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ON POTENT ELEMENTS AND PSEUDO CLEAN RINGS

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Abstract: In this paper, potent index of an element and pseudo clean rings are considered. Some properties and examples of pseudo clean rings are given. We also show that $\mathbb{Z}_m$ is pseudo clean for every $2 \leq m \in \mathbb{Z}$ and pseudo clean rings are clean. Furthermore, we prove pseudo clean rings are directly finite and have stable range one.

Keywords: clean rings; strongly J-clean rings; pseudo clean rings; potent elements

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1 Introduction

All rings in this paper are assumed to be associative with identity. For a ring R, we denoted the group of units, the set of idempotents, the set of nilpotent, and Jacobson radical by $U(R)$, $E(R)$, $N(R)$, and $J(R)$, respectively. $M_n(R)$ and $T_n(R)$ stand for the $n \times n$ matrix ring and upper triangular matrix ring over R, respectively.

A ring R is said to be exchanged[1] if for every $a \in R$, there exists an idempotent $e \in Ra$ such that $1-e \in R(1-a)$. An element $a \in R$ is called clean if it is the sum of an idempotent and a unit and Nicholson said that R is clean if every element of R is clean in[1]. He also proved clean rings and exchange rings are equivalent for abelian rings(i.e. all idempotents are central).

Recall that an element a of R is strongly π-regular if there exists a positive integer n such that $a^n \in a^{n+1}R \cap Ra^{n+1}$ and R is called strongly π-regular[2] if every element of R is strongly π-regular. A ring R is reduced if $N(R) = 0$. A ring R is said to be weakly abel[3] if $eR(1-e) \in J(R)$ for all $e \in E(R)$(equivalently, $R/J(R)$ is abelian). R is directly finite if given $a, b \in R$, $ab = 1$ implies $ba = 1$. It is well known that $\{\text{reduced rings}\} \subseteq \{\text{abelian rings}\} \subseteq \{\text{weakly abel rings}\} \subseteq \{\text{directly finite rings}\}$. Recall that a ring R is said to have stable range one[4] if $aR + bR = R$ with $a, b \in R$ implies that there exists $y \in R$ such that $a + by \in U(R)$. The stable range one condition is especially interesting because of Evans'

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Theorem, which states that a module cancels from direct sums whenever it has stable range one. In [1], Ara proved that every strongly π-regular ring has stable range one.

In 1999, Nicholson called an element is strongly clean[5] if it can be written the sum of an idempotent and an unit which commutate. A ring R is strongly clean provided that every element is strongly clean. He also proved strongly π-regular elements are strongly clean and so strongly π-regular rings are strongly clean. There are some open questions whether every strongly clean ring is directly finite or has stable range one. Some other results can be found in [6–8].

An element e is called potent[9] if there exists some integer $k \geq 2$ such that $e^k = e$. In this paper, some properties on potent elements are given. We also define a ring R is pseudo clean provided that every element of R can be written as the sum of a potent element and an element in its Jacobson radical. It is proved every pseudo clean clean rings is clean and directly finite and have stable range one. An example is also given to show clean rings need not to be pseudo clean(see Example 3.2). Let $\mathbb{Z}_n$ stand for the ring which the integer ring $\mathbb{Z}$ module $n\mathbb{Z}$ and $\mathbb{Z}_n^*$ stand for the group whose elements consist of $U(\mathbb{Z}_n)$. It is well known that $\mathbb{Z}_n^*$ is a cyclic group if and only if $n \in \{2, 4, p^l, 2p^l\}$ where p is odd prime number and l is positive integer(by elementary number theory). We prove $\mathbb{Z}_n$ is pseudo clean for every $2 \leq n \in \mathbb{Z}$. Furthermore, Some properties of pseudo clean rings are given.

2 Potent Elements

Let R be a ring, we denote all potent elements of R by $Pot(R)$. In this section, some properties of potent elements are given.

Proposition 2.1 Let R be a ring. The following results hold:
(1) $Pot(R) \cap U(R) = \sqrt{1}^*$, where$\sqrt{1}^* \doteq \{e | e^k = 1 \text{ for some integer } k \geq 1\}$.
(2) $Pot(R) \cap J(R) = \{0\}$.
(3) $Pot(R) \cap N(R) = \{0\}$.
(4) $Pot(R) \cap \{1 + N(R)\} = \{1\}$ if and only if R is reduce or $Char(R) = 0$, where $1 + N(R) = \{1 + x | x \in N(R)\}$.
(5) If $Char(R) = p$ where p is a prime, then $\{1 + N(R)\} \subseteq \sqrt{1}^*$.
(6) If $Char(R) = 2^n - 2$ for some integer $n \geq 2$, then $1 + E(R) \subseteq Pot(R)$, where $1 + E(R) = \{1 + x | x \in E(R)\}$.

