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DYNAMIC ANALYSIS OF A TYPE OF FINANCIAL RISK CONTAGION MODEL INVOLVING IMMUNITY PERIOD AND SELF-RESCUE

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Abstract: In this paper, we study the dynamics of a Susceptible-Exposed-Infectious-Recovered (SEIR) financial risk contagion model with time delay. Using stability theory and Hopf bifurcation theory, equilibria stability and Hopf bifurcation are analyzed in detail. Based on the epidemic model, we improve it by taking prior prevention and self-rescue into consideration, conclude preventive intensity and self-rescue capabilities effect the number of infections. At the same time, the analytical conditions for Hopf bifurcation are obtained, and the relevant results are verified by numerical simulations.

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1 Introduction

In recent years, financial liberalization and economic integration have continued to develop. However, with the expansion of international financial markets, the frequency and harm of financial risks are also increasing. Comparing the previous situation, we found that global risks always broke out in one country and then spread to other countries through trade, finance and other mechanisms. The process was obviously contagious and did huge harm to infected areas. In our country, the possibility of suffering financial risks and the harm caused by risks have been increasing in recent years. Therefore, the report of the 20th National Congress of the Communist Party of China also pointed out that “it is necessary to improve the financial regulatory system and maintain the bottom line of preventing systemic financial risks.”How to effectively monitor and prevent financial risks has become a hot issue studied by scholars.

In order to study the contagion mechanisms of different financial systems and reduce economic losses, many scholars have applied various methods to study financial risks in recent

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years. For example, some statistical models have been used to empirically study the characteristics of risk contagion among different financial markets in [1–3]. Some scholars have also introduced complex networks into financial systems, studied the topological structures and evolutionary rules of different financial markets, and described financial risk contagion from the perspective of network structure [4–6].

Due to the similarities in the spread of infectious diseases and financial risks [7], the epidemic model has provided new ideas for the study of financial risk contagion. Many scholars have started to use the epidemic model to consider different propagation mechanisms of financial risks. In this regard, Garas et al. [8] first applied a SIR model to study global financial crisis contagion, and obtained 12 central susceptible sources that may trigger the global financial crisis. On the basis of Garas, Toivanen [9] applied a SIR model and simulations to further study the risk contagion among European banks, obtained the relevant factors that easily cause risk contagion. Subsequently, many scholars studied financial risks by establishing different epidemic models. For example, Bucci et al. [10] believed that banks would only be affected by speculation, and infected banks would never develop immunity after recovering from speculation, so they established a susceptible-infected-susceptible (SIS) model of financial risk contagion, which truly describes the contagion process among banks. Since exposed individuals exist in the financial risk system, Chenyu B et al. [11] considered the strictness of government supervision, established a SEIR model by dividing infected institutions into two types, and concluded that supervision stringency affected the stability of system. Xu Kai [12] considered the existence of latent subject (H) based on the SIR model, constructed a SHIR model of associated credit risk contagion, and analyzed the relevant factors that affected associated credit risk contagion. In addition, considering the impact of time delay and the characteristics of Internet P2P lending platforms, Zhao et al. [13] constructed a SEIR model with time delay, used equilibrium and thresholds to analyze the effect of different factors on credit risk contagion behavior. To sum up, we can see that existing research cannot better describe the complex financial systems. Therefore, based on the SEIR model, the financial risk contagion model is improved in this paper.

We arrange the paper as follows: In Section 2, the construction process of the risk contagion model is explained in detail. In Section 3, equilibria stability and Hopf bifurcation analysis are investigated, and the corresponding numerical simulations are given in Section 4. Finally, relevant conclusions are drawn, the theoretical results provide a theoretical basis for governments and enterprises to monitor and prevent financial risks.

2 Model Building

2.1 Model Assumptions and Parameter Settings

In this paper, we construct a risk contagion model by taking into account the prior prevention of regulatory agencies, the self-rescue of individuals, and temporary immunity period. We consider four types of subjects who are susceptible to infect risks (S), exposed to risk shocks and in a latent state (E), infected with risks (I), and recovered from infection (R). Based on [11], the assumptions made in this paper are as follows:

Assumption 1: The system network is not closed but limited, and each financial individual is a node in the network system. Financial individuals enter or exit the system at the same proportion during the risk contagion process.

Assumption 2: There are differences among financial individuals. After being infected, some individuals can become recovered groups through various preventive measures, while some individuals exit the financial system because they cannot acquire immunity.

