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WELL-POSEDNESS AND PEAKON SOLUTIONS FOR A HIGHER ORDER CAMASSA-HOLM TYPE EQUATION

CHEN shuang

(*School of Mathematics and Statistics, Ningbo University, Ningbo 315211, China*)

Abstract: In this paper, we delve into a generalized higher order Camassa-Holm type equation, (or, an ghmCH equation for short). We establish local well-posedness for this equation under the condition that the initial data u_0 belongs to the Sobolev space $H^s(\mathbb{R})$ for some $s > \frac{7}{2}$. In addition, we obtain the weak formulation of this equation and prove the existence of both single peakon solution and a multi-peakon dynamic system.

Keywords: Generalized higher order Camassa-Holm type equation; Local well-posedness; Peakon

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1 Introduction

In this paper, we focus on the initial-value problem associated with a generalized higher order modified Camassa-Holm equation

$$\begin{cases} m_t + \left((u^2 - u_x^2)^n m\right)_x = 0, & t > 0,\ x \in \mathbb{R}, \\ u\left(x, 0\right) = u_0\left(x\right), & t > 0,\ x \in \mathbb{R}, \\ m = \left(1 - \alpha^2 \partial_x^2\right)\left(1 - \beta^2 \partial_x^2\right) u, & t > 0,\ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where t represents time, x denotes a single spatial variable and α, β are two parameters. For convenienve, we call the equation (1.1) the ghmCH equation. Without loss of generality, we only consider the case $\alpha > \beta > 0$, similar results can be derived when α and β are both negative by substituting α and β with their absolute values.

The Camassa-Holm equation (CH)

$$m_t + 2mu_x + m_x u = 0, \quad m = u - u_{xx} \tag{1.2}$$

was originally introduced in the context of hereditary symmetries investigated by Fuchssteiner and Fokas [1], subsequently derived by Camassa and Holm as a model for the unidirectional propagation of long waves in shallow water [2]. The CH equation (1.2) has been

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Biography: Chen shuang (1999–), female, Manchu , born at Chengde, Hebei, postgraduate, major in soliton and integrable systems. E-mail: chenshuang19990815@163.com.

extensively explored due to its complete integrability [2], multi-soliton solutions [2], well-posedness [3, 4], wave breaking [5, 6], etc. The most notable feature of the CH equation (1.2) is to admit peakon solutions $u(t,x) = ae^{-|x-ct|}$ [2], where a peakon is a weak solution in a Sobolev space with a corner at its peak.

Apart from the CH equation (1.2), another integrable model with peakon solutions has been discovered. The modified Camassa-Holm equation (mCH)

$$m_t + \left(\left(u^2 - u_x^2\right)m\right)_x = 0, \quad m = u - u_{xx} \tag{1.3}$$

was proposed by Fuchssteiner [1], Olver and Rosenau [7] by applying the tri-Hamiltonian duality method to the bi-Hamiltonian representation of the modified Korteweg-deVries equation. Qiao delved into the integrability and structure aspects of solutions for the mCH equation in [8, 9]. The existence of single peakon solution and multi-peakon solutions of mCH equation (1.3) was proved in [10]. The local well-posedness for the Cauchy problem of the mCH equation (1.3) in Besov spaces was established in [11–13].

Subsequently, a generalized Fokas-Olver-Resenau-Qiao equation

$$m_t + \left(\left(u^2 - u_x^2\right)^Q m\right)_x = 0, \quad m = u - u_{xx} \tag{1.4}$$

was introduced by Recio and Anco [14], who not only proposed the model but also demonstrated the existence of single peakon solution and multi-peakon solutions. Moreover, they proved that this equation admits a Hamiltonian structure. In [15], they established the local well-posedness in the critical space $B^{\frac{5}{2}}_{2,1}(\mathbb{R})$ and showed the continuality of the data-to-solution mapping.

Recently, a similar model, the generalized fifth order CH equation (FOCH),

$$m_t + m_x u + bmu_x = 0, \quad m = \left(1 - \alpha^2\partial_x^2\right)\left(1 - \beta^2\partial_x^2\right)u, \quad \alpha \neq \beta, \quad \alpha\beta \neq 0 \tag{1.5}$$

was introduced by Liu and Qiao [16], who obtained numerous solutions. The local and global existence of the solution to the FOCH equation (1.5) was established in [17]. For the case $\alpha = \beta = 1$ and $b = 2$, McLachlan and Zhang obtained a global weak solution for the FOCH equation (1.5) and established its local well-posedness in H^s with $s > \frac{7}{2}$ on the circle [18]. Tang and Liu verified the Cauchy problem related to this equation is locally well-posed in the critical Besov space $B^{7/2}_{2,1}$ and successfully attained the peak solution [19].

In the past few decades, there has been a surge of interest in the CH equation (1.2), which has sparked extensive research into CH-type equations especially those featuring peakons. Examples of such equations include the CH (1.2) and the mCH (1.3), all of which are integrable systems with both single peakon and multi-peakons. Additionally, single peakon and multi-peakons are discovered in some non-integrable systems. Since the equation studied in this paper has higher order nonlinear terms, and the mCH equation only has cubic nonlinearity, it is expected that the ghmCH equation (1.1) should also possess peakon solitons.

The structure of this paper is organized as follows. In section 2, we introduce some fundamental results, which will be applied in subsequent sections. In section 3, we obtain the local well-posedness of the initial-value problem associated with the ghmCH equation (1.1). In section 4, we first obtain the weak formulation of the ghmCH equation (1.1). Then, we demonstrate the existence of single peakon solution and multi-peakon dynamic system.

2 Preliminaries

We first recall some lemmas and definitions, which will be used in the following sections.

Definition 2.1 [1] Assume $0 < s < \infty$ and $u \in L^2(\mathbb{R}^n)$, we define the Sobolev spaces $H^s(\mathbb{R}^n)$ by

$$H^s(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) \mid (1+|\xi|^2)^{\frac{s}{2}}\hat{u}(\xi) \in L^2(\mathbb{R}^n)\}.$$

Lemma 2.1 ([6]) Let v be a vector field such that $\bigtriangledown v \in L^1((0,T); H^{s-1}(\mathbb{R}))$, where $s > \frac{3}{2}$. Suppose $f_0 \in H^s(\mathbb{R})$, $F \in L^1((0,T); H^s(\mathbb{R}))$, and $f \in L^\infty((0,T); H^s(\mathbb{R})) \cap C((0,T); \mathscr{S}'(\mathbb{R}))$ solves the one-dimensional linear transport equation

$$\begin{cases} \partial_t f + v \cdot \bigtriangledown f = F, \\ f \mid_{t=0} = f_0. \end{cases}$$

Then, there is a constant C that depends only on s, such that for any $0 < t < T$, the following two estimates hold:

$$\|f\|_{H^s} \leq \|f_0\|_{H^s} + \int_0^t \|F(\mu)\|_{H^s}\ d\mu + C\int_0^t V(\mu)\|f(\mu)\|_{H^s}\ d\mu,$$

$$\|f\|_{H^s} \leq e^{C\int_0^t V(\mu)\ d\mu}\Big(\|f_0\|_{H^s} + \int_0^t e^{-C\int_0^\tau V(\mu)\ d\mu}\|F(\mu)\|_{H^s}\ d\mu\Big),$$

where $V(\mu) := \| \bigtriangledown v(\mu) \|_{H^{s-1}}$.