Proof (1)For any $x \in Pot(R) \cap U(R)$, write $x^k = x$ for some integer $k \geq 2$. Hence $x^{k-1} = 1$. Clearly, $\sqrt{1}^* \subseteq Pot(R) \cap U(R)$. Thus $Pot(R) \cap U(R) = \sqrt{1}^*$.

(2)For every $x \in Pot(R) \cap J(R)$, there exists $k \geq 2$ such that $x^k = x$. Note that $1 - x^{k-1} \in U(R)$, we have $x = 0$. Hence $Pot(R) \cap J(R) = \{0\}$.

(3)Similar to the proof of (2).

(4) $\Rightarrow$ Suppose $N(R) \neq 0$ and $Char(R) = m < \infty$. There exist nonzero $n \in N(R)$ such that $n^2 = 0$. We have $(1 + n)^m = \sum\limits_{i=0}^{m} C_m^i n^i = 1$, hence $1 + n \in Pot(R) \cap \{1 + N(R)\}$. Contradictorily,

$\Leftarrow$ If R is reduce, it is clear $Pot(R)\cap\{1+N(R)\}=\{1\}$. If $Char(R)=0$ and there exist nonzero $n\in N(R)$ such that $1+n\in Pot(R)\cap\{1+N(R)\}$, suppose $n^l=0$ but $n^{l-1}\neq 0$ for some integer $l\geq 2$. Note that $(1+n)^k=1+n$ for some integer $k\geq 2$, we get $\sum\limits_{i=1}^{k}C_k^i n^i=n$. Hence $(k-1)n+\sum\limits_{i=2}^{k}C_k^i n^i=0$. Thus $(k-1)n^{l-1}=n^{l-2}((k-1)n+\sum\limits_{i=2}^{k}C_k^i n^i)=0$. We get $n^{l-1}=0$, contradictorily.

(5)It follows $(1+x)^p=1+x^p$ for all $x\in R$.

(6) For any $e\in E(R)$, we have $(1+e)^n=1+e+(2^n-2)e=1+e$ for all $n\in\mathbb{Z}$. Hence $1+e\in Pot(R)$.

The next lemma is very useful.

Lemma 2.2 Let $e^k=e$ for some integer $k\geq 2$. Then $e^{a(k-1)+1}=e$ for any integer $a\geq 1$.

Proof It follows from $e^{a(k-1)+1}=e^{(a-1)(k-1)+1}=\cdots=e^{k-1+1}=e$.

Proposition 2.3 Let R be a ring and $e_1,e_2\in Pot(R)$. There exists an integer $k\geq 2$ such that $e_1^k=e_1,e_2^k=e_2$.

Proof Assume $e_1^{k_1}=e_1,e_2^{k_2}=e_2$ for $k_1\geq 2$, $k_2\geq 2$. Take $k=(k_1-1)(k_2-1)+1$, by Lemma 2.2, we have $e_1^k=e_1,e_2^k=e_2$.

Proposition 2.4 Let R_1 and R_2 be rings and $e_1\in Pot(R_1),e_2\in Pot(R_2)$. There exists an integer $k\geq 2$ such that $e_1^k=e_1,e_2^k=e_2$.

Proof Similar to the proof of the above proposition.

Theorem 2.5 Let $I=\{1,2,\cdots,n\}$ and R_i be rings ($i\in I$).

(1) If $e_1,e_2\in Pot(R_1)$ and $e_1e_2=e_2e_1$, then $e_1e_2\in Pot(R_1)$.

(2) If $e_1,e_2\in Pot(R_1)$ and $e_1e_2=e_2e_1=0$, then $e_1+e_2\in Pot(R_1)$.

(3) If $e_i\in Pot(R_i)$ ($i\in I$), then $(e_1,e_2,\cdots,e_n)\in Pot(\prod_{i=1}^n R_i)$.

(4) If $e_1,e_1,\cdots,e_s\in Pot(R_1)$, then $diag(e_1,e_1,\cdots,e_s)\in M_s(R_1)$.