Assumption 3: All susceptible individuals can establish lending relationships or hold common assets with infected individuals or exposed individuals, which further spread financial risks through connection relationships.

Assumption 4: Exposed individuals can perceive the existence of risks and control the spread of financial risks through their own financing capabilities or external policy support.

Based on the above assumptions, the financial risk contagion model established in this paper is as follows:

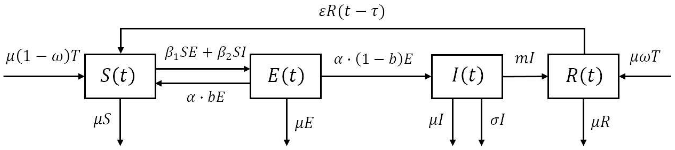


Fig.1. Flow chart of risk contagion.

$$\begin{cases} \dfrac{dS(t)}{dt} = \mu(1-\omega)T - \beta_1 S(t)E(t) - \beta_2 S(t)I(t) + \alpha b E(t) + \varepsilon R(t-\tau) - \mu S(t), \\ \dfrac{dE(t)}{dt} = \beta_1 S(t)E(t) + \beta_2 S(t)I(t) - \alpha E(t) - \mu E(t), \\ \dfrac{dI(t)}{dt} = \alpha(1-b)E(t) - \sigma I(t) - mI(t) - \mu I(t), \\ \dfrac{dR(t)}{dt} = \mu\omega T + mI(t) - \varepsilon R(t-\tau) - \mu R(t), \end{cases} \tag{2.1}$$

where the initial conditions of Eq.(2.1) are $S(0) > 0$, $E(0) > 0$, $I(0) > 0$, $R(0) > 0$, and let $N(0)$ be the total number of individuals when $t = 0$, i.e.

$$N(0) = S(0) + E(0) + I(0) + R(0) > 0. \tag{2.2}$$

The parameters used in Eq.(2.1) and the meanings are shown in the following table.

Table 1 Parameter Settings

Parameter	Range	Practical significance
ω	$[0,1]$	The intensity of prior prevention
β_1	$[0,1]$	The infection rate of S come into contact with E
β_2	$[0,1]$	The infection rate of S come into contact with I
b	$[0,1]$	The rescue effect of individuals themselves or external policies
α	$[0,1]$	The transfer rate of exposed individuals
σ	$[0,1]$	The bankruptcy rate of infected individuals
m	$[0,1]$	The probability of infected individuals regaining immunity
μ	$[0,1]$	The exit rate of individuals in each state
ε	$[0,1]$	The probability that individuals lose immunity
τ	$[0,+\infty)$	Temporary immunity period
T	$[0,+\infty)$	The number of new individuals entering the system

2.2 Positive Invariant Set of the Model

In order to ensure that Eq.(2.1) has practical significance, in this part we will discuss the positive invariant set of Eq.(2.1) to illustrate its positivity and boundedness.

Proposition 2.1 The positive invariant set of Eq.(2.1) is $\Omega=\{(S,E,I,R)|S,\ E,\ I,\ R\geq 0,\ S+E+I+R\leq T\}$.

Proof The initial conditions of (2.1) are $S(0)>0$, $E(0)>0$, $I(0)>0$, $R(0)>0$.

When $t\geq 0$,

$$\begin{aligned}&\frac{dS}{dt}|_{S=0}=\mu(1-\omega)T+\alpha bE+\varepsilon R(t-\tau)>0,\\&\frac{dE}{dt}|_{E=0}=\beta_2 SI>0,\\&\frac{dI}{dt}|_{I=0}=\alpha(1-b)E>0,\\&\frac{dR}{dt}|_{R=0}=\mu\omega T+mI>0.\end{aligned}$$

Since $S(t)$, $E(t)$, $I(t)$, $R(t)$ are continuous functions about t, all solutions in Eq.(2.1) are non-negative when $t\geq 0$. So $\Omega_1=\{(S,E,I,R)|S,\ E,\ I,\ R\geq 0\}$ is positive and invariant to (2.1).

Then, we set $N(t)=S(t)+E(t)+I(t)+R(t)$.

According to (2.1) we can get

$$\frac{dN}{dt}=\mu T-\mu N-\sigma I\leq \mu T-\mu N. \tag{2.3}$$

So, there is $0\leq N\leq T$.