Lemma 2.2 ([12]) For any $s_1 \in \mathbb{R}$, if there exists a constant $\varepsilon > 0$ such that $s_2 \geq max\{\frac{1}{2}+\varepsilon, -s_1+\varepsilon, s_1\}$, $f \in H^{s_1}$ and $g \in H^{s_2}$, then

$$\|fg\|_{H^{s_1}} \leq C\|f\|_{H^{s_1}}\|g\|_{H^{s_2}}.$$

3 Local Well-Posedness

In this section, we discuss the local well-posedness of the initial-value problem (1.1) with $n = 2$. Our main result in this section is as follows:

Theorem 3.1 Suppose that $s > \frac{7}{2}$ and $u_0 \in H^s(\mathbb{R})$. Then there exists a $T = T(u_0) > 0$ such that initial-value problem (1.1) has a unique solution

$$u \in C((0,T); H^s(\mathbb{R})) \cap C^1((0,T); H^{s-1}(\mathbb{R})).$$

In addition, the solution map:

$$u_0 \longmapsto u : H^s(\mathbb{R}) \longrightarrow C((0,T); H^s(\mathbb{R})) \cap C^1((0,T); H^{s-1}(\mathbb{R}))$$

is continuous.

It is worth noting that the operator $P:=(1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)$ can be determined through the convolution form of Green's function

$$u=\Big(\left(1-\alpha^2\partial_x^2\right)\left(1-\beta^2\partial_x^2\right)\Big)^{-1}m=G*m, \tag{3.1}$$

where $G(x)$ is the Green's function for the operator P. Then, from $(1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)G=\delta(x)$ and inverse Fourier transform, we find that,

$$G(x)=\frac{1}{2(\alpha^2-\beta^2)}\left(\alpha e^{-\frac{1}{\alpha}|x|}-\beta e^{-\frac{1}{\beta}|x|}\right). \tag{3.2}$$

In order to use Lemma 2.1, we need to rewrite the ghmCH equation (1.1) with $n=2$ into the following equivalent form:

$$u_t+\left(u^4u_x-\frac{2}{3}u^2u_x^3+\frac{1}{5}u_x^5\right)+G*H_1(u)+\partial_xG*H_2(u)+\partial_x^2G*H_3(u)=0, \tag{3.3}$$

where

$$\begin{aligned}
H_1(u)&:=4u^4u_x+\frac{2}{3}u^2u_x^3-\frac{1}{5}u_x^5,\\
H_2(u)&:=\left(4\left(\alpha^2+\beta^2\right)-2\right)u^3u_x^2+4\alpha^2\beta^2u^3u_{xx}^2-24\alpha^2\beta^2u^2u_x^2u_{xx}\\
&\quad-2\alpha^2\beta^2u^2u_{xx}^3-\left(\frac{4}{3}\left(\alpha^2+\beta^2\right)+24\alpha^2\beta^2-1\right)uu_x^4\\
&\quad-16\alpha^2\beta^2uu_x^2u_{xx}^2+\frac{8}{3}\alpha^2\beta^2u_x^4u_{xx}+6\alpha^2\beta^2u_x^2u_{xx}^3,\\
H_3(u)&:=-16\alpha^2\beta^2u^3u_xu_{xx}+6\alpha^2\beta^2u^2u_xu_{xx}^2+\frac{40}{3}\alpha^2\beta^2uu_x^3u_{xx}-6\alpha^2\beta^2u_x^3u_{xx}^2.
\end{aligned}$$

Before proving Theorem 3.1, we first give the following lemma, which is a direct result on the uniqueness and continuity of the initial data.

Lemma 3.1 Let $s>\frac{5}{2}$, suppose that $u^{(1)},\ u^{(2)}\in L^\infty((0,T);H^s(\mathbb{R}))\cap C((0,T);\mathscr{S}'(\mathbb{R}))$ are two solutions of the initial-value problem (1.1) with initial data $u_0^{(1)},u_0^{(2)}\in H^s$, respectively. Then for every $t\in[0,T]$, we have

$$\left\|\left(u^{(1)}-u^{(2)}\right)(t)\right\|_{H^{s-1}}\le\left\|u_0^{(1)}-u_0^{(2)}\right\|_{H^{s-1}}e^{C\int_0^t(||u^{(1)}(\tau)||_{H^s}^4+||u^{(2)}(\tau)||_{H^s}^4)\ d\tau}. \tag{3.4}$$

Proof Denote $u^{(12)}:=u^{(1)}-u^{(2)}$. Apparently

$$u^{(12)}\in L^\infty((0,T);H^s(\mathbb{R}))\cap C((0,T);\mathscr{S}'(\mathbb{R})).$$

It can be seen from (3.3) that $u^{(12)}$ satisfies the transport equation

$$\begin{aligned}
&\partial_tu^{(12)}+\Big[\left(u^{(1)}\right)^4-\frac{2}{3}\left(u^{(1)}\right)^2\left(\left(u_x^{(1)}\right)^2+u_x^{(1)}u_x^{(2)}+\left(u_x^{(2)}\right)^2\right)+\frac{1}{5}\Big(\left(u_x^{(1)}\right)^4\\
&+\left(u_x^{(1)}\right)^3u_x^{(2)}+\left(u_x^{(1)}\right)^2\left(u_x^{(2)}\right)^2+u_x^{(1)}\left(u_x^{(2)}\right)^3+\left(u_x^{(2)}\right)^4\Big)\Big]u_x^{(12)}=h\left(u^{(12)},u^{(1)},u^{(2)}\right)
\end{aligned} \tag{3.5}$$

with

$$h\big(u^{(12)},u^{(1)},u^{(2)}\big):=-\left[\left(\left(u^{(1)}\right)^2+\left(u^{(2)}\right)^2\right)\left(u^{(1)}+u^{(2)}\right)u_x^{(2)}-\frac{2}{3}\left(u_x^{(2)}\right)^3\left(u^{(1)}+u^{(2)}\right)\right]$$
$$\times u^{(12)}+G*F_1+\partial_x G*F_2+\partial_x^2 G*F_3.$$

