

THE SPECTRAL RADIUS OF UNIFORM HYPERGRAPH DETERMINED BY THE SIGNLESS LAPLACIAN MATRIX

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Abstract: This paper studies the problem of the spectral radius of the uniform hypergraph determined by the signless Laplacian matrix. The upper bound of the spectral radius of a uniform hypergraph is obtained by using Rayleigh principle and the perturbation of the spectral radius under moving the edge operation, and the extremal hypergraphs are characterized for both supertree and unicyclic hypergraphs. The spectral radius of the graph is generalized.

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1 Introduction

The goal of spectral graph theory is to study the relation between combinatorial properties of graphs and the eigenvalues of matrices associated to the graph, such as the adjacency matrix, the Laplacian matrix and the signless Laplacian matrix. In 2008, Lim ([1]) proposed to study the spectral hypergraph theory via eigenvalues of tensors. In 2012, Cooper and Dutle ([2]) defined the eigenvalues of the adjacency tensor for studying the spectra of hypergraphs, and proved a number of basic results in spectral graph theory. Recently, a lot of results have been developed on the spectral hypergraph theory via tensors ([3-7]). However, it is hard to obtain eigenvalues of tensors because most tensor problems are NP-hard ([8]). Therefore, it remains important to study the spectral hypergraph theory through eigenvalues of matrices.

In ([9]), Kauê introduced the definition of the signless Laplacian matrix of hypergraph, which is another approach to deal with spectral properties of hypergraphs. Let H be a hypergraph whose incidence matrix is B . The signless Laplacian matrix of H is defined as $Q(H) = BB^T$. The signless Laplacian spectral radius of H , denoted by $\rho(Q(H))$ or $\rho(H)$, is

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the largest eigenvalue of the signless Laplacian matrix $Q(H)$. In this paper, we shall study the signless Laplacian spectral radius with respect to connected k -uniform hypergraphs.

A k -uniform hypergraph $H = (V, E)$ consists of a vertex set $V = V(H)$ and a hyperedge (or edge) set $E = E(H)$, where each edge $e \in E(H)$ is a subset of $V(H)$ containing exactly k vertices. In particular, a 2-uniform hypergraph is a graph. The incidence matrix of H is defined to be a matrix $B = (b_{ij})$ whose rows and columns are indexed by the vertices and edges of H , respectively. The (i, j) -entry of B is 1 if $v_i \in e_j$ and 0 otherwise.

Let $E_v = E_v(H)$ be the set of edges of H containing v . For a vertex $v \in V(H)$ the degree $d_{\mathcal{H}}(v)$ (or $d(v)$) is defined as $d_H(v) = |E_v(H)|$. Vertex with degree one is called pendent vertex, and an edge e is called a pendent edge if e contains exactly $k - 1$ pendent vertices. We denote by $N(v)$ the neighborhood of a vertex $v \in V(H)$. A walk of length l in H is a sequence of vertices and edges: $v_0 e_1 v_1 e_2 \dots e_l v_l$ satisfying that $\{v_i, v_{i+1}\} \subseteq e_i$ for $i = 0, 1, \dots, l - 1$. A walk is called a path if no vertices or edges are repeated. We say a walk is closed if $v_0 = v_l$. A closed walk is called a cycle if all vertices and edges in the walk are distinct. For any $u, v \in V(H)$, if there is a path from u to v , then we say that H is connected. A connected k -uniform hypergraph with n vertices and m edges is r -cyclic if $n - 1 = (k - 1)m - r$. For $r = 0, 1$ or 2 , it is called a supertree, a unicyclic hypergraph or a bicyclic hypergraph. We say a hypergraph is linear if any two edges share at most one common vertex. For other undefined notations and terminology from graph theory, the readers can refer to ([10]).

2 Preliminaries and Some Lemmas

Let $H = (V, E)$ be a connected hypergraph. The clique graph \mathcal{C} of H is a multigraph whose vertex set $V(\mathcal{C}) = V(H)$, and the number of edges between two vertices of \mathcal{C} is equal to the number of hyperedges containing them in H ([11]). If H is a graph, then the clique graph of H is itself. The line graph \mathcal{L} of H is obtained by transforming the hyperedges of H into the vertices of \mathcal{L} , and the number of edges between two vertices of \mathcal{L} is equal to the number of common vertices in the two corresponding hyperedge.

Example 1. The clique graph \mathcal{C} and line graph \mathcal{L} of H are illustrate in Figure 1, where $V(H) = \{1, 2, 3, 4, 5\}$, $E(H) = \{(1, 2, 3), (1, 4, 5), (3, 4, 5)\}$.