Therefore, the positive invariant set of Eq.(2.1) is $\Omega=\{(S,E,I,R)|S,\ E,\ I,\ R\geq 0,$ $0\leq S+E+I+R\leq T\}$.

3 Dynamic Analysis

In the financial risk system, equilibria of Eq.(2.1) are the key to risk contagion process. Therefore, in this section we will analyze the existence and dynamic characteristics of equilibria in the positive invariant set Ω.

3.1 Existence of Equilibria

In this part we will discuss the existence of equilibria in Eq.(2.1).

Due to the non-negative characteristics of Eq.(2.1), we can get that Eq.(2.1) has two equilibria: the risk-free equilibrium $P_0(S_0, E_0, I_0, R_0)$ and the risky equilibrium $P^*(S^*, E^*, I^*, R^*)$.

First, for the risk-free equilibrium P_0, the solution can be obtained as $S_0 = \frac{\mu(1-\omega)T+\varepsilon T}{\varepsilon+\mu}$, $E_0 = 0$, $I_0 = 0$, $R_0 = \frac{\mu\omega T}{\varepsilon+\mu}$. Since T, ε, μ, $\varepsilon+\mu > 0$ and $0 \le \omega \le 1$, P_0 exists.

Before analyzing the existence of P^*, we first discuss the basic reproduction number ($\widetilde{R}_0$). The basic reproduction number is a critical indicator that reflects whether risks are contagious among individuals, and is of great significance to our study of risk contagion.

For the calculation of $\widetilde{R}_0$, we use the next generation matrix method in [14] to obtain

$$\widetilde{R}_0 = \frac{\beta_1 S_0(\sigma+m+\mu) + \beta_2 S_0 \alpha(1-b)}{(\alpha+\mu)(\sigma+m+\mu)}. \tag{3.1}$$

Next we will discuss the existence of P^*. According to the expression of $\widetilde{R}_0$, we get that the existence of P^* satisfies the following theorem.

Theorem 3.1 When $\widetilde{R}_0 > 1$, the risky equilibrium $P^*(S^*, E^*, I^*, R^*)$ of Eq.(2.1) exists; when $\widetilde{R}_0 \le 1$, P^* does not exist.

Proof Solving the risky equilibrium of Eq.(2.1), we can get

$$I^* = \frac{\alpha(1-b)}{\sigma+m+\mu}E^*,\ R^* = \frac{mI^* + \mu\omega T}{\varepsilon+\mu}.$$

Substituting $I^* = \frac{\alpha(1-b)}{\sigma+m+\mu}E^*$ into $\beta_1 S^* E^* + \beta_2 S^* I^* - \alpha E^* - \mu E^* = 0$, we can get

$$S^* = \frac{(\alpha+\mu)(\sigma+m+\mu)}{\beta_1(\sigma+m+\mu) + \beta_2\alpha(1-b)}. \tag{3.2}$$

Substituting Eq.(3.2) into $\mu(1-\omega)T - \beta_1 S^* E^* - \beta_2 S^* I^* + \alpha b E^* + \varepsilon R^* - \mu S^* = 0$, we get

$$E^* = \frac{(\sigma+m+\mu)[(\varepsilon+\mu)\mu(1-\omega)T + \varepsilon\mu\omega T - \mu(\varepsilon+\mu)S^*]}{(\sigma+m+\mu)(\varepsilon+\mu)(\mu+\alpha(1-b)) + \varepsilon m\alpha(1-b)}. \tag{3.3}$$

$S^* > 0$ is obviously established. Next we prove that $E^* > 0$.

$$\begin{aligned}
&E^* > 0\\
\Leftrightarrow\ &(\varepsilon+\mu)\mu(1-\omega)T + \varepsilon\mu\omega T - \mu(\varepsilon+\mu)S^* > 0\\
\Leftrightarrow\ &(\varepsilon+\mu)S_0 - (\varepsilon+\mu)S^* > 0\\
\Leftrightarrow\ &\beta_1 S_0(\sigma+m+\mu) + \beta_2 S_0\alpha(1-b) > (\alpha+\mu)(\sigma+m+\mu)\\
\Leftrightarrow\ &\widetilde{R}_0 > 1
\end{aligned}$$

Therefore, if and only if $\widetilde{R}_0 > 1$, the risky equilibrium $P^*(S^*, E^*, I^*, R^*)$ exists.

3.2 Stability and Hopf Bifurcation Analysis of P_0

Since the stability of P_0 determines whether financial risks will eventually be eliminated, in this part we will analyze the stability of P_0 and the analytical conditions for Hopf bifurcation. And the relevant results are shown in the following theorem.