Since H^{s-1} is an Banach algebra for $s>\frac{3}{2}$, we can get the following result,

$$\begin{aligned}&\Big\|\left(u^{(1)}\right)^4-\frac{2}{3}\left(u^{(1)}\right)^2\left(\left(u_x^{(1)}\right)^2+u_x^{(1)}u_x^{(2)}+\left(u_x^{(2)}\right)^2\right)+\frac{1}{5}\left(\left(u_x^{(1)}\right)^4\right.\\&\left.+\left(u_x^{(1)}\right)^3u_x^{(2)}+\left(u_x^{(1)}\right)^2\left(u_x^{(2)}\right)^2+u_x^{(1)}\left(u_x^{(2)}\right)^3+\left(u_x^{(2)}\right)^4\right)\Big\|_{H^{s-1}}\\ \le &C\left(\left\|u^{(1)}\right\|_{H^s}^4+\left\|u^{(2)}\right\|_{H^s}^4\right).\end{aligned}$$

Note that convolution with the derivatives of the fundamental solution $\partial_x^k G$ can increase the regularity in the Sobolev spaces by $(4-k)$ orders. Thus, using Lemma 2.2, we get

$$\left\|h\big(u^{(12)},u^{(1)},u^{(2)}\big)\right\|_{H^{s-1}}\le C\left(\left\|u^{(1)}\right\|_{H^s}^4+\left\|u^{(2)}\right\|_{H^s}^4\right)\left\|u^{(12)}\right\|_{H^{s-1}}.$$

Then, apply Lemma 2.1 to the equation (3.5), we get $u^{(12)}\in C((0,T)\,;\,H^{s-1}(\mathbb{R}))$ and

$$\begin{aligned}\left\|u^{(12)}(t)\right\|_{H^{s-1}}\le C\int_0^t&\left(\left\|u^{(1)}(\tau)\right\|_{H^s}^4+\left\|u^{(2)}(\tau)\right\|_{H^s}^4\right)\left\|u^{(12)}(\tau)\right\|_{H^{s-1}}\,d\tau\\&+\left\|u^{(12)}(0)\right\|_{H^{s-1}}.\end{aligned}\tag{3.6}$$

Accordingly, applying Grönwall's inequality to (3.6), we obtain (3.4) and the proof of Lemma 3.1 is complete.

Now, let us start the proof of Theorem 3.1, which is motivated by the proof of local existence theorem about the mCH (1.3) equation. Firstly, we shall use the classical Friedrichs regularization method to construct the approximate solution of the ghmCH equation (1.1).

Lemma 3.2 Suppose $u_0\in H^s(\mathbb{R})$, $s>\frac{7}{2}$. Starting from $u^{(0)}=0$ and by induction, there exists a sequence of smooth functions $\{u^{(n)}\}_{n\in\mathbb{N}}\in C(\mathbb{R}^+;H^\infty)$ solving the linear transport equation iteratively:

$$\begin{cases}\left(\partial_t+\left(\left(u^{(n)}\right)^2-\left(u_x^{(n)}\right)^2\right)^2\partial_x\right)m^{(n+1)}=-\left(\left(u^{(n)}\right)^2-\left(u_x^{(n)}\right)^2\right)_x^2 m^{(n)}, & t>0,\ x\in\mathbb{R},\\ m^{(n+1)}=\left(1-\alpha^2\partial_x^2\right)\left(1-\beta^2\partial_x^2\right)u^{(n+1)}, & t>0,\ x\in\mathbb{R},\\ u^{(n+1)}\mid_{t=0}=u_0^{(n+1)}(x)=S_{n+1}u_0, & x\in\mathbb{R}.\end{cases}\tag{3.7}$$

In addition, there exists a constant $T>0$, such that the following properties hold:
(i) $\{u^{(n)}\}_{n\in\mathbb{N}}$ is uniformly bounded in $C((0,T)\,;H^s(\mathbb{R}))\cap C^1((0,T)\,;H^{s-1}(\mathbb{R}))$.
(ii) $\{u^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C((0,T)\,;H^{s-1}(\mathbb{R}))$.

Proof Notice that H^{s-1} is an Banach algebra for $s>\frac{3}{2}$, thus we have

$$\left\|\left(\left(u^{(n)}\right)^2-\left(u_x^{(n)}\right)^2\right)^2\right\|_{H^{s-4}}\le C\left\|\left(u^{(n)}\right)^2-\left(u_x^{(n)}\right)^2\right)^2\Big\|_{H^{s-1}}\le C\left\|u^{(n)}\right\|_{H^s}^4.\tag{3.8}$$

Since $||u^{(n+1)}||_{H^s}$ is equivalent to $||m^{(n+1)}||_{H^{s-4}}$, utilizing Lemma 2.2 with $s_1 = s-4$, $s_2 = s-3$, we can obtain that

$$\begin{aligned}\Big\|\Big(\big(u^{(n)}\big)^2-\big(u_x^{(n)}\big)^2\Big)_x^2 m^{(n)}\Big\|_{H^{s-4}} &\leq C\Big\|\Big(\big(u^{(n)}\big)^2-\big(u_x^{(n)}\big)^2\Big)_x^2\Big\|_{H^{s-3}}\Big\|m^{(n)}\Big\|_{H^{s-4}}\\ &\leq C\Big\|u^{(n)}\Big\|_{H^s}^5.\end{aligned} \tag{3.9}$$

Further, applying Lemma 2.1 to equation (3.7), and utilizing (3.8) and (3.9), we get

$$\begin{aligned}&\Big\|u^{(n+1)}(t)\Big\|_{H^s} \leq \Big\|m^{(n+1)}(t)\Big\|_{H^{s-4}}\\ \leq & Ce^{C\int_0^t ||u^{(n)}(\tau)||_{H^s}^4 \, d\tau}\cdot\|u_0\|_{H^s} + C\int_0^t e^{C\int_\tau^t ||u^{(n)}(\eta)||_{H^s}^4 \, d\eta}\cdot\Big\|u^{(n)}(\tau)\Big\|_{H^s}^5 \, d\tau,\end{aligned} \tag{3.10}$$

where we need to use $\|m^{n+1}(0)\|_{H^{s-4}} \leq C\|u^{n+1}(0)\|_{H^s} \leq C\|S_{n+1}u_0\|_{H^s} \leq C\|u_0\|_{H^s}$.