Proof It is clear by Proposition 2.3 and Proposition 2.4.

Corollary 2.6 Let R be a ring. Then $a\in Pot(R)$ if and only if $-a\in Pot(R)$.

Proof Since $-1\in Pot(R)$, we get the result easily by Theorem 2.5 (1).

For convenience, we define the concept which is called potent index of a potent element and a ring.

Definition 2.7 A potent element $e\in Pot(R)$ in a ring R is said to have potent index k if k is the smallest positive integer such that $e^k=e(k\geq 2)$. The potent index of e denoted by $Pind_R(e)$. A ring R is said to have potent index k (possibly ∞) if k is the smallest positive integer such that $e^k=e(k\geq 2)$ for all $e\in Pot(R)$. The potent index of R denoted by $Pind(R)$.

Example 2.8 (1) Let R be a ring. $e\in E(R)$ if and only if $Pind_R(e)=2$. Particularly, $Pind_R(0)=Pind_R(1)=2$.

(2) Let R be a ring and $Char(R)=m\neq 2$. Then $Pind_{T_2(R)}\ (\begin{pmatrix}1&1\\0&1\end{pmatrix})=m+1$, since

$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^n = \begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$ for all n.

(3) Let $R = \mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$ where p is a prime. Then $Pind_R(a) = p$ where a is primitive root modulo p. So $Pind(R) = p$.

(4) Let R be a boolean ring. Then $Pind(R) = 2$.

(5) Let I be a set consisting of all prime and $R = \prod_{p\in I}\mathbb{Z}_p$. Then $Pind(R) = \infty$.

Theorem 2.9 Let $I = \{1, 2, \cdots, n\}$ and R_i be rings $(i \in I)$.

(1) If $e \in Pot(R_1)$ and $e^k = e$ for some integer $k \geq 2$, then $Pind_{R_1}(e) - 1|k - 1$. In particular, $Pind_{R_1}(e) - 1|Pind(R_1) - 1$.

(2)If $e \in Pot(R_1)$ and $e^{r+k} = e^k$ for some integer $k \geq 1$, then $Pind_{R_1}(e) - 1|r$.

(3) If $e_1, e_2 \in Pot(R_1)$ and $e_1e_2 = e_2e_1 = 0$, then $Pind_{R_1}(e_1 + e_2) = k + 1$, where $k = [Pind_{R_1}(e_1) - 1, Pind_{R_1}(e_2) - 1]$.

(4) If $e_i \in Pot(R_i)$ $(i \in I)$, then $Pind_{\prod_{i=1}^n R_i}(e_1, e_2, \cdots, e_n) = k+1$, where $k = [Pind_{R_1}(e_1) - 1, Pind_{R_2}(e_2) - 1, \cdots, Pind_{R_n}(e_n) - 1]$.

(5) If $e_1, e_2, \cdots, e_s \in Pot(R_1)$, then $Pind_{M_s(R_1)}(diag(e_1, e_2, \cdots, e_s)) = k + 1$, where $k = [Pind_{R_1}(e_1) - 1, Pind_{R_1}(e_2) - 1, \cdots, Pind_{R_1}(e_s) - 1]$.

(6) If $Pot(R_1) = \{e_1, e_2, \cdots, e_s\}$, then $Pind(R_1) = k+1$ where $k = [Pind_{R_1}(e_1)-1, Pind_{R_1}(e_2)-1, \cdots, Pind_{R_1}(e_s) - 1]$. In this case, $Pind(R_1) < \infty$.

Proof (1) Assume there exist l, r such that $k - 1 = l(Pind_{R_1}(e)) + r$, where $0 \leq r < Pind_{R_1}(e) - 1$. we have $e = e^k = e^{l(Pind_{R_1}(e))+r+1} = e^{r+1}$. Hence $r = 0$.

(2) Without loss of generality, we can suppose $0 \leq k \leq Pind_{R_1} - 1$. Assume $r = l(Pind_{R_1}(e)) + h$, where $0 \leq h < Pind_{R_1}(e) - 1$. Then $e^k = e^{r+k} = e^{l(Pind_{R_1}(e))+h} = e^{h+k}$. Hence $e = e^{Pind_{R_1}(e)} = e^{Pind_{R_1}(e)-k+k} = e^{Pind_{R_1}(e)-h}$, we get $h = 0$. Therefore $Pind_{R_1}(e) - 1|r$.