Theorem 3.2 When $\widetilde{R}_0 < 1$ and $\varepsilon \le \mu$, $P_0(S_0, E_0, I_0, R_0)$ of Eq.(2.1) is locally asymptotically stable. When $\varepsilon > \mu$, Eq.(2.1) undergoes a Hopf bifurcation at $\tau = \tau_1$, and when $\tau \in (0, \tau_1)$, P_0 is locally asymptotically stable; when $\tau > \tau_1$, a cluster of periodic solutions bifurcate from P_0.

Proof For Eq.(2.1), the Jacobian matrix of P_0 is

$$J_0 = \begin{bmatrix} -\mu & -\beta_1 S_0 + \alpha b & -\beta_2 S_0 & \varepsilon e^{-\lambda\tau} \\ 0 & \beta_1 S_0 - \alpha - \mu & \beta_2 S_0 & 0 \\ 0 & \alpha(1-b) & -(\sigma + m + \mu) & 0 \\ 0 & 0 & m & -\mu - \varepsilon e^{-\lambda\tau} \end{bmatrix}.$$

Calculating its characteristic equation to get

$$(\lambda + \mu)(\lambda + \mu + \varepsilon e^{-\lambda\tau})[\lambda^2 + a_1\lambda + a_2] = 0, \tag{3.4}$$

where $a_1 = (\sigma + m + \mu) - (\beta_1 S_0 - \alpha - \mu)$, $a_2 = -(\sigma + m + \mu)(\beta_1 S_0 - \alpha - \mu) - \beta_2 S_0 \alpha(1-b)$.

Obviously, $\lambda_1 = -\mu$ has a negative real part.

For

$$\lambda^2 + a_1\lambda + a_2 = 0. \tag{3.5}$$

From Eq.(3.1), we can obtain when $\widetilde{R}_0 < 1$, then $\beta_1 S_0 < \alpha + \mu$, thus $a_1 > 0$; when $\widetilde{R}_0 < 1$, then $\beta_1 S_0(\sigma + m + \mu) + \beta_2 S_0 \alpha(1-b) < (\alpha + \mu)(\sigma + m + \mu)$, thus $a_2 > 0$.

Therefore, $a_1 a_2 > 0$ is established. According to the Routh-Hurwitz criteria, all roots of Eq.(3.5) have negative real parts.

For

$$\lambda + \mu + \varepsilon e^{-\lambda\tau} = 0. \tag{3.6}$$

Let $\lambda = i\eta (\eta > 0)$ be the root of Eq.(3.6), then substituting it into Eq.(3.6) we obtain

$$i\eta + \mu + \varepsilon\cos\eta\tau - i\varepsilon\sin\eta\tau = 0. \tag{3.7}$$

Separating the real and imaginary parts of Eq.(3.7) we can get

$$\begin{cases} \eta - \varepsilon\sin\eta\tau = 0, \\ \mu + \varepsilon\cos\eta\tau = 0. \end{cases} \tag{3.8}$$

From Eq.(3.8), we can see that $\eta^2 = \varepsilon^2 - \mu^2$, so η exists when $\varepsilon > \mu$.

When $\varepsilon > \mu$, the time delay corresponding to Eq.(3.7) is

$$\tau_1^{(j)} = \frac{1}{\eta}\arccos(-\frac{\mu}{\varepsilon}) + \frac{2j\pi}{\eta}, j = 0, 1, 2, ... \tag{3.9}$$

Let $\tau_1 = \min\left\{\tau_1^{(j)}\right\}, j = 0, 1, 2, ...$, then differentiating two sides of Eq.(3.6) with respect to τ, we can obtain

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = -\frac{1}{\lambda(\lambda+\mu)} - \frac{\tau}{\lambda}. \tag{3.10}$$

And then

$$Re\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_1} = \frac{1}{\eta^2+\mu^2} \neq 0. \tag{3.11}$$

Therefore, using the Hopf bifurcation theorem in [15], we can get that the Hopf bifurcation occurs in Eq.(2.1) when $\tau = \tau_1$.

To sum up, we can conclude that when $\widetilde{R}_0 < 1$ and $\varepsilon \leq \mu$, the risk-free equilibrium of Eq.(2.1) is asymptotically stable, and risks will eventually disappear; when $\widetilde{R}_0 > 1$, the risk-free equilibrium P_0 will lose stability, and the risky equilibrium P^* will occur.