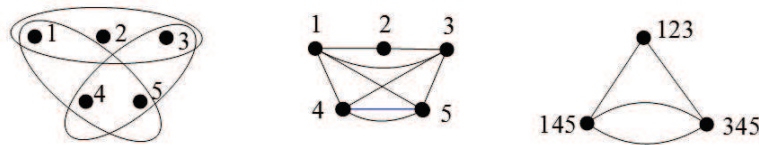


Figure 1 Hypergraph H , clique graph \mathcal{C} and line graph \mathcal{L}

Lemma 2.1 Let H be a k -uniform hypergraph, B its incidence matrix, D its degree matrix, A_C and A_L the adjacency matrices of its clique graph and line graph, respectively. Then $BB^T = D + A_C$, $B^T B = kI + A_L$.

It is easy to see that $Q = BB^T = D + A$ if H is a simple graph, where A is the adjacency matrix of H . That is, the definition of Q for hypergraphs coincides with the signless Laplacian matrix introduced by graph theory. For $k \geq 2$, let H be a k -uniform hypergraph with $V(H) = [n]$, where $[n] = \{1, 2, \dots, n\}$. A column vector $x = (x_1, \dots, x_n)^T$ can be considered as a function defined on $V(H)$ which maps vertex i to x_i , i.e., $x(i) = x_i$ for $i = 1, \dots, n$. For a subset $U \in [n]$, we write $x(U) = x_U = \sum_{i \in U} x_i$ and $d_i = d(i)$. By Lemma 2.1, using the product of matrices, we have

$$(Qx)_i = d_i x_i + \sum_{j \in N(i)} x_j = \sum_{e \in E_i} x(e). \quad (2.1)$$

Let H be a k -uniform hypergraph with $V(H) = [n]$. For $x \in R^n$, by Lemma 2.1 we get

$$x^T Qx = \sum_{i \in V} d_i x_i^2 + 2 \sum_{e \in E} \sum_{\{i, j\} \subset e} x_i x_j = \sum_{e \in E} [x(e)]^2. \quad (2.2)$$

Let H be a k -uniform hypergraph and $Q = (q_{ij})$ be a matrix with row sums $r_i (i = 1, \dots, n)$, according to Lemma 2.1, we have $r_i = \sum_{j=1}^n q_{ij} = kd_i$. By matrix theory, we get

$$k\delta \leq \rho(H) \leq k\Delta. \quad (2.3)$$

Where δ and Δ are the minimum degree and maximum degree of H , respectively. It is clear that the signless Laplacian matrix Q is symmetric, non-negative and semi-definite positive. Furthermore, if H is connected, then Q is irreducible. These properties allow us to apply the Rayleigh principle and Perron-Frobenius theorem for the matrix Q .

Theorem 2.2 (Rayleigh principle ([12])) Let H be a k -uniform hypergraph, and $R_+^n = \{x \in R^n | x \geq 0\}$. If λ_1 is the largest eigenvalue of Q , then

$$\lambda_1 = \rho(H) = \max\{x^T Qx | x \in R_+^n, \|x\| = 1\}.$$

Further, $x \in R_+^n$ with $\|x\| = 1$ is an eigenvector of Q corresponding to $\rho(H)$ if and only if it is an optimal solution of the maximization problem.

By Perron-Frobenius theorem, for irreducible nonnegative matrix Q , it has a unique positive eigenvector x with $\|x\| = 1$ corresponding to $\rho(H)$, and we call x the principal eigenvector of Q . Let x be the principal eigenvector of Q . By Eq.2.1, the eigenequation of Q for $\rho(H)$ can be written as follows:

$$\rho(H)x_i = d_i x_i + \sum_{j \in N(i)} x_j = \sum_{e \in E_i} x(e). \quad (2.4)$$

Lemma 2.3 Suppose that A and B are $m \times n$ and $n \times m$ matrices, respectively. Then AB and BA have the same non-zero eigenvalues.

Proof According to the matrix operations, we can get the following equality

$$\begin{pmatrix} I_m & -A \\ 0 & I_n \end{pmatrix} \begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix} \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}.$$

It is clear that

$$\begin{pmatrix} I_m & -A \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & I_n \end{pmatrix}.$$

It implies that the matrix $\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix}$ is similar to $\begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$. Thus they have the same characteristic polynomial, that is $\lambda^n \det(\lambda I_m - AB) = \lambda^m \det(\lambda I_n - BA)$. Obviously, λ is a non-zero eigenvalue of AB if and only if it is one of BA .

According to Lemma 2.3, BB^T and $B^T B$ have the same non-zero eigenvalues. Therefore the following theorem is given.

Theorem 2.4 Let H be a k -uniform hypergraph with n vertices and m edges, and $L(H)$ be the line graph of H , then

$$\rho(H) = \rho(L(H)) + k.$$

Applying this theorem, we can obtain $\rho(H)$ from the adjacency spectral radius of $L(H)$. Now we state another method to prove the result.