(3) follows from (2) and (4),(5), (6) by (1).

Remark 2.10 Let R be a ring and $e_1, e_2 \in Pot(R)$. If $e_1e_2 = e_2e_1$, then

$$(e_1e_2)^{[Pind_{R_1}(e_1)-1, Pind_R(e_2)-1]+1} = e_1e_2.$$

Hence $Pind_R(e_1e_2) - 1|[Pind_R(e_1) - 1, Pind_R(e_2) - 1]$ by Theorem 2.9(1). Generally,

$$Pind_R(e_1e_2) \neq [Pind_R(e_1) - 1, Pind_R(e_2) - 1] + 1.$$

Example 2.11 Let R be a ring and $Char(R) \neq 2$. Then $Pind_{R^2}((-1, 0)) = 3$, $Pind_{R^2}((0, -1)) = 3$, hence $[Pind_{R^2}((-1, 0)) - 1, Pind_{R^2}((0, -1)) - 1] + 1 = 3$, however, $Pind_{R^2}((-1, 0)(0, -1)) = Pind_{R^2}(0, 0) = 2$.

3 Pseudo Clean Rings and Examples

In this section we define the concept of pseudo clean rings and show that the class of pseudo clean rings is a proper subset the class of clean rings. We also provide some examples of pseudo clean rings.

Definition 3.1 Let R be a ring. An element $a \in R$ is called pseudo clean if there exist $e \in Pot(R)$ and $w \in J(R)$ such that $a = e + w$. A ring R is pseudo clean if every element of R is pseudo clean.

Proposition 3.2 Every pseudo clean ring is clean.

Proof Assume R is pseudo clean ring, for any $a \in R$, write $a = e+w$ where $e \in Pot(R)$, $w \in J(R)$. Note that e is strongly π-regular, there exist $f \in E(R)$ and $u \in U(R)$ such that $e = f + u$. Thus $a = f + (u + w)$ is the clean expression of a.

The following example shows that the converse of Proposition 3.2 is not true, that is, clean rings not be pseudo clean.

Example 3.3 Let $\mathbb{Z}_{(p)} = \{\frac{m}{n} | m, n \in \mathbb{Z}, p \nmid n\}$ where p is a prime. Then $\mathbb{Z}_{(p)}$ is a local ring with the Jacobson Radical $J(\mathbb{Z}_{(p)}) = p\mathbb{Z}_{(p)}$. As every local ring is clean, so is $\mathbb{Z}_{(p)}$. Clearly, $Pot(\mathbb{Z}_p) = \{-1, 0, 1\}$, hence $\mathbb{Z}_p$ is not pseudo clean when $p \geq 5$.

Theorem 3.4 Let $R_i (i = 1, 2, \cdots, n)$ be pseudo clean rings. Then $\prod_{i=1}^n R_i$ is pseudo clean.

Proof For any $(a_1, a_2, \cdots, a_n) \in \prod_{i=1}^n R_i$, write $a_i = e_i + w_i$ for some $e_i \in Pot(R_i)$ and $w_i \in J(R_i)$ since R_i is pseudo clean. Hence $(a_1, a_2, \cdots, a_n) = (e_1, e_2, \cdots, e_n) + (w_1, w_2, \cdots, w_n)$. Clearly, $(e_1, e_2, \cdots, e_n) \in Pot(\prod_{i=1}^n R_i)$ and $(w_1, w_2, \cdots, w_n) \in J(\prod_{i=1}^n R_i)$. Therefore, we complete the proof.

As the application of Theorem 3.4 , next, we prove $\mathbb{Z}_n$ is pseudo clean for every $2 \leq n \in \mathbb{Z}$.

Lemma 3.5 Let p be a prime number. Then $\mathbb{Z}_{p^l}$ is pseudo clean where $1 \leq l \in \mathbb{Z}$.

Proof Obviously, $J(\mathbb{Z}_{p^l}) = p\mathbb{Z}_{p^l}$. If $p = 2$, it is clear that $\mathbb{Z}_{2^l}$ is pseudo clean. If $p \geq 3$, then $\mathbb{Z}_{p^l} = J(\mathbb{Z}_{p^l}) \cup U(\mathbb{Z}_{p^l}) = J(\mathbb{Z}_{p^l}) \cup \mathbb{Z}^*_{p^l}$. It is well known that $\mathbb{Z}^*_n$ is a cyclic group. Thus $\mathbb{Z}_{p^l}$ is pseudo clean.

