x|^a}dx} - \frac{\int_{\mathbb{R}^N} f(t_\epsilon v_\epsilon)v_\epsilon dx}{t_\epsilon \int_{\mathbb{R}^N} \frac{|v_\epsilon|^{2^*(a)}}{|x|^a}dx} \leq \frac{\|v_\epsilon\|^2}{\int_{\mathbb{R}^N} \frac{|v_\epsilon|^{2^*(a)}}{|x|^a}dx}.$$

Therefore we have

$$t_\epsilon \leq \Big(\frac{\|v_\epsilon\|^2}{\int_{\mathbb{R}^N} \frac{|v_\epsilon|^{2^*(a)}}{|x|^a}dx}\Big)^{\frac{1}{2^*(a)-2}} \leq A_2 < +\infty \quad \text{if } \epsilon \text{ is sufficiently small.}$$

Step 2. We want to estimate $J(t_\varepsilon v_\epsilon)$. By Lemma (3.1), we can obtain that

$$\begin{aligned} J(t_\varepsilon v_\epsilon) &= \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N} (|\nabla v_\epsilon|^2 + v_\epsilon^2)dx - \int_{\mathbb{R}^N} F(t_\epsilon v_\epsilon)dx - \frac{t_\epsilon^{2^*(a)}}{2^*(a)}\int_{\mathbb{R}^N} \frac{|v_\epsilon|^{2^*(a)}}{|x|^a}dx \\ &= \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N} |\nabla v_\epsilon|^2 dx - \frac{t_\epsilon^{2^*(a)}}{2^*(a)}\int_{\mathbb{R}^N} \frac{|v_\epsilon|^{2^*(a)}}{|x|^a}dx + \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N} |v_\epsilon|^2 dx - \int_{\mathbb{R}^N} F(t_\epsilon v_\epsilon)dx \\ &\leq \Big(\frac{t_\varepsilon^2}{2} - \frac{t_\epsilon^{2^*(a)}}{2^*(a)}\Big)(A_a)^{\frac{N-a}{2-a}} + O(\epsilon^{N-2}) + \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N} |v_\epsilon|^2 dx - \int_{\mathbb{R}^N} F(t_\epsilon v_\epsilon)dx. \end{aligned}$$

Since the function $Q(t) = \frac{t_\varepsilon^2}{2} - \frac{t_\epsilon^{2^*(a)}}{2^*(a)}$ has only maximum at $t = 1$, we can obtain

$$\begin{aligned} J(t_\varepsilon v_\epsilon) &\leq \Big(\frac{1}{2} - \frac{1}{2^*(a)}\Big)(A_a)^{\frac{N-a}{2-a}} + O(\epsilon^{N-2}) + \frac{A_2^2}{2}\int_{\mathbb{R}^N} |v_\epsilon|^2 dx - \int_{\mathbb{R}^N} F(t_\epsilon v_\epsilon)dx \\ &= \frac{2-a}{2(N-a)}(A_a)^{\frac{N-a}{2-a}} + O(\epsilon^{N-2}) + C\int_{\mathbb{R}^N} |v_\epsilon|^2 dx \\ &\quad - \int_{\mathbb{R}^N} \Big[\frac{1}{2}\Big(|t_\epsilon v_\epsilon|^2 - |G^{-1}(t_\epsilon v_\epsilon)|^2\Big) + H(G^{-1}(t_\epsilon v_\epsilon))\Big]dx. \end{aligned}$$

It follows from Lemma 2.1(2) that

$$J(t_\varepsilon v_\epsilon) \leq \frac{2-a}{2(N-a)}(A_a)^{\frac{N-a}{2-a}} + O(\epsilon^{N-2}) + C\int_{\mathbb{R}^N} |v_\epsilon|^2 dx - \int_{\mathbb{R}^N} H(G^{-1}(t_\epsilon v_\epsilon))dx.$$

By Lemma 2.1(4) we can obtain

$$\mu G'(t)H(t) \leq G(t)H'(t) \Rightarrow H(G^{-1}(s)) \geq s^{\mu} \quad \text{for all} s \geq 0.$$

Therefore

$$\begin{aligned} J(t_{\varepsilon}v_{\epsilon}) &\leq \frac{2-a}{2(N-a)}(A_a)^{\frac{N-a}{2-a}} + O(\epsilon^{N-2}) + C\int_{\mathbb{R}^N}|v_{\epsilon}|^2dx - \int_{\mathbb{R}^N}|t_{\epsilon}v_{\epsilon}|^{\mu}dx \\ &\leq \frac{2-a}{2(N-a)}(A_a)^{\frac{N-a}{2-a}} + O(\epsilon^{N-2}) + C\int_{\mathbb{R}^N}|v_{\epsilon}|^2dx - A_1^{\mu}\int_{\mathbb{R}^N}|v_{\epsilon}|^{\mu}dx. \end{aligned}$$