Proof For the k -uniform hypergraph, let vertex set $V(H) = [n]$ and edge set $E(H) = e_1, e_2, \dots, e_m$. Let $L(H)$ be the line graph of H with vertex set $V(L(H)) = e_1, e_2, \dots, e_m$ and edge set $E(L(H))$ as defined above. We denote by d_i and d_{e_i} the degree of a vertex i in H and the degree of a vertex e_i in $L(H)$, respectively. Suppose the positive vector x is the principal eigenvector corresponding to the spectral radius $\rho(H)$. Let $e_t = \{t_1, t_2, \dots, t_k\}$ be an edge of H . For each vertex t_i in e_t , combining with (2.1), we have

$$\rho(H)x_{t_i} = \sum_{e \in E_{t_i}} x(e).$$

Summing over all vertices in e_t , we obtain that

$$\begin{aligned} \sum_{i=1}^k \rho(H)x_{t_i} &= \sum_{i=1}^k \sum_{e \in E_{t_i}} x(e) = kx(e_t) + \sum_{j \neq t} |e_j \cap e_t| x(e_j). \\ \rho(H)x(e_t) &= kx(e_t) + \sum_{j \neq t} |e_j \cap e_t| x(e_j). \\ \rho(H) &= k + \sum_{j \neq t} |e_j \cap e_t| \frac{x(e_j)}{x(e_t)}. \end{aligned} \tag{2.5}$$

Where $|e_j \cap e_t|$ is the number of common vertices of the edges e_j and e_t in H . Notice that the line graph $L(H)$ may be a multiple graph, and the number of edge $e_j e_t$ in $L(H)$ is exactly $|e_j \cap e_t|$. Define two positive vectors $a = (a_1, a_2, \dots, a_m)^T$ and $y = (y_1, y_2, \dots, y_m)^T$, where $a_i = x(e_i)$ and $y_i = \frac{a_i}{\|a\|}$. It is easy to verify that $\|y\| = 1$. Then we have

$$\begin{aligned} \rho(L(H))y_t &= \sum_{e_j e_t \in E(L(H))} y_j = \sum_{j \neq t} |e_j \cap e_t| y_j, \\ \rho(L(H)) &= \sum_{j \neq t} |e_j \cap e_t| \frac{y_j}{y_t} = \sum_{j \neq t} |e_j \cap e_t| \frac{a_j}{a_t} = \sum_{j \neq t} |e_j \cap e_t| \frac{x(e_j)}{x(e_t)}. \end{aligned} \quad (2.6)$$

Combining (2.5) with (2.6), we have $\rho(H) = \rho(L(H)) + k$.

Lemma 2.5 ([9]) Let H be a connected uniform hypergraph. If H_0 is a subgraph of H , then

$$\rho(H_0) \leq \rho(H).$$

The following inequality is easy to verify, which will be used in the following proof.

$$f(x) = ax^2 + b(c-x)^2 \geq \frac{abc^2}{a+b}. \quad (2.7)$$

For a connected irregular graph G with n vertices, maximum degree Δ and diameter d , Ning et al. ([13]) showed that $\rho(G) < 2\Delta - \frac{1}{(d-\frac{1}{4})n}$, and Guo and Zhou ([14]) showed that $\rho_\alpha(G) < \Delta - \frac{2(1-\alpha)}{(2d-\alpha)n}$. Where $\rho(G)$ and $\rho_\alpha(G)$ are the signless Laplacian spectral radius and α -spectral radius of G , respectively. Motivated by ([13,14]), we give an upper bound for the signless Laplacian spectral radius of irregular hypergraphs.

Theorem 2.6 Let H be a connected irregular k -uniform hypergraph with n vertices, m edges, maximum degree Δ and diameter d , we have $\rho(H) < k\Delta - \frac{k\Delta(n\Delta - km)}{2dmk(n\Delta - km) + n\Delta}$.

Proof Let x be the principal eigenvector of $Q(H)$, and u, v be the vertices of H such that $x_u = \max\{x_i : 1 \leq i \leq n\}$ and $x_v = \min\{x_i : 1 \leq i \leq n\}$. Because $k\Delta \geq \rho(H)$, we consider the difference $k\Delta - \rho(H)$. By Eq. 2.2, we have

$$\begin{aligned} k\Delta - \rho(H) &= k \sum_{i \in V} \Delta x_i^2 - x^T Q x \\ &= k \sum_{i \in V} \Delta x_i^2 - \sum_{i \in V} d_i x_i^2 - 2 \sum_{e \in E} \sum_{\{i,j\} \subset e} x_i x_j \\ &= k \sum_{i \in V} (\Delta - d_i) x_i^2 + (k-1) \sum_{i \in V} d_i x_i^2 - 2 \sum_{e \in E} \sum_{\{i,j\} \subset e} x_i x_j \\ &= k \sum_{i \in V} (\Delta - d_i) x_i^2 + \sum_{e \in E} \sum_{\{i,j\} \subset e} (x_i - x_j)^2. \end{aligned}$$