Theorem 3.6 Then $\mathbb{Z}_n$ is pseudo clean for all $2 \leq n \in \mathbb{Z}$.

Proof It is clear by Chinese remainder theorem, Theorem 3.4 and Lemma 3.5.

Recall that a ring R is a strongly J-clean ring[10] if for every $a \in R$, there exist $e \in E(R)$ and $w \in J(R)$ such that $a = e + w$ and $ew = we$. A ring R is said to be weakly J-clean[11] if for every $a \in R$, there exist $e \in E(R)$ and $w \in J(R)$ such that $a = w + e$ or $a = w - e$. Clearly, strongly J-clean rings are weakly J-clean and weakly J-clean rings are pseudo clean. Note that $\mathbb{Z}_n$ is strongly J-clean if and only if $n = 2^l (1 \leq l \in \mathbb{Z})$ and $\mathbb{Z}_n$ is weakly J-clean if and only if $n = 2^l 3^k (0 \leq l, k \in \mathbb{Z}, n \geq 2)$. Thus we conclude $\{\text{strolgly J}-\text{clean rings}\} \subsetneqq \{\text{weakly J}-\text{clean rings}\} \subsetneqq \{\text{pseudo clean rings}\} \subsetneqq \{\text{Clean rings}\}$.

Example 3.7 Here are some examples:

(1) The class of pseudo clean rings is closed under homomorphic images. This follows from the fact that $\varphi(Pot(R)) \subset Pot(\varphi(R))$ and $\varphi(J(R)) \subset J(\varphi(R))$ for all ring homomorphism φ of R. However, the inverse of this statement need not be true. For example, let p be a prime, then $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ is pseudo clean, but $\mathbb{Z}$ is not.

(2) $R[x]$ is certainly never a pseudo clean ring as it is not clean.

(3) $R[[x]]$ is pseudo clean if and only if so is R. The reason is that R is the homomorphic

image of $R[[x]]$ and $J(R[[x]]) = \{a_0 + a_1x + a_2x^2 + \cdots | a_0 \in J(R), a_1, a_2, \cdots \in R\}$.
(4) The upper(lower) triangular matrix ring over R is pseudo clean if and only if so is R. Indeed, it is trivial R is pseudo clean if the upper(lower) triangular matrix ring over R is pseudo clean. Conversely, let $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$ be an element of the upper triangular matrix ring over R where all $a_{ik} \in R$. Since R is pseudo clean, there exist $e_{ii} \in Pot(R)$ and $w_{ii} \in J(R)$ for any $a_{ii} (i = 1, 2, \cdots, n)$. We have the pseudo clean expression

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} e_{11} & 0 & \cdots & 0 \\ 0 & e_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e_{nn} \end{pmatrix} + \begin{pmatrix} w_{11} & a_{12} & \cdots & a_{1n} \\ 0 & w_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & w_{nn} \end{pmatrix}.$$

It can directly verify that the former is potent and the latter belongs to the Jacobson radical of the upper triangular matrix ring over R.
(5) $R[x]/(x^n)$ is pseudo clean if and only if so is R. It follows that there exists an isomorphism from $R[x]/(x^n)$ to a subring of the upper(lower) triangular matrix ring over R.
(6) R is pseudo clean if and only if so is the trivial extension $T(R, R)$.
(7) Let A, B be rings and M be a A-B-bimodule. Then the formal triangular matrix $T(A, B, M) = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is a ring with the usual matrix addition and multiplication. $T(A, B, M)$ is pseudo clean if and only if so is R.
(8) Let R be a ring and V be a R-R-bimodule (possibly with no unity) satisfied $(vw)r = v(wr), (vr)w = v(rw)$ and $(rv)w = r(vw)$ for every $v, w \in V$ and $r \in R$. The ideal extension of R by V is a ring, denoted by $I(R; V) = R \bigoplus V$ with the following addition and multiplication

$$(r_1, v_1) + (r_2, v_2) = (r_1 + r_2, v_1 + v_2),$$
$$(r_1, v_1)(r_2, v_2) = (r_1r_2, r_1v_2 + v_1r_2 + v_1v_2).$$

If R is pseudo clean and for any $v \in V$, there exists $w \in V$ such that $v + w + vw = 0$, then $I(R; V)$ is a pseudo clean ring. In fact, for all $(r, v) \in I(R; V)$, write $r = e + w$, where $e \in Pot(R), w \in J(R)$ since R is pseudo clean. Following [9], we have $(r, v) - (e, 0) = (w, v) \in J(I(R; V))$. Hence $I(R; V)$ is pseudo clean.