3.3 Stability and Hopf Bifurcation Analysis of P^*

Before studying the stability of P^* and the analytical conditions for Hopf bifurcation, we first discuss the roots of the fourth degree polynomial.

For

$$v^4 + l_{11}v^3 + l_{12}v^2 + l_{13}v + l_{14} = 0. \tag{3.12}$$

We assume that $h(v) = v^4+l_{11}v^3+l_{12}v^2+l_{13}v+l_{14}$, $p_1 = \frac{l_{12}}{2} - \frac{3}{16}l_{11}^2$, $q_1 = \frac{l_{11}^3}{32} - \frac{l_{11}l_{12}}{8} + l_{13}$, $D = (\frac{q_1}{2})^2 + (\frac{p_1}{3})^3$, then the roots of Eq.(3.12) satisfy the following lemma.

Lemma 3.1 [16] Let v_1, v_2, v_3 be the roots of $h'(v) = 0$, and then

(1) when $l_{14} < 0$, Eq.(3.12) has at least one positive root;

(2) when $l_{14} \geq 0$:

(i) if $D \geq 0$, Eq.(3.12) has positive roots if and only if $v_1 > 0$ and $h(v_1) < 0$;

(ii) if $D < 0$, Eq.(3.12) has positive roots if and only if there is at least one $v^* \in \{v_1, v_2, v_3\}$ such that $v^* > 0$ and $h(v^*) \leq 0$.

Then, through Lemma 3.1 and related analysis, we can conclude the following theorem.

Theorem 3.3 For the conditions:

(H_{10}) $\widetilde{R}_0 > 1$;

(H_{11}) One of the following cases holds:

$a: l_{14} < 0$;

$b: l_{14} \geq 0$, $D \geq 0$ and $v_1 > 0$, $h(v_1) < 0$;

$c: l_{14} \geq 0$, $D < 0$ and there exists $v^* \in \{v_1, v_2, v_3\}$ such that $v^* > 0$, $h(v^*) \leq 0$;

(H_{12}) $h'(v_{10}) \neq 0$.

If $(H_{10})-(H_{12})$ hold, then when $\tau \in (0, \tau_2)$, P^* is locally asymptotically stable; when $\tau = \tau_2$, Eq.(2.1) undergoes a Hopf bifurcation and a cluster of periodic solutions bifurcate from P^*.

Proof First, we linearize Eq.(2.1) at P^* to get

$$\begin{cases} \dfrac{d\hat{S}(t)}{dt} = a_{11}\hat{S}(t) + a_{12}\hat{E}(t) + a_{13}\hat{I}(t) + a_{14}\hat{R}(t-\tau), \\ \dfrac{d\hat{E}(t)}{dt} = a_{21}\hat{S}(t) + a_{22}\hat{E}(t) + a_{23}\hat{I}(t), \\ \dfrac{d\hat{I}(t)}{dt} = a_{31}\hat{E}(t) + a_{32}\hat{I}(t), \\ \dfrac{d\hat{R}(t)}{dt} = a_{41}\hat{I}(t) + a_{42}\hat{R}(t) + a_{43}\hat{R}(t-\tau), \end{cases} \tag{3.13}$$

where

$$\hat{S} = S - S^*,\ \hat{E} = E - E^*,\ \hat{I} = I - I^*,\ \hat{R} = R - R^*.$$

$$a_{11} = -(\beta_1 E^* + \beta_2 I^* + \mu),\ a_{12} = -\beta_1 S^* + \alpha b,\ a_{13} = -\beta_2 S^*,\ a_{14} = \varepsilon,$$

$$a_{21} = \beta_1 E^* + \beta_2 I^*,\ a_{22} = \beta_1 S^* - \alpha - \mu,\ a_{23} = \beta_2 S^*,$$

$$a_{31} = \alpha(1-b),\ a_{32} = -(\sigma + m + \mu),\ a_{41} = m,\ a_{42} = -\mu,\ a_{43} = -\varepsilon.$$