Note that $\sum_{i \in V} d_i = km$, then we have

$$k\Delta - \rho(H) \geq k(n\Delta - km)x_v^2 + \sum_{e \in E} \sum_{\{i,j\} \subset e} (x_i - x_j)^2. \quad (2.8)$$

Let $P = (v_0, e_1, v_1, e_2, \dots, v_{r-1}, e_r, v_r)$ be a shortest path between u and v with $v_0 = u$ and $v_r = v$. We have

$$\begin{aligned}
 \sum_{e \in E} \sum_{\{i,j\} \subset e} (x_i - x_j)^2 &\geq \sum_{e \in E(P)} \sum_{\{i,j\} \subset e} (x_i - x_j)^2 \\
 &\geq \sum_{i=1}^r (x_{v_{i-1}} - x_{v_i})^2 + \sum_{i=1}^r \sum_{t \in e_i \setminus \{v_{i-1}, v_i\}} ((x_{v_{i-1}} - x_t)^2 + (x_t - x_{v_i})^2) \\
 &\geq \sum_{i=1}^r (x_{v_{i-1}} - x_{v_i})^2 + \sum_{i=1}^r \sum_{t \in e_i \setminus \{v_{i-1}, v_i\}} \frac{(x_{v_{i-1}} - x_{v_i})^2}{2} \\
 &= \sum_{i=1}^r (x_{v_{i-1}} - x_{v_i})^2 + \frac{k-2}{2} \sum_{i=1}^r (x_{v_{i-1}} - x_{v_i})^2 \\
 &= \frac{k}{2} \sum_{i=1}^r (x_{v_{i-1}} - x_{v_i})^2 \geq \frac{k}{2r} (x_{v_0} - x_{v_r})^2 \geq \frac{k}{2d} (x_u - x_v)^2. \tag{2.9}
 \end{aligned}$$

In the above inequalities, the first and the second ones are obvious, the third and the sixth ones follow from the Cauchy–Schwarz inequality, and $r \leq d$ implies the final one. By combining (2.8) and (2.9), and using (2.7), we get

$$k\Delta - \rho(H) \geq k(n\Delta - km)x_v^2 + \frac{k}{2d}(x_u - x_v)^2 \geq \frac{k(n\Delta - km)\frac{k}{2d}x_u^2}{k(n\Delta - km) + \frac{k}{2d}}. \tag{2.10}$$

On the other hand, by the Cauchy–Schwarz inequality, we have

$$\rho(H) = \sum_{e \in E} [x(e)]^2 \leq k \sum_{i \in V} d_i x_i^2 \leq k^2 m x_u^2.$$

Therefore, we get $x_u^2 \geq \frac{\rho(H)}{mk^2}$. If $x_u^2 = \frac{\rho(H)}{mk^2}$, it means $x_u = x_i$ for $i \in V$. In this case, we can conclude that $x = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})$. It implies H is a regular hypergraph. This is a contradiction. Thus,

$$x_u^2 > \frac{\rho(H)}{mk^2}. \tag{2.11}$$

According to (2.10) and (2.11), we obtain that $\rho(H) < k\Delta - \frac{k\Delta(n\Delta - km)}{2dmk(n\Delta - km) + n\Delta}$.

Theorem 2.7 Let H be a k -uniform linear hypergraph with n vertices and m edges and $L(H)$ be the line graph of H , then

$$\rho(H) \leq 2\sqrt{\Delta_L - 1} + k.$$

where Δ_L is the maximal degree of $L(H)$. And the equality is achieved if and only if H is an unicyclic hypergraph without pendent edges.

Proof If H is a k -uniform linear hypergraph with m edges, then the line graph $L(H)$ is a tree or unicyclic graph with m vertices. It should be noted that if H is a hypergraph rather than a linear hypergraph, then the graph $L(H)$ might be a bicyclic graph. If $L(H)$ is a tree, then it is a spanning subgraph of a unicyclic graph G , and we have $\rho(L(H)) \leq \rho(G)$. Thus we only consider that $L(H)$ is a unicyclic graph. Let y be the principal eigenvector corresponding to $\rho(L(H))$. Assume that C is the unique cycle of $L(H)$ and k is its length. We label the vertices of $L(H)$ such that $V(L(H)) = \{v_1, \dots, v_m\}$ and $V(C) = \{v_1, \dots, v_k\}$. For $t \in V(L(H))$, denote by $d(t, C)$ the minimum distance between t and vertices of C . We give an orientation to the graph $L(H)$ in which the edges of C are arcs $(v_1, v_2), \dots, (v_{k-1}, v_k), (v_k, v_1)$ and an edge uv outside C as (u, v) if $d(u, C) > d(v, C)$. Thus, for any $i = 1, \dots, m$, there is a unique arc from v_i to some other vertex v_j , where v_i and v_j are called the tail and head of the arc, respectively. For the oriented graph, the number of v_j as head of arcs appearing in the arc set is equal to $d_{L(H)}(v_j) - 1$. From Cauchy-Schwarz inequality and $d_{L(H)}(v_j) \leq \Delta_L$, we have