Let $T>0$ such that $320C^5\|u_0\|_{H^s}^4 T \leq 1$ and suppose by induction that for every $0<t<T$

$$\Big\|u^{(n)}(t)\Big\|_{H^s} \leq \frac{2C\|u_0\|_{H^s}}{\big(1-64C^5\|u_0\|_{H^s}^4 t\big)^{\frac14}}. \tag{3.11}$$

Apparently, the expression (3.11) holds for $t=0$ forever. Notice the first term of (3.10), and using (3.11), after integration by parts we can get

$$C\int_0^t \Big\|u^{(n)}(\tau)\Big\|_{H^s}^4 \, d\tau \leq C\int_0^t \frac{16C^4\|u_0\|_{H^s}^4}{1-64C^5\|u_0\|_{H^s}^4\tau}\, d\tau = -\frac14\ln\big(1-64C^5\|u_0\|_{H^s}^4 t\big). \tag{3.12}$$

Plugging (3.12) into (3.10), we can deduce that

$$\begin{aligned}\Big\|u^{(n+1)}(t)\Big\|_{H^s} &\leq \frac{C\|u_0\|_{H^s}}{\big(1-64C^5\|u_0\|_{H^s}^4 t\big)^{\frac14}}\left(\int_0^t \frac{32C^5\|u_0\|_{H^s}^4}{\big(1-64C^5\|u_0\|_{H^s}^4\tau\big)^{\frac54}}\, d\tau - 1\right)\\ &\leq \frac{2C\|u_0\|_{H^s}}{\big(1-64C^5\|u_0\|_{H^s}^4 t\big)^{\frac14}}.\end{aligned}$$

Therefore, $\{u^{(n)}\}_{n\in\mathbb{N}}$ is uniformly bounded in $C((0,T);\,H^s(\mathbb{R}))$.

Now, we begin to estimate the H^{s-5} norm of the following two terms. Using Lemma 2.2, choose $s_1 = s-5$, $s_2 = s-2$, we can obtain

$$\begin{aligned}&\Big\|\Big(\big(u^{(n)}\big)^2-\big(u_x^{(n)}\big)^2\Big)^2\partial_x m^{(n+1)}\Big\|_{H^{s-5}}\\ \leq & C\Big\|\Big(\big(u^{(n)}\big)^2-\big(u_x^{(n)}\big)^2\Big)^2\Big\|_{H^{s-2}}\Big\|\partial_x m^{(n+1)}\Big\|_{H^{s-5}}\\ \leq & C\Big(\Big\|u^{(n)}\Big\|_{H^{s-1}}^4+\Big\|u^{(n)}u_x^{(n)}\Big\|_{H^s}^2+\Big\|u_x^{(n)}\Big\|_{H^{s-1}}^4\Big)\Big\|m^{(n+1)}\Big\|_{H^{s-4}}\\ \leq & C\Big\|u^{(n)}\Big\|_{H^s}^4\Big\|u^{(n+1)}\Big\|_{H^s}.\end{aligned}$$

In a similar manner, using Lemma 2.2, choose $s_1 = s-4$, $s_2 = s-2$, we have

$$\Big\|\Big(\big(u^{(n)}\big)^2-\big(u_x^{(n)}\big)^2\Big)_x^2 m^{(n)}\Big\|_{H^{s-5}} \leq C\Big\|\Big(\big(u^{(n)}\big)^2-\big(u_x^{(n)}\big)^2\Big)_x^2\Big\|_{H^{s-2}}\Big\|m^{(n)}\Big\|_{H^{s-4}} \leq C\Big\|u^{(n)}\Big\|_{H^s}^5.$$

Therefore, according to equation (3.7), we can get that $\partial_t u^{(n+1)} \in C((0,T);H^{s-1}(\mathbb{R}))$ is uniformly bounded, which means that the sequence $\{u^{(n)}\}_{n\in\mathbb{N}}$ is uniformly bounded in $C((0,T);H^s(\mathbb{R})) \cap C^1((0,T);H^{s-1}(\mathbb{R}))$.

Next we will prove that $\{u^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C((0,T);H^{s-1}(\mathbb{R}))$. In order to achieve this goal, it is necessary to construct the form of $u^{(n+l)} - u^{(n)}$. Thus, according to equation (3.7), we obtain that, for all $n, l \in \mathbb{N}$

$$\left[\partial_t + \left(\left(u^{(n+l)}\right)^2 - \left(u_x^{(n+l)}\right)^2\right)^2 \partial_x\right]\left(m^{n+l+1} - m^{n+1}\right) = f^{(n,l)}, \tag{3.13}$$

where

$$\begin{aligned} f^{(n,l)} = & \left[\left(\left(u^{(n)}\right)^2 - \left(u_x^{(n)}\right)^2\right)^2 - \left(\left(u^{(n+l)}\right)^2 - \left(u_x^{(n+l)}\right)^2\right)^2\right]\partial_x m^{(n+1)} \\ & - \left(\left(u^{(n+l)}\right)^2 - \left(u_x^{(n+l)}\right)^2\right)^2_x \left(m^{(n+l)} - m^{(n)}\right) \\ & - \left[\left(\left(u^{(n+l)}\right)^2 - \left(u_x^{(n+l)}\right)^2\right)^2 - \left(\left(u^{(n)}\right)^2 - \left(u_x^{(n)}\right)^2\right)^2\right]_x m^{(n)}. \end{aligned}$$

Set $\tilde{v} := u^{(n+l+1)} - u^{(n+1)}$. Then (3.13) is equivalent to

$$\left[\partial_t + \left(\left(u^{(n+l)}\right)^2 - \left(u_x^{(n+l)}\right)^2\right)^2 \partial_x\right] P\tilde{v} = f^{(n,l)}.$$

Further by transformation we can get

$$P\left[\left(\partial_t + \left(\left(u^{(n+l)}\right)^2 - \left(u_x^{(n+l)}\right)^2\right)^2 \partial_x\right)\tilde{v}\right] = g^{(n,l)}, \tag{3.14}$$

where

$$\begin{aligned} g^{(n,l)} = & \partial_x \tilde{v}\left[\alpha^2\beta^2\partial_x^4\left(\left(u^{(n+l)}\right)^2 - \left(u_x^{(n+l)}\right)^2\right)^2 - \left(\alpha^2+\beta^2\right)\partial_x^2\left(\left(u^{(n+l)}\right)^2 - \left(u_x^{(n+l)}\right)^2\right)^2\right] \\ & + \partial_x^2\tilde{v}\left[4\alpha^2\beta^2\partial_x^3\left(\left(u^{(n+l)}\right)^2 - \left(u_x^{(n+l)}\right)^2\right)^2 - 2\left(\alpha^2+\beta^2\right)\partial_x\left(\left(u^{(n+l)}\right)^2 - \left(u_x^{(n+l)}\right)^2\right)^2\right] \\ & + \partial_x^3\tilde{v}\left[6\alpha^2\beta^2\partial_x^2\left(\left(u^{(n+l)}\right)^2 - \left(u_x^{(n+l)}\right)^2\right)^2\right] + \partial_x^4\tilde{v}\left[4\alpha^2\beta^2\partial_x\left(\left(u^{(n+l)}\right)^2 - \left(u_x^{(n+l)}\right)^2\right)^2\right] \\ & + P\tilde{v}_t + \left(\left(u^{(n+l)}\right)^2 - \left(u_x^{(n+l)}\right)^2\right)^2 P\partial_x\tilde{v}. \end{aligned}$$