The characteristic equation of Eq.(3.13) at P^* is

$$\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4 + (d_1\lambda^3 + d_2\lambda^2 + d_3\lambda + d_4)e^{-\lambda\tau} = 0, \tag{3.14}$$

where

$$b_1 = a_{32} + a_{22},\ b_2 = a_{32}a_{22} - a_{23}a_{31},$$

$$c_1 = -b_1 - a_{42} - a_{11},\ c_2 = b_2 + b_1(a_{42} + a_{11}) + a_{42}a_{11} - a_{21}a_{12},$$

$$c_3 = -b_2(a_{42} + a_{11}) - b_1 a_{42}a_{11} + a_{21}a_{12}(a_{42} + a_{32}) - a_{21}a_{31}a_{13},$$

$$c_4 = b_2 a_{42}a_{11} - a_{21}a_{12}a_{42}a_{32} + a_{21}a_{31}a_{13}a_{42},$$

$$d_1 = -a_{43},\ d_2 = b_1 a_{43} + a_{43}a_{11},\ d_3 = -b_2 a_{43} - b_1 a_{43}a_{11} + a_{21}a_{12}a_{43},$$

$$d_4 = b_2 a_{11}a_{43} - a_{21}a_{12}a_{32}a_{43} + a_{21}a_{31}a_{13}a_{43} - a_{21}a_{31}a_{41}a_{14}.$$

Let $\lambda = i\theta(\theta > 0)$ be the root of Eq.(3.14), then substituting it into Eq.(3.14) we obtain

$$\theta^4 - ic_1\theta^3 - c_2\theta^2 + ic_3\theta + c_4 + (-id_1\theta^3 - d_2\theta^2 + id_3\theta + d_4)(\cos\theta\tau - i\sin\theta\tau) = 0. \tag{3.15}$$

Separating the real and imaginary parts of Eq.(3.15) we can get

$$\begin{cases} E_{11}(\theta)\cos\theta\tau + E_{12}(\theta)\sin\theta\tau = E_{13}(\theta), \\ E_{12}(\theta)\cos\theta\tau - E_{11}(\theta)\sin\theta\tau = E_{14}(\theta), \end{cases} \tag{3.16}$$

where

$$E_{11}(\theta) = d_4 - d_2\theta^2,\ E_{12}(\theta) = d_3\theta - d_1\theta^3,\ E_{13}(\theta) = -\theta^4 + c_2\theta^2 - c_4,\ E_{14}(\theta) = c_1\theta^3 - c_3\theta.$$

Simplifying Eq.(3.16) we can get the following equation about θ:

$$\theta^8 + l_{11}\theta^6 + l_{12}\theta^4 + l_{13}\theta^2 + l_{14} = 0, \tag{3.17}$$

where

$l_{11}=c_1^2-2c_2-d_1^2$, $l_{12}=2c_4+c_2^2-2c_1c_3-d_2^2+2d_1d_3$, $l_{13}=c_3^2-2c_2c_4+2d_2d_4-d_3^2$, $l_{14}=c_4^2-d_4^2$.

Let $\theta^2=v$, then Eq.(3.17) can be reduced to

$$v^4+l_{11}v^3+l_{12}v^2+l_{13}v+l_{14}=0. \tag{3.18}$$

According to Lemma 3.1, we can get that when the condition (H_{11}) holds, Eq.(3.18) will have at least one positive root. Without loss of generality, we assume that Eq.(3.18) has four positive roots, denoted v_{11}, v_{12}, v_{13} and v_{14} respectively, then Eq.(3.17) has four positive roots $\theta_k=\sqrt{v_{1k}}$, $k=1,2,3,4$.

Therefore, for each fixed θ_k, the time delay corresponding to Eq.(3.15) is

$$\tau_{2k}^{(j)}=\frac{1}{\theta_k}\arccos\left[\frac{E_{11}(\theta_k)E_{13}(\theta_k)+E_{12}(\theta_k)E_{14}(\theta_k)}{E_{11}^2(\theta_k)+E_{12}^2(\theta_k)}\right]+\frac{2j\pi}{\theta_k},\ k=1,2,3,4.\ j=0,1,2,... \tag{3.19}$$

Let $\tau_2=\min\left\{\tau_{2k}^{(0)}\right\}$, $v_{10}=\theta_{10}^2$, $\theta_{10}=\theta_k|_{\tau=\tau_2}$, $k=1,2,3,4$.
Differentiating two sides of Eq.(3.14) with respect to τ, we can obtain

$$\left[\frac{d\lambda}{d\tau}\right]^{-1}=-\frac{4\lambda^3+3c_1\lambda^2+2c_2\lambda+c_3}{\lambda(\lambda^4+c_1\lambda^3+c_2\lambda^2+c_3\lambda+c_4)}+\frac{3d_1\lambda^2+2d_2\lambda+d_3}{\lambda(d_1\lambda^3+d_2\lambda^2+d_3\lambda+d_4)}-\frac{\tau}{\lambda}. \tag{3.20}$$

And then

$$Re\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_2}=\frac{h'(v_{10})}{E_{11}^2(\theta_{10})+E_{12}^2(\theta_{10})}. \tag{3.21}$$

So when the condition $(H_{12}): h'(v_{10})\neq 0$ holds, $Re\left[\frac{d\lambda}{d\tau}\right]_{\tau=\tau_2}\neq 0$.