$$\begin{aligned} \rho(L(H)) &= y^T A_L y = 2 \sum_{v_i v_j \in E(L(H))} x_i x_j = 2 \sum_{i=1}^n x_i x_j \\ &\leq 2 \sqrt{\sum_{i=1}^n x_i^2 \sum_{j=1}^n (d_{L(H)}(v_j) - 1) x_j^2} \\ &= 2 \sqrt{\sum_{j=1}^n (d_{L(H)}(v_j) - 1) x_j^2} \\ &\leq 2 \sqrt{\sum_{j=1}^n (\Delta_L - 1) x_j^2} = 2\sqrt{\Delta_L - 1}. \end{aligned}$$

By Theorem 2.4, we have $\rho(H) = \rho(L(H)) + k \leq 2\sqrt{\Delta_L - 1} + k$. If $\rho(L(H)) = 2\sqrt{\Delta_L - 1}$, it implies that $L(H)$ is regular, and $L(H)$ is a cycle. Thus, H is an unicyclic hypergraph without pendent edges, and the reverse is also true.

3 The Operations of Moving Edges

Let $H = (V, E)$ be a k -uniform hypergraph with $u \in V(H)$ and $e_1, \dots, e_r \in E(H)$ such that $u \notin e_i$ for $1 \leq i \leq r$. Suppose that $v_i \in e_i$ and write $e'_i = (e_i \setminus \{v_i\}) \cup \{u\}$ for $i = 1, \dots, r$. Let $H' = (V, E')$ be the hypergraph with $E' = (E \setminus \{e_1, \dots, e_r\}) \cup \{e'_1, \dots, e'_r\}$. Then we say that H' is obtained from H by moving edges (e_1, \dots, e_r) from (v_1, \dots, v_r) to u .

Theorem 3.8 Let $r \geq 1$, H be a connected k -uniform hypergraph, H' be the hypergraph obtained from H by moving edges (e_1, \dots, e_r) from (v_1, \dots, v_r) to u . Let x be the principal eigenvector of Q corresponding to $\rho(H)$, and suppose that $x_u \geq \max\{x_{v_1}, \dots, x_{v_r}\}$, then $\rho(H') > \rho(H)$.

Proof Let Q' be the signless Laplacian matrix of H' . By Theorem 2.2, we have $\rho(H) = x^T Q x$ and $\rho(H') \geq x^T Q' x$. By the hypothesis, one can see that $x(e'_i) - x(e_i) = x_{u_i} - x_{v_i} \geq 0$ for $i = 1, \dots, r$. Thus

$$\begin{aligned} \rho(H') - \rho(H) &\geq x^T Q' x - x^T Q x \\ &= \sum_{e \in E'} (x(e))^2 - \sum_{e \in E} (x(e))^2 \\ &= \sum_{i=1}^r ((x(e'_i))^2 - (x(e_i))^2) \\ &= \sum_{i=1}^r (x(e'_i) + x(e_i))(x(e'_i) - x(e_i)) \geq 0. \end{aligned}$$

Suppose that $\rho(H') = \rho(H)$. Applying eigenvalue equations and equation (2.4), we have

$$0 = \rho(H')x_u - \rho(H)x_u = \sum_{e \in E'_u} x(e) - \sum_{e \in E_u} x(e) = \sum_{i=1}^r x(e'_i) > 0.$$

It is a contradiction. It follows that $\rho(H') > \rho(H)$.

Theorem 3.9 Let H be a connected k -uniform hypergraph with $e, f \in E$ and $e \cap f = \emptyset$. Let $U_1 \subset e$, $U_2 = e \setminus U_1$, $V_1 \subset f$, $V_2 = f \setminus V_1$, and $1 \leq |U_1| = |V_1| \leq k - 1$. Let $e' = U_1 \cup V_2$ and $f' = U_2 \cup V_1$. Suppose that x is the principal eigenvector of $Q(H)$. Let $H' = H - \{e, f\} + \{e', f'\}$, and $e', f' \notin E(H)$. If $x_{U_1} \geq x_{V_1}$, $x_{U_2} \leq x_{V_2}$ and one is strict, then $\rho(H') > \rho(H)$.