Let

$$I = O(\epsilon^{N-2}) + C\int_{\mathbb{R}^N}|v_{\epsilon}|^2dx - A_1^{\mu}\int_{\mathbb{R}^N}|v_{\epsilon}|^{\mu}dx,$$

then we only need to prove $I < 0$ for small ϵ. It follows from Lemma (3.1) that

$$I = C\epsilon^{N-2} + C\begin{cases} O(\epsilon^2) & for \quad N \geq 5, \\ O(\epsilon^2|\ln\epsilon|) & for \quad N = 4, \\ O(\epsilon^{N-2}) & for \quad N = 3. \end{cases} - C\begin{cases} O(\epsilon^{N-\frac{\mu(N-2)}{2}}) & for \quad \mu > \frac{2^*}{2}, \\ O(\epsilon^{\frac{\mu(N-2)}{2}}|\ln\epsilon|) & for \quad \mu = \frac{2^*}{2}, \\ O(\epsilon^{\frac{\mu(N-2)}{2}}) & for \quad \mu < \frac{2^*}{2}. \end{cases}$$

When $N \geq 4, \mu > 2$, we can verify that $I < 0$ as $\epsilon > 0$ sufficiently small. Therefore (2.20) is naturally true. And then the proof is done.

Define

$$b = \inf_{v\in\mathcal{N}} J(v),$$

where

$$\mathcal{N} = \Big\{u \in H^1(\mathbb{R}^N)\setminus\{0\} | \int_{\mathbb{R}^N}\Big(|\nabla v|^2 + \frac{G^{-1}(v)}{g(G^{-1}(v))}v - \frac{(v^+)^{2^*(a)}}{|x|^a} - \frac{h(G^{-1}(v))}{g(G^{-1}(v))}v\Big)dx = 0\Big\}$$

It is clear that $\mathcal{N} \neq \varnothing$ since the equation (1.9) has at least one positive solution.

Lemma 3.2 Assume that (g_1), and $(h_1)-(h_3)$ hold, then we have

$$b = c.$$

Proof It is obvious that for all $u \in \mathbb{N}$, there exists $t^* > 0$, such that

$$J(t^*u) = \sup_{t>0} J(tu) \quad \text{and} \quad \frac{dJ(tu)}{dt}|_{t=t^*} = 0.$$

Set

$$k_1 = \int_{\mathbb{R}^N}|\nabla u|^2dx, \quad k_2 = \int_{\mathbb{R}^N}\frac{(u^+)^{2^*(a)}}{|x|^a}dx,$$

and then

$$\begin{aligned}\frac{dJ(tu)}{dt} &= t\int_{\mathbb{R}^N}|\nabla u|^2dx+\int_{\mathbb{R}^N}Q_1(tu)udx-t^{2^*(a)-1}\int_{\mathbb{R}^N}\frac{(u^+)^{2^*(a)}}{|x|^a}dx-\int_{\mathbb{R}^N}Q_2(tu)udx\\ &= k_1t-k_2t^{2^*(a)-1}+\int_{\mathbb{R}^N}Q_1(tu)udx-\int_{\mathbb{R}^N}Q_2(tu)udx,\end{aligned}$$

where $Q_1(s), Q_2(s)$ are given fuctions by Lemma 2.1(6). Denote

$$\gamma(t)=k_1-k_2t^{2^*(a)-2}+\int_{\mathbb{R}^N}\frac{Q_1(tu)u}{t}dx-\int_{\mathbb{R}^N}\frac{Q_2(tu)u}{t}dx.$$

By Lemma 2.1(6), we can obtain

$$\begin{aligned}\gamma'(t) &= -(2^*(a)-2)k_2t^{2^*(a)-3}+\int_{\mathbb{R}^N}\frac{tuQ_1'(tu)u-Q_1(tu)u}{t^2}dx\\ &\quad -\int_{\mathbb{R}^N}\frac{tuQ_2'(tu)u-Q_2(tu)u}{t^2}dx\\ &< 0\end{aligned}$$

for $t>0$. Therefore $\gamma(t)$ has at most one zero point in $(0,\infty)$. It follows from $u\in\mathcal{N}$, $\left.\frac{dJ(tu)}{dt}\right|_{t=t^*}=0$ and $\left.\frac{dJ(tu)}{dt}\right|_{t=1}=0$ that $t^*=1$. Thus

$$b=\inf_{u\in\mathbb{N}}J(u)=\inf_{u\in\mathbb{N}}\sup_{t>0}J(tu)\geq c.$$

On the other hand, since c is a critical value of $J(v)$, we can obtain that $b\leq c$ by the definition of b. Thus

$$b=c.$$

And then the proof is done.

By Lemma (3.3), we know that the functional $J(v)$ can be achieved by a function $w\in\mathcal{N}$ and w is a positive ground state solution of (1.9).

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拟线性薛定谔方程的正基态解

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摘 要: 本文研究一类含hardy型项的拟线性薛定谔方程的正基态解. 利用变分法和变量代换法, 得到了给定拟线性薛定谔方程的正基态解.

关 键 词: 拟线性方程; 薛定谔方程; 正基态解; 变分法

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