Based on above discussion and the Hopf bifurcation theorem in [15], we can get that the Hopf bifurcation occurs when $\tau=\tau_2$.

Therefore, the critical value of temporary immunity period is not infinite, but there is a threshold. When temporary immunity time is within the critical value, the risk will eventually stabilize, and when it exceeds the critical value, the system will eventually become unstable.

4 Numerical Simulations

In this section, we will give specific parameter values to support the theoretical analysis in Section 3.

4.1 Simulation Results of P_0

From Theorem 3.2, we conclude that when $\widetilde{R}_0<1$ and $\varepsilon\leq\mu$, $P_0(S_0,E_0,I_0,R_0)$ is locally asymptotically stable. Therefore, we take $\mu=\varepsilon=0.05$, $\omega=0.8$, $T=6$, $\beta_1=0.01$, $\beta_2=0.02$, $\alpha=0.06$, $b=0.5$, $\sigma=0.02$, $m=0.08$ here. The calculation shows that $\widetilde{R}_0=0.3535<1$, and the system only has a risk-free equilibrium $P_0(3.6,0,0,2.4)$. At this time, the number of infected individuals will continue to decrease, which is shown in Fig.2, financial risks will eventually disappear.

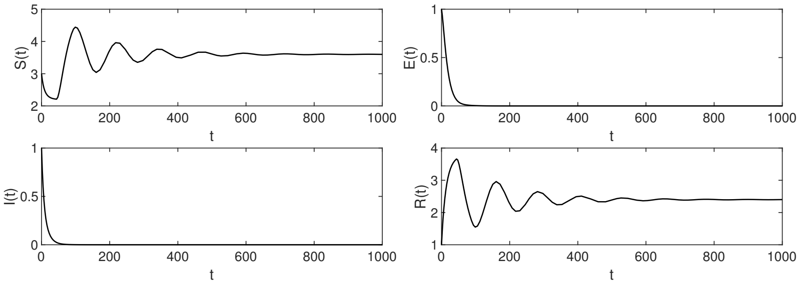


Fig.2. $\widetilde{R}_0 < 1$ and $\varepsilon \le \mu$, P_0 is locally asymptotically stable.

Next, we take $\varepsilon = 0.05$, $\mu = 0.02$, and keep other parameters unchanged. At this time $\varepsilon > \mu$, we can get $\widetilde{R}_0 = 0.6364 < 1$, and the system only has a risk-free equilibrium $P_0(4.6286, 0, 0, 1.3714)$. According to Eq.(3.8), we can conclude that $\eta = 0.0458$ exists, which satisfies the conditions for Hopf bifurcation, and the time delay critical value $\tau_1 = 43.2576$.

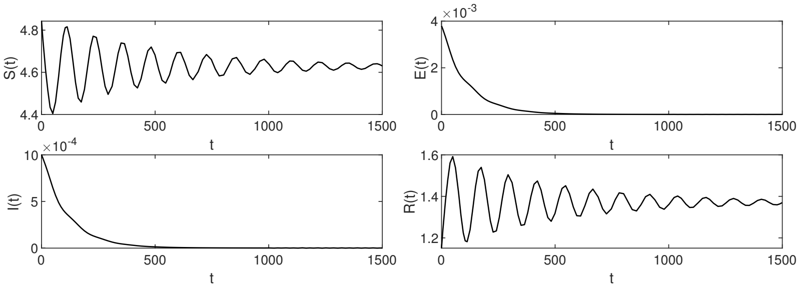


Fig.3. $\tau = 37.6 < \tau_1$, P_0 is locally asymptotically stable.

Fig.3 shows the case of $\tau = 37.6 < \tau_1$. It can be seen that the system produces damped oscillation and finally converges to P_0. This means that under these parameter values, the risk system will eventually stabilize and the risk will disappear.