Proof By Theorem 2.2, we have

$$\begin{aligned} \rho(H') - \rho(H) &\geq x^T Q(H')x - x^T Q(H)x = \sum_{e \in E(H')} (x(e))^2 - \sum_{e \in E(H)} (x(e))^2 \\ &= (x(e'))^2 + (x(f'))^2 - (x(e))^2 - (x(f))^2 \\ &= (x_{U_1} + x_{V_2})^2 - (x_{U_1} + x_{U_2})^2 + (x_{U_2} + x_{V_1})^2 - (x_{V_1} + x_{V_2})^2 \\ &= 2(x_{V_2} - x_{U_2})(x_{U_1} - x_{V_1}) \geq 0. \end{aligned}$$

Suppose that $\rho(H') = \rho(H)$. Without loss of generality, we assume $x_{U_2} < x_{V_2}$. For $u \in V_1$, applying eigenvalue equations, we have

$$\begin{aligned} 0 &= \rho(H')x_u - \rho(H)x_u = \sum_{e \in E'_u} x(e) - \sum_{e \in E_u} x(e) \\ &= x(f') - x(f) = x_{V_1} + x_{U_2} - x_{V_1} - x_{V_2} < 0. \end{aligned}$$

It is a contradiction. It follows that $\rho(H') > \rho(H)$.

Corollary 3.10 Suppose that H is a connected k -uniform hypergraph, and $e, f \in E(H)$ such that $\{u_1, u_2\} \subset e$ and $\{v_1, v_2\} \subset f$. Let x be the principal eigenvector of $Q(H)$,

and $x_{u_1} > x_{v_1}, x_{u_2} \leq x_{v_2}$. Then there exists k -subsets e', f' of $V(H)$ with $\{u_1, v_2\} \subset e', \{u_2, v_1\} \subset f'$ such that $\rho(H') > \rho(H)$, where $H' = H - \{e, f\} + \{e', f'\}$.

Proof If $x_{e \setminus \{u_1, u_2\}} \geq x_{f \setminus \{v_1, v_2\}}$, then $x_{e \setminus \{u_2\}} > x_{f \setminus \{v_2\}}$. Let $e' = (e \setminus \{u_2\}) \cup \{v_2\}$, $f' = (f \setminus \{v_2\}) \cup \{u_2\}$. By theorem 3.9, we get $\rho(H') > \rho(H)$.

If $x_{e \setminus \{u_1, u_2\}} \leq x_{f \setminus \{v_1, v_2\}}$, then $x_{f \setminus \{v_1\}} \geq x_{e \setminus \{u_1\}}$. Let $e' = (f \setminus \{v_1\}) \cup \{u_1\}$, $f' = (e \setminus \{u_1\}) \cup \{v_1\}$. By theorem 3.9, we have $\rho(H') > \rho(H)$.

Let P be a path from u to v in a hypergraph H . If $d_H(u), d_H(v) \geq 3$ and $d_H(w) = d_P(w)$ for any $w \in V(P) \setminus \{u, v\}$, then P is called an internal path of H . If $d_H(u) \geq 3$ and $d_H(w) = d_P(w)$ for any $w \in V(P) \setminus \{u\}$, then P is called a pendant path of H . We denote by P_s the path with length s . Given a connected k -uniform hypergraph H with two vertices (not necessarily distinct) $u, v \in V$. Let $H_{u,v}(p, q)$ be the hypergraph obtained from H by attaching the pendant paths P_p to u and P_q to v .

Theorem 3.11 Let u, v be two non-pendant vertices of the connected k -uniform hypergraph H . Suppose that P_s is an internal path between u and v with length s . If $p \geq q \geq 1$ and $p - q + 1 \geq s \geq 0$, then $\rho(H_{u,v}(p + 1, q - 1)) < \rho(H_{u,v}(p, q))$.

Proof Assume that $(u, e_1, u_1, \dots, u_p, e_{p+1}, u_{p+1})$ and $(v, f_1, v_1, \dots, v_{q-2}, f_{q-1}, v_{q-1})$ are two pendant paths of $H_{u,v}(p + 1, q - 1)$ at u and v , respectively. Let $u_0 = u, v_0 = v$, and x be the principal eigenvector of $Q(H_{u,v}(p + 1, q - 1))$.

Suppose that $\rho(H_{u,v}(p + 1, q - 1)) \geq \rho(H_{u,v}(p, q))$, we can get the following claim.

Claim 1. $x_{u_{p-i}} > x_{v_{q-i-1}}$ for $i = 0, \dots, q - 1$.