4 Some Properties of Pseudo Clean Rings

In this section, some properties of pseudo clean rings are given.

We use $\overline{x}$ to stand for $x + J(R)$, for $x \in R$. We call an element $\overline{x} \in R/J(R)$ can be lifted modulo $J(R)$ in case there exists a potent element $e \in R$ such $x - e \in J(R)$.

Proposition 4.1 Let R be a ring. Then R is pseudo clean if and only if $R/J(R) = Pot(R/J(R))$ and each potent element can be lifted modulo $J(R)$.

Proof $\Rightarrow$ Clearly $R/J(R) = Pot(R/J(R))$. Let $\overline{x} \in R/J(R)$. By hypothesis, there exists $e \in Pot(R)$ such that $x - e \in J(R)$. Thus $\overline{x}$ can be lifted modulo $J(R)$.

$\Leftarrow$ For every $x \in R$, we have $\overline{x}$ is potent. By assumption, there exist $e \in Pot(R)$ such that $x - e \in J(R)$. Set $w = x - e \in J(R)$. Then $x = e + w$, therefore R is pseudo clean.

Proposition 4.2 Let R be a pseudo clean ring. Then $U(R) = \sqrt{1}^* + J(R)$.

Proof Clearly, $\sqrt{1}^* + J(R) \subseteq U(R)$. Suppose $x \in U(R)$, there exist $e \in Pot(R)$ and $w \in J(R)$ such that $x = e + w$ as R is pseudo clean. Hence $e = x - w \in Pot(R) \cap U(R) = \sqrt{1}^*$ by Proposition 2.1. Thus $U(R) \subseteq \sqrt{1}^* + J(R)$.

Corollary 4.3 Let R be a local ring. Then R is pseudo clean if and only if $U(R) = \sqrt{1}^* + J(R)$.

Lemma 4.4 Let R be a ring and $a \in R$. The following statements are equivalent for $n \geq 1$:
(1) $a = a(ua)^n$ for some $u \in U(R)$.
(2) $a = ve$ for some $e^{n+1} = e$ and some $v \in U(R)$.
(3) $a = fw$ for some $f^{n+1} = f$ and some $w \in U(R)$.

Proof See Lemma 4.3 of [12].

Proposition 4.5 Let R be a pseudo clean ring and $a \in R$. Then either (i) $a \in \sqrt{1}^* + J(R)$ or (ii) both $(a+1)R$ and $R(a+1)$ contain nonzero idempotents.

Proof Since R is pseudo clean, there exist $e \in Pot(R)$ and $w \in J(R)$ such that $x = e + w$. Suppose $Pind_R(e) = k$, we have $(a+1)e^{k-1} = e + (w+1)e^{k-1}$. So $(a+1)(1-e^{k-1}) = (w+1)(1-e^{k-1})$. Note that $1 - e^{k-1} \in E(R)$, we have $(w+1)(1-e^{k-1}) = fu$ for some $f^2 = f$ and some $u \in U(R)$ by Lemma 4.4. So $f = (a+1)(1-e^{k-1})u^{-1} \in (a+1)R$. Suppose (i) does not hold. Then $a - e^{k-1} \neq 0$, hence $f \neq 0$. Thus, $(a+1)R$ contains a nonzero idempotent. Similarly, $R(a+1)$ contains a nonzero idempotent.

Corollary 4.6 Let R be a pseudo clean ring and $a \in R$. Then either (i) both aR and Ra contain nonzero idempotents or (ii) both $(a+1)R$ and $R(a+1)$ contain nonzero idempotents.

Let R be a ring and $a \in R$. Let $ann_l = \{r \in R | ra = 0\}$ and $ann_r = \{r \in R | ar = 0\}$.

Proposition 4.7 Let R be a ring and $a \in R$. If $a = e + w$ and $ew = we$ where $e \in Pot(R)$ and $w \in J(R)$, then $ann_r(a) \in R(1-e)$ and $ann_l(a) \in (1-e)R$.

Proof Given $x \in ann_r(a)$, we have $x(e+w) = 0$. Hence $x(1-e) = x(w+1)$. $ew = we$ implies $(1-e)(w+1) = (w+1)(1-e)$. So $x = x(1-e)(w+1)^{-1} = x(w+1)^{-1}(1-e) \in R(1-e)$. Similarly, we can get $ann_l(a) \in (1-e)R$ easily.