Notice that $P\tilde{v}_t + ((u^{(n+l)})^2 - (u_x^{(n+l)})^2)^2 P\partial_x\tilde{v} = f^{(n,l)}$, thus we assume that

$$g^{(n,l)} := N_1 + N_2 + N_3 + f^{(n,l)}. \tag{3.15}$$

Applying the operator P^{-1} to (3.14), which gives rise to

$$\left(\partial_t + \left(\left(u^{(n+l)}\right)^2 - \left(u_x^{(n+l)}\right)^2\right)^2 \partial_x\right)\tilde{v} = P^{-1}g^{(n,l)}. \tag{3.16}$$

Due to $||P^{-1}g^{(n,l)}||_{H^{s-1}}$ is equivalent to $||g^{(n,l)}||_{H^{s-5}}$, we just need to estimate the H^{s-5} norm of $||g^{(n,l)}||$ and the H^{s-1} norm of $((u^{(n+l)})^2 - (u_x^{(n+l)})^2)^2$. First of all, it can be obtained by direct calculation

$$\left\|\left(\left(u^{(n+l)}\right)^2 - \left(u_x^{(n+l)}\right)^2\right)^2\right\|_{H^{s-1}} \le C\left\|u^{(n+l)}\right\|^4_{H^s}.$$

Before calculating the H^{s-5} norm of $g^{(n,l)}$ in (3.15), we first estimate the H^{s-5} norm of $f^{(n,l)}$. Using Lemma 2.2, the boundedness of the $f^{(n,l)}$ is obvious,

$$\left\|f^{(n,l)}\right\|_{H^{s-5}} \le C\left\|\left(u^{(n+l)} - u^{(n)}\right)\right\|_{H^{s-1}}\left(\left\|u^{(n)}\right\|^4_{H^s} + \left\|u^{(n+1)}\right\|^4_{H^s} + \left\|u^{(n+l)}\right\|^4_{H^s}\right).$$

Finally, using Lemma 2.2 and choosing the appropriate s_1 and s_2, we can obtain the H^{s-5} norm of N_j for $j=1,2,3,4$. Plugging $\|N_j\|_{H^{s-5}}$ and $\|f^{(n,l)}\|_{H^{s-5}}$ into (3.15), we have

$$\begin{aligned} g^{(n,l)} \leq & C\left\|\left(u^{(n+l)}-u^{(n)}\right)\right\|_{H^{s-1}}\left(\left\|u^{(n)}\right\|_{H^s}^4+\left\|u^{(n+1)}\right\|_{H^s}^4\right. \\ & \left.+\left\|u^{(n+l)}\right\|_{H^s}^4\right)+\|\tilde{v}\|_{H^{s-1}}\left\|u^{(n+l)}\right\|_{H^s}^4. \end{aligned} \tag{3.17}$$

Applying Lemma 2.1 to (3.14) and using (3.17), we deduce that

$$\begin{aligned} \|\tilde{v}(t)\|_{H^{s-1}} \leq & e^{C\int_0^t \|((u^{(n+l)})^2-(u_x^{(n+l)})^2)^2(\tau)\|_{H^{s-1}}\, d\tau} \cdot \|\tilde{v}(0)\|_{H^{s-1}} \\ & +\int_0^t e^{C\int_\tau^t \|((u^{(n+l)})^2-(u_x^{(n+l)})^2)^2(\eta)\|_{H^{s-1}}\, d\eta} \cdot \|g^{(n,l)}(\tau)\|_{H^{s-5}}\, d\tau \\ \leq & e^{C\int_0^t \|u^{(n+l)}(\tau)\|_{H^s}^4\, d\tau} \cdot \|\tilde{v}(0)\|_{H^{s-1}} \\ & +C\int_0^t e^{C\int_\tau^t \|u^{(n+l)}(\eta)\|_{H^s}^4\, d\eta} \cdot \|\tilde{v}(\tau)\|_{H^{s-1}} \cdot \|u^{(n+l)}(\tau)\|_{H^s}^4\, d\tau \\ & +C\int_0^t e^{C\int_\tau^t \|u^{(n+l)}(\eta)\|_{H^s}^4\, d\eta} \cdot \|u^{(n+l)}(\tau)-u^{(n)}(\tau)\|_{H^{s-1}} \\ & \cdot\left(\|u^{(n)}(\tau)\|_{H^s}^4+\|u^{(n+1)}(\tau)\|_{H^s}^4+\|u^{(n+l)}(\tau)\|_{H^s}^4\right) d\tau. \end{aligned}$$

By the previous proof we know that $\{u^{(n)}\}_{n\in\mathbb{N}}$ is uniformly bounded in $C((0,T);H^s(\mathbb{R})) \cap C^1((0,T);H^{s-1}(\mathbb{R}))$. Based on Littlewood-Paley decomposition, we have

$$\tilde{v}_0=u_0^{(n+l+1)}-u_0^{(n+1)}=S_{n+l+1}u_0-S_{n+1}u_0=\sum_{q=n+1}^{n+l+1}\Delta_q u_0,$$

and there exists a constant C_T independent of n and l such that for every $0<t<T$,

$$\|\tilde{v}(t)\|_{H^{s-1}} \leq C_T\left(2^{-n}+\int_0^t\left\|u^{(n+l)}(\tau)-u^{(n)}(\tau)\right\|_{H^{s-1}} d\tau\right).$$

By making argument inductive with respect to n, we can easily obtain that

$$\|\tilde{v}(t)\|_{L_T^\infty H^{s-1}} \leq \frac{(TC_T)^{n+1}}{(n+1)!}\left\|u^{(l)}\right\|_{L_T^\infty(H^s)}+C_T\sum_{k=0}^n 2^{-(n-k)}\frac{(TC_T)^k}{k!}.$$

Similarly $||u^{(l)}||_{L_T^\infty(H^s)}$ can be bounded independently of l. We conclude that there exist some new constant C_T' independent of n and l such that

$$\|\tilde{v}(t)\|_{L_T^\infty(H^{s-1})} \leq 2^{-n}C_T'.$$

Therefore, $\{u^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C((0,T);H^{s-1}(\mathbb{R}))$.