We use the mathematical induction to prove the claim. For $i = 0$, suppose that $x_{u_p} \leq x_{v_{q-1}}$. Let H' be the hypergraph obtained from $H_{u,v}(p + 1, q - 1)$ by moving e_{p+1} from u_p to v_{q-1} . Obviously, $H' \cong H_{u,v}(p, q)$. By Theorem 3.8, we get $\rho(H_{u,v}(p, q)) > \rho(H_{u,v}(p + 1, q - 1))$, it is a contradiction. Therefore $x_{u_p} > x_{v_{q-1}}$.

Now assume that $x_{u_{p-i}} > x_{v_{q-i-1}}$ for $i = 1, \dots, q - 2$, and let us show that $x_{u_{p-(i+1)}} > x_{v_{q-(i+1)-1}}$. Suppose that $x_{u_{p-(i+1)}} \leq x_{v_{q-(i+1)-1}}$. Then, there exist e', f' such that $\{u_{p-i}, v_{q-i-2}\} \subset e', \{u_{p-i-1}, v_{q-i-1}\} \subset f'$. Obviously, $H' = H_{u,v}(p + 1, q - 1) - \{e_{p-i}, f_{q-i-1}\} + \{e', f'\} \cong H_{u,v}(p, q)$. According to Corollary 3.10, we have $\rho(H_{u,v}(p + 1, q - 1)) < \rho(H_{u,v}(p, q))$, which is a contradiction. Thus $x_{u_{p-(i+1)}} > x_{v_{q-(i+1)-1}}$. The proof of claim 1 is complete.

Let $P_s = (w_0, m_1, w_1, \dots, w_{s-1}, m_s, w_s)$ be the internal path from v to u , where $w_0 = v$ and $w_s = u$. Suppose that G is the hypergraph obtained from $H_{u,v}(p + 1, q - 1)$ by moving $E_v \setminus \{m_1, f_1\}$ from v to u_{p-q+1} . Because $x_{u_{p-q+1}} > x_v$, it implies $\rho(G) > \rho(H_{u,v}(p + 1, q - 1))$.

Case 1. If $s = 0$ or $s = p - q + 1$, then $G \cong H_{u,v}(p, q)$. We have $\rho(H_{u,v}(p, q)) > \rho(H_{u,v}(p + 1, q - 1))$. This is a contradiction.

Case 2. If $1 \leq s \leq p - q$. In this case, let y be the principal eigenvector of $Q(G)$, we can prove $y_{w_i} < y_{u_{p-q+1-i}}$ for $i = 0, 1, \dots, s$, and the method of proof is similar to

Claim 1. Let G' be the hypergraph obtained from G by moving $E_u(G) \setminus \{m_s, e_1\}$ from u to $u_{p-q+1-s}$. Obviously, $G' \cong H_{u,v}(p, q)$ and $\rho(G') > \rho(G)$. Therefore, $\rho(H_{u,v}(p, q)) > \rho(G) > \rho(H_{u,v}(p+1, q-1))$. This is a contradiction.

4 The Extremal Spectral Radii of Supertrees and Unicyclic Hypergraphs

Let H be a k -uniform supertree with n vertices, if H is a path $(v_0, e_1, v_1, \dots, v_{m-1}, e_m, v_m)$ such that the vertices v_1, \dots, v_{m-1} are of degree two, and all the other vertices of H are of degree one, then H is called a loose path, denoted by $P_{n,k}$. If H is a supertree such that all edges share a common vertex, then H is called a hyperstar, denoted by $S_{n,k}$.

Theorem 4.12 If T is a connected k -uniform supertree with n vertices, then $2\cos\frac{(k-1)\pi}{n+k-2} + k \leq \rho(T) \leq \frac{n+k^2-2k}{k-1}$. Where the left equality holds if and only if $T \cong P_{n,k}$, and the right equality holds if and only if $T \cong S_{n,k}$.

Proof Let $V_2(T)$ be the vertex set in T with degrees at least two, and x be the principal eigenvector of T . We use the mathematical induction on the number of $V_2(T)$. If $|V_2(T)| = 1$, then T is the hyperstar $S_{n,k}$. Now assume that $|V_2(T)| \geq 2$, namely T is not a hyperstar. Let $u, v \in V_2(T)$, as T is connected, there exists a path P connecting u and v . Without loss of generality, let $x_u \geq x_v$. Moving the edges incident with v , which are not included in $E(P)$, from v to u , we will get a new supertree T' . Then by Theorem 3.8 we have $\rho(T) < \rho(T')$. On the other hand, because of the fact that $|V_2(T')| < |V_2(T)|$, by the inductive hypothesis we have $\rho(T') \leq \rho(S_{n,k})$. Combining the above two relations we get $\rho(T) \leq \rho(S_{n,k})$, and the right equality holds if and only if $H \cong S_{n,k}$.