Proposition 4.8 Let R be a pseudo clean ring. Then $N(R) \subseteq J(R)$.

Proof As R is pseudo clean, for any $x \in N(R)$, write $x = e + w$ for some $e \in E(R)$, $w \in J(R)$. Assume $x^n = 0$ for some n. Then there exist $l \geq n$ such that $Pind_R(e) - 1 | l - 1$. Note that $e = e^l = e^l - x^l \in Pot(R) \cap J(R)$, we have $e = 0$. Hence $x = w \in J(R)$ and $N(R) \subseteq J(R)$, as desired.

Remark 4.9 Let R be a ring. Then $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in N(R)$ but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin J(R)$. Hence $M_2(R)$ is not pseudo clean even if R is pseudo clean. However, the matrix ring over clean rings is clean[13].

Corollary 4.10 Let R be a pseudo clean ring. Then $M_n(R)$ is pseudo clean if and only if $n = 1$.

Proposition 4.11 Let R be a pseudo clean ring and $Pind(R) < \infty$. Then $a^{Pind(R)} - a \in J(R)$ for all $a \in R$.

Proof Take $Pind(R)$ denote by k, we write $a = e + j$ for any $a \in R$ where $e \in Pot(R)$ and $j \in J(R)$. Hence

$$a^k = (e + j)^k = e^k + w = e + w$$

where $w \in J(R)$. Thus, $a^k - a = w - j \in J(R)$ and we were done.

Next, we stated the main result in this paper.

Theorem 4.12 Let R be a pseudo clean ring. Then

(1) R is weakly abel.

(2) R is directly finite.

(3) R has stable range one.

Proof Clearly, $R/J(R)$ is strongly π-regular. This implies that $R/J(R)$ is abelian and has stable range one[14, Theorem 4]. Thus we have the above results.

Remark 4.13 In fact, if R be a pseudo clean ring, then $R/J(R)$ is a potent ring (i.e. its every element is potent). Thus the Jacosbon's Theorem[15] implies $R/J(R)$ is a commutative ring. More results about Jacosbon's Theorem see[16, 17].

If e is an idempotent of a ring R, then eRe is a subring of R with identity e. eRe is called the corner ring of R. It is proved that if eRe and $(1-e)R(1-e)$ are clean, then so is R [13]. For pseudo clean rings, we have the following result.

Theorem 4.14 Let R be a ring and e be an idempotent of R such that

$$eR(1-e) + (1-e)Re \in J(R).$$

If eRe and $(1-e)R(1-e)$ are pseudo clean, then so is R.

Proof Set $f = 1 - e$. For any $x \in R$, write $x = a + b + c + d$ where $a \in eRe$, $b \in eRf$, $c \in fRe$ and $d \in fRf$. Since eRe and $(1-e)R(1-e)$ are pseudo clean, then there exist $e_1 \in Pot(eRe)$, $w_1 \in J(eRe)$, $e_2 \in Pot(fRf)$, and $w_2 \in J(fRf)$ such that $a = e_1 + w_1$, $d = e_2 + w_2$. Hence $x = (e_1 + e_2) + (w_1 + b + c + w_2)$. Note that $e_1e_2 = e_2e_1 = 0$ and $eR(1-e) + (1-e)Re \in J(R)$, we have $e_1 + e_2 \in Pot(R)$ and $w_1 + b + c + w_2 \in J(R)$. Therefore R is a pseudo clean ring.

Remark 4.15 Note that both $eR(1-e)$ and $(1-e)Re$ are nilpotent, we have that the the condition $eR(1-e) + (1-e)Re \in J(R)$ is not superfluous by proposition 4.8.

Corollary 4.16 Let R be a ring and e be an central idempotent of R. Then eRe and $(1-e)R(1-e)$ are pseudo clean if and only if so is R.

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Potent元及伪clean环

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摘要: 本文研究了元素的potent指数和伪clean环,给出了一些伪clean 环的性质的例子,证明了$\mathbb{Z}_m$($2 \leq m \in \mathbb{Z}$)是伪clean环以及伪clean环是clean 环.此外，本文也证明了伪clean环是直有限的且有稳定秩一.

关键词: clean环; 强J-clean环; 伪clean环; potent元

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