Proof of Theorem 3.1 With the help of Lemma 3.2, we know that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C((0,T),H^{s-1}(\mathbb{R}))$, thus there is a function $u\in C((0,T);H^{s-1}(\mathbb{R}))$ that serves as the limit of $\{u_n\}_{n\in\mathbb{N}}$. Now we need to prove that u belongs to $C((0,T);H^s(\mathbb{R}))\cap C^1((0,T);H^{s-1}(\mathbb{R}))$ and solves the Cauchy problem (1.1). According to Lemma 3.2, $\{u_n\}_{n\in\mathbb{N}}$

is uniformly bounded in $L^{\infty}\left((0,T);H^{s}\left(\mathbb{R}\right)\right)$. The Fatou property guarantees that u also belongs to $L^{\infty}\left((0,T);H^{s}\left(\mathbb{R}\right)\right)$.

Additionally, since $\{u_n\}_{n\in\mathbb{N}}$ converges to u in $C((0,T);H^{s-1}\left(\mathbb{R}\right))$, an interpolation argument guarantees that the convergence also holds in $C((0,T);H^{s'}\left(\mathbb{R}\right))$ for any $s'<s$. By taking the limit in the equation (3.7), we can conclude that u is indeed a solution to the Cauchy problem (1.1). Considering that u belongs to $L^{\infty}\left((0,T);H^{s}\left(\mathbb{R}\right)\right)$, where $s>\frac{7}{2}$, the right-hand side of the equation

$$\partial_t m+\left(u^2-u_x^2\right)^2\partial_x m=-\left(u^2-u_x^2\right)_x^2 m$$

belongs to $L^{\infty}((0,T);H^{s-4}\left(\mathbb{R}\right))$. It is evident that $u\in C((0,T);H^{s'}\left(\mathbb{R}\right))$ for any $s'<s$. Finally, by observing the equation we can deduce that $\partial_t u\in C((0,T);H^{s-1}\left(\mathbb{R}\right))$. Furthermore, a standard use of a sequence of viscosity approximate solution $\left(u_{\xi}\right)_{\xi>0}$ for the Cauchy problem (1.1) which converges uniformly in

$$C\left((0,T);H^{s}\left(\mathbb{R}\right)\right)\cap C^1((0,T);H^{s-1}\left(\mathbb{R}\right))$$

gives the continuity of the solution u in $C\left((0,T);H^{s}\left(\mathbb{R}\right)\right)\cap C^1((0,T);H^{s-1}\left(\mathbb{R}\right))$.

4 Peakon Solitons

In this section, we initially derive the weak formulation of the ghmCH equation (1.1), and utilize it to demonstrate the existence of single peakon solution, and subsequently establish a general N-peakon dynamic system.

4.1 Weak Formulation

Firstly, by substituting m expressed in terms of u into the ghmCH equation (1.1), we can yield a fully nonlinear partial differential equation. By applying the operator $P^{-1}=((1-\alpha^2\partial_x^2)\left(1-\beta^2\partial_x^2\right))^{-1}$ to this equation, we can easily obtain that

$$\begin{aligned}u_t=&-\partial_x P^{-1}\Big(\left(u-\left(\alpha^2+\beta^2\right)u_{xx}\right)\left(u^2-u_x^2\right)^n-2n\alpha^2\beta^2u_x\\&\times\left(u-u_{xx}\right)u_{xxx}\left(u^2-u_x^2\right)^{n-1}\Big)-\alpha^2\beta^2\partial_x^2P^{-1}\left(\left(u^2-u_x^2\right)^n u_{xxx}\right).\end{aligned}$$

An equivalent form can be obtained by using (3.1),

$$\begin{aligned}u_t=&-\partial_x G*\Big(\left(u-\left(\alpha^2+\beta^2\right)u_{xx}\right)\left(u^2-u_x^2\right)^n-2n\alpha^2\beta^2u_x\\&\times\left(u-u_{xx}\right)u_{xxx}\left(u^2-u_x^2\right)^{n-1}\Big)-\partial_x^2G*\left(\left(u^2-u_x^2\right)^n u_{xxx}\right).\end{aligned}\tag{4.1}$$

Subsequently, we multiply (4.1) by a test function $\varphi\left(t,x\right)$-a smooth function with compact support, and integrate over $-\infty<x<\infty$ and $0\le t<\infty$. Then, the weak formulation of the ghmCH equation (1.1) can be obtained through integration by parts,

$$\int_0^{\infty}\int_{-\infty}^{\infty} u\varphi_t + G*\Big(\big(u-\left(\alpha^2+\beta^2\right)u_{xx}\big)\left(u^2-u_x^2\right)^n - 2n\alpha^2\beta^2 u_x\left(u-u_{xx}\right)\left(u^2-u_x^2\right)^{n-1}$$
$$\times u_{xxx}\Big)\varphi_x - \alpha^2\beta^2 G*\left(\left(u^2-u_x^2\right)^n u_{xxx}\right)\varphi_{xx}\ dxdt + \int_{-\infty}^{\infty} u_0\left(x\right)\varphi\left(0,x\right)\ dx = 0. \tag{4.2}$$

The weak equation is crucial for deriving both the single peak solution and multi-peakon dynamic system of the ghmCH (1.1) equation. In the forthcoming work, we shall leverage the weak equation to deduce the single peakon solution and multi-peakon dynamic system.

4.2 Single Peakon Solution

It is known that a remarkable property to the mCH equation (1.3) is the existence of single peakon solution, which is given by $u_a\left(t,x\right)=ae^{-|x-ct|},\ c=\frac{2}{3}a^2$. Now we are concerned with the existence of single peakon solution to the ghmCH equation (1.1).

Definition 4.1 Given initial data $u_0\left(x\right)$, the function $u\left(t,x\right)$ is said to be a weak solution to the initial-value problem (1.1) if it satisfies the weak formulation for any smooth test function $\varphi\left(t,x\right)\in\mathbb{C}_c^{\infty}\left(\left[0,T\right)\times\mathbb{R}\right)$. If $u\left(t,x\right)$ is a weak solution on $[0,T)$ for every $T>0$, then it is called a global weak solution.

Theorem 4.1 For any $b\neq 0$, the peaked functions of the form

$$u=\frac{b}{2\left(\alpha^2-\beta^2\right)}\left(\alpha e^{-\frac{1}{\alpha}|x-ct|}-\beta e^{-\frac{1}{\beta}|x-ct|}\right), \tag{4.3}$$

where

$$c=-\frac{b^2\left(-2\alpha^5+2\alpha^4\beta+4\alpha^3\beta^2-4\alpha^2\beta^3-2\alpha\beta^4+2\beta^5\right)}{8\left(\alpha^2-\beta^2\right)^3\left(\alpha+\beta\right)}$$

is a single peakon solution to the ghmCH equation (1.1) in the case of $n=1$.

Proof The derivatives of the expression (4.3) with respect to the variables x and t do not exist in the traditional sense of the straight line $x=ct$. Consequently, the conventional calculus method fails to prove that (4.3) is the solution of ghmCH equation (1.1) in the case of $n=1$.