According to Theorem 2.4, $\rho(S_{n,k}) = \rho(L(S_{n,k})) + k$. The line graph $L(S_{n,k})$ is a complete graph with m vertices, where $m = \frac{n-1}{k-1}$, and $\rho(L(S_{n,k})) = m - 1$. Therefore $\rho(S_{n,k}) = m - 1 + k = \frac{n+k^2-2k}{k-1}$.

If T is a supertree minimizing the spectral radius, we have the following two claims.

Claim 1. There are at most two vertices on each edge of T with degrees at least 2.

Otherwise, there exists such edges e_i which include at least 3 vertices with degrees at least 2. Let $U_2(T)$ be the vertex set of the edges e_i with degrees at least 2. Let w be a vertex of T , and $t \in U_2(T)$ be the furthest vertex to w . Suppose that e_t is the edge including the vertex t , then there are pendant paths starting from t and another vertex $v \in e_t$, respectively. By Theorem 3.11, we use the operations of moving edges and obtain a new supertree T' . One can see that $\rho(T') < \rho(T)$, this is a contradiction.

Claim 2. $V(T)$ does not contain the vertex with degrees at least three.

Otherwise, let $V_3(T)$ be the vertex set in T with degrees at least three, where $|V_3(T)| \geq 1$. Let u be a vertex of T , and $v \in V_3(T)$ be the furthest vertex to u . Then there are at least $(d(v) - 1)$ pendant paths starting from v . By Theorem 3.11, we get a new supertree T' with

at least $(d(t) - 2)$ pendant paths starting from t . We have $\rho(T') < \rho(T)$, it is a contradiction.

Combining the above claims, we arrive our desired result, that is $\rho(P_{n,k}) \leq \rho(T)$.

By Theorem 2.4, $\rho(P_{n,k}) = \rho(L(P_{n,k})) + k$. The line graph $L(P_{n,k})$ is a path with m vertices, where $m = \frac{n-1}{k-1}$, and $\rho(L(P_{n,k})) = 2\cos\frac{\pi}{m+1} = 2\cos\frac{(k-1)\pi}{n+k-2}$. Thus $\rho(P_{n,k}) = 2\cos\frac{(k-1)\pi}{n+k-2} + k$.

Let U_1 be a unicyclic k -uniform hypergraphs consisting of two edges sharing two common vertices. Suppose that U_2 is obtained from the hypergraph U_1 by attaching a hyperstar with its center at one vertex of degree 2.

Theorem 4.13 If U is a connected unicyclic k -uniform hypergraph with n vertices and $m(m \geq 3)$ edges, then $\rho(U) \leq \frac{(2k+m-1) + \sqrt{m^2 - 2m + 9}}{2}$, where the equality holds if and only if $U \cong U_2$.

Proof Assume that U_3 is a connected unicyclic k -uniform hypergraph maximizing the spectral radius, then U_3 contains a vertex adjacent to all other vertices. Otherwise, let x be a the principal eigenvector of U_3 . We take one vertex u such that $x_u = \max\{x_v : v \in V(U_3)\}$. If there exists a vertex w not adjacent to u , then there exists a path connecting u and w , say $ue_1u_1 \cdots u_{r-1}e_rw$, where $r \geq 2$. Moving the edge e_r from u_{r-1} to u , by Theorem 3.8, we will get a new unicyclic hypergraph U' which satisfies $\rho(U') > \rho(U_3)$. It is a contradiction. Considering this property, it is easy to see that $U_3 \cong U_2$.

Let x_1 be the entry of x corresponding to the vertex of degree 2 in U_2 . Denote by x_2 the entry of x corresponding to the vertex of degree at least 3. By symmetry, the entry of x corresponding to the vertex of pendent edge with degree 1 is denoted by x_4 , and the other vertices of degree 1 are denoted by x_3 . Let $\rho = \rho(U_2)$. By Eq.2.4, we have $(\rho - 2)x_1 = 2(k - 2)x_3 + 2x_2$, $(\rho - k + 2)x_3 = x_1 + x_2$, $(\rho - m)x_2 = 2(k - 2)x_3 + 2x_1 + (m - 2)(k - 1)x_4$, $(\rho - k + 1)x_4 = x_2$. By simplifying the equations, we have

$$\rho^2 - (2k + m - 1)\rho + k^2 + (m - 1)k - 2 = 0, \quad \rho = \frac{(2k + m - 1) + \sqrt{m^2 - 2m + 9}}{2}.$$

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无符号拉普拉斯矩阵确定的一致超图的谱半径

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摘要: 本文研究了无符号拉普拉斯矩阵所确定的一致超图的谱半径问题. 利用Rayleigh原理和超图在移边操作下谱半径的变化规律, 得到了一致超图的谱半径的上界, 刻画了超树和单圈一致超图谱半径达到界时对应的超图. 推广了关于一般图的谱半径结果.

关键词: 谱半径; 一致超图; 无符号拉普拉斯矩阵

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