In the following proof, we consider only the case of $x>ct$, and the case of $x<ct$ will be proved in the same way. Now, we will consider each term in the equation (4.2) for the case of $n=1$. Firstly, we consider the term related to the t-derivative of the equation (4.2) in the case of $n=1$. When $x>ct$, a direct calculation leads to

$$\begin{aligned}&\int_0^{\infty}\int_{-\infty}^{\infty} u\varphi_t\ dxdt+\int_{-\infty}^{\infty}u_0\left(x\right)\varphi\left(0,x\right)\ dx\\ =&\int_0^{\infty}\int_{-\infty}^{\infty}\frac{-bc}{2\left(\alpha^2-\beta^2\right)}sgn\left(x-ct\right)\left(e^{-\frac{1}{\alpha}|x-ct|}-e^{-\frac{1}{\beta}|x-ct|}\right)\varphi\ dxdt\\ =&\int_0^{\infty}\int_{-\infty}^{\infty}\frac{-bc}{2\left(\alpha^2-\beta^2\right)}\left(e^{-\frac{x}{\alpha}+\frac{ct}{\alpha}}-e^{-\frac{x}{\beta}+\frac{ct}{\beta}}\right).\end{aligned} \tag{4.4}$$

Next, we deal with the second term in the equation (4.2) in the case of $n = 1$ by utilizing the method of integration by parts

$$\begin{aligned}&\int_0^\infty \int_{-\infty}^\infty G * \Big(\big(u - \left(\alpha^2 + \beta^2\right) u_{xx}\big) \left(u^2 - u_x^2\right) - 2\alpha^2\beta^2 u_x \big(u - u_{xx}\big) u_{xxx}\Big)\varphi_x \, dxdt \\ =& - \int_0^\infty \int_{-\infty}^\infty \partial_x G * \Big(\big(u - \left(\alpha^2 + \beta^2\right) u_{xx}\big) \left(u^2 - u_x^2\right) - 2\alpha^2\beta^2 u_x \big(u - u_{xx}\big) u_{xxx}\Big)\varphi \, dxdt.\end{aligned} \tag{4.5}$$

We split up the integral interval $\mathbb{R}$ into three parts: $(-\infty, ct)$, (ct, x) and $(x, +\infty)$ when $x > ct$. Then we define

$$\partial_x G * \Big(\big(u - \left(\alpha^2 + \beta^2\right) u_{xx}\big) \left(u^2 - u_x^2\right) - 2\alpha^2\beta^2 u_x \left(u - u_{xx}\right) u_{xxx}\Big)\varphi =: I_1 + I_2 + I_3.$$

Finally, we deal with the remaining terms in equation (4.2) in the case of $n = 1$,

$$\int_0^\infty \int_{-\infty}^\infty G * \big(\left(u^2 - u_x^2\right) u_{xxx}\big) \varphi_{xx} \, dxdt = \int_0^\infty \int_{-\infty}^\infty \partial_x^2 G * \big(\left(u^2 - u_x^2\right) u_{xxx}\big) \varphi \, dxdt. \tag{4.6}$$

Using the same method as above, we define

$$\partial_x^2 G * \big(\left(u^2 - u_x^2\right) u_{xxx}\big) =: II_1 + II_2 + II_3.$$

Plugging (4.4), (4.5) and (4.6) into (4.2), we can conclude that,

$$\begin{aligned}&\int_0^\infty \int_{-\infty}^\infty \Big(u_t + \partial_x G * \Big(\big(u - \left(\alpha^2 + \beta^2\right) u_{xx}\big) \left(u^2 - u_x^2\right) \\ &- 2\alpha^2\beta^2 u_x \left(u - u_{xx}\right) u_{xxx}\Big) + \alpha^2\beta^2 \partial_x^2 G * \big(\left(u^2 - u_x^2\right) u_{xxx}\big)\Big)\varphi \, dxdt = 0\end{aligned}$$

holds for any test function $\varphi(t, x)$. Hence, when $x > ct$, a direct computation gives rise to

$$\begin{aligned}&I_1 + I_2 + I_3 + \alpha^2\beta^2 \left(II_1 + II_2 + II_3\right) \\ =&\frac{b^3 \left(-2\alpha^5 + 2\alpha^4\beta + 4\alpha^3\beta^2 - 4\alpha^2\beta^3 - 2\alpha\beta^4 + 2\beta^5\right)}{16 \left(\alpha^2 - \beta^2\right)^4 (\alpha + \beta)} \times \left(e^{-\frac{x}{\alpha} + \frac{ct}{\alpha}} - e^{-\frac{x}{\beta} + \frac{ct}{\beta}}\right).\end{aligned} \tag{4.7}$$

In view of (4.4) and (4.7), we find that

$$c = -\frac{b^2 \left(-2\alpha^5 + 2\alpha^4\beta + 4\alpha^3\beta^2 - 4\alpha^2\beta^3 - 2\alpha\beta^4 + 2\beta^5\right)}{8 \left(\alpha^2 - \beta^2\right)^3 (\alpha + \beta)},$$

which completes the proof of Theorem 4.1.

Moreover, in a similar manner, we can get

$$u = \frac{h}{2\left(\alpha^2 - \beta^2\right)} \left(\alpha e^{-\frac{1}{\alpha}|x - ct|} - \beta e^{-\frac{1}{\beta}|x - ct|}\right), \quad \text{where} \quad c = \frac{h^4}{(\alpha + \beta)^4}$$

is a single peakon solution to ghmCH equation (1.1) in the case of $n = 2$.

Using the above method, we can obtain the single peakon solution of the ghmCH equation (1.1) when n takes different integers. Since the method is similar and the calculation process is cumbersome, we only give the single peakon solution when $n=1,2$.

4.3 Multi-Peakon Dynamic System

Now, we shift our focus to derive the multi-peakon dynamic system of the ghmCH equation (1.1). Since the multi-peak solution is a linear superposition of the given peak traveling waves, we assume

$$u(t,x)=\sum_{i=1}^{N}\frac{\alpha p_i(t)}{2(\alpha^2-\beta^2)}e^{-\frac{1}{\alpha}|x-q_i|}-\frac{\beta p_i(t)}{2(\alpha^2-\beta^2)}e^{-\frac{1}{\beta}|x-q_i|},\quad N\geq 2 \tag{4.8}$$

with time-dependent amplitudes $p_i(t)$ and positions $q_i(t)$.

Theorem 4.2 The multi-peakon (4.8) is a solution of the ghmCH equation (1.1) if and only if $p_i(t)$ and $q_i(t)$ satisfy the following system of 2n ODE's

$$\begin{cases} p_{i,t}=0\\ q_{i,t}=\Big[\Big(\displaystyle\sum_{j=1}^{N}\frac{p_j}{2(\alpha^2-\beta^2)}\big(\alpha e^{-\frac{1}{\alpha}|q_i-q_j|}-\beta e^{-\frac{1}{\beta}|q_i-q_j|}\big)\Big)^2\\ \qquad -\Big(\displaystyle\sum_{j=1}^{N}\frac{p_j}{2(\alpha^2-\beta^2)}\big(-sgn\big(q_i-q_j\big)\big(e^{-\frac{1}{\alpha}|q_i-q_j|}-e^{-\frac{1}{\beta}|q_i-q_j|}\big)\big)\Big)^2\Big]^n. \end{cases} \tag{4.9}$$

Proof We can substitute the expression of the multi-peakon solutions (4.8) into the weak formulation (4.2). Without loss of generality, we will assume that $q_1(t)<q_2(t)<\cdots<q_N(t)$ at a fixed $t>0$ and segregate the integral over x into corresponding intervals. Then by employing integration by parts, we derive the expressions of $p_i(t)$ and $q_i(t)$. While the calculation process resembles that of a single peakon but is more lengthy, we will adopt another approach as utilized in [14]:

Let $y(x)$ be a distribution with singular support at a finite number of points $x=x_i$ in $\mathbb{R}$, and define its non-singular part

$$\langle y(x)\rangle=\begin{cases} y(x), & x\neq x_i,\\ \dfrac{1}{2}\left(y(x_i^+)+y(x_i^-)\right), & x=x_i, \end{cases} \tag{4.10}$$

at its jump discontinuities

$$[y]_{x_i}=y(x_i^+)-y(x_i^-)=[\langle y(x)\rangle]_{x_i}. \tag{4.11}$$

Firstly, we need to rewrite the expression (4.2) in the following equivalent form,

$$\begin{aligned}&\int_0^\infty\int_{-\infty}^\infty \left(u_t-\left(\alpha^2+\beta^2\right)u_{xxt}\right)\varphi-\left(u-\left(\alpha^2+\beta^2\right)u_{xx}\right)\left(u^2-u_x^2\right)^n\varphi_x\\ &+\alpha^2\beta^2\left(u^2-u_x^2\right)_x^n u_{xxx}\varphi_x+\alpha^2\beta^2\left(u^2-u_x^2\right)^n u_{xxx}\varphi_{xx}+\alpha^2\beta^2u_t\varphi_{xxxx}\,dxdt=0.\end{aligned} \tag{4.12}$$

Notice from (4.10) that,

$$u_t - (\alpha^2+\beta^2)u_{xxt} = -\alpha^2\beta^2 \Big\langle \langle u_{txxx}\rangle_x \Big\rangle.$$

We start by dealing with the terms in (4.12) related to the t derivative of u,

$$\begin{aligned}&\int_0^\infty \int_{-\infty}^\infty \left(u_t - \left(\alpha^2+\beta^2\right) u_{xxt}\right)\varphi + \alpha^2\beta^2 u_t \varphi_{xxxx}\, dxdt \\ =& -\alpha^2\beta^2 \sum_{i=1}^N \int_0^\infty \left(\varphi_{xxx}\left[u_t\right]_{q_i} - \varphi_{xx}\left[u_{tx}\right]_{q_i} + \varphi_x \left[u_{txx}\right]_{q_i} - \varphi\left[\langle u_{txxx}\rangle\right]_{q_i}\ \right)dt.\end{aligned} \tag{4.13}$$

Next, we consider the remaining terms independent of the t-derivative of u in (4.12),

$$\begin{aligned}&\int_0^\infty \int_{-\infty}^\infty -\left(u - \left(\alpha^2+\beta^2\right)u_{xx}\right)\left(u^2-u_x^2\right)^n \varphi_x \\ &+ \alpha^2\beta^2\left(u^2-u_x^2\right)^n_x u_{xxx}\varphi_x + \alpha^2\beta^2\left(u^2-u_x^2\right)^n u_{xxx}\varphi_{xx}\, dxdt \\ =& -\alpha^2\beta^2 \sum_{i=1}^N \int_0^\infty \varphi_x \left[\left(u^2-u_x^2\right)^n u_{xxx}\right]_{q_i}\ dt,\end{aligned} \tag{4.14}$$

where we use $u - (\alpha^2+\beta^2)\, u_{xx} = -\alpha^2\beta^2 \langle u_{xxxx}\rangle$.

Combining the expression (4.13) and (4.14), we conclude that

$$\begin{aligned}&\sum_{i=1}^N \int_0^\infty \Big(-\varphi\left[\langle u_{txxx}\rangle\right]_{q_i} + \varphi_x\left[\left(u^2-u_x^2\right)^n u_{xxx}\right]_{q_i} + \varphi_x\left[u_{txx}\right]_{q_i} - \varphi_{xx}\left[u_{tx}\right]_{q_i} + \varphi_{xxx}\left[u_t\right]_{q_i}\Big)\, dt \\ =&0.\end{aligned} \tag{4.15}$$

The jump terms involving t-derivatives of u are given by

$$\left[u_t\right]_{q_i} = 0, \quad \left[u_{tx}\right]_{q_i} = 0, \quad \left[u_{txx}\right]_{q_i} = -\frac{1}{\alpha^2\beta^2} p_i q_{i,t}, \quad \left[\langle u_{txxx}\rangle\right]_{q_i} = \frac{1}{\alpha^2\beta^2} p_{i,t}, \tag{4.16}$$

which can be derived from (4.11). The equation (4.15) will hold for all test function $\varphi(t,x)$ if and only if

$$\left[u_{txx}\right]_{q_i} + \left[\left(u^2-u_x^2\right)^n u_{xxx}\right]_{q_i} = 0, \quad \left[\langle u_{txxx}\rangle\right]_{q_i} = 0. \tag{4.17}$$

Furthermore, a straightforward computation gives

$$\left[\left(u^2-u_x^2\right)^n u_{xxx}\right]_{q_i} = \frac{p_i}{\alpha^2\beta^2}\left(u^2-u_x^2\right)^n.$$

Substituting (4.17) and (4.16) into (4.15) yields the multi-peakon dynamic system (4.9). In Theorem 4.2, $p_{i,t} = 0$ implies that the peak amplitude does not change along with the time t.

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高阶Camassa-Holm类方程的适定性和Peakon解

陈 爽

(宁波大学数学与统计学院, 浙江 宁波 315211)

摘要: 本文研究了广义的高阶Camassa-Holm类方程, 简记为ghmCH方程. 在$s>\frac{7}{2}$, 初值u_0属于$H^s(\mathbb{R})$的条件下, 我们建立了该方程的局部适定性. 除此之外, 我们获得了该方程的弱形式, 并证明了单峰解和多峰动力系统的存在性.

关键词: 广义的高阶Camassa-Holm类方程; 局部适定性; 尖峰孤子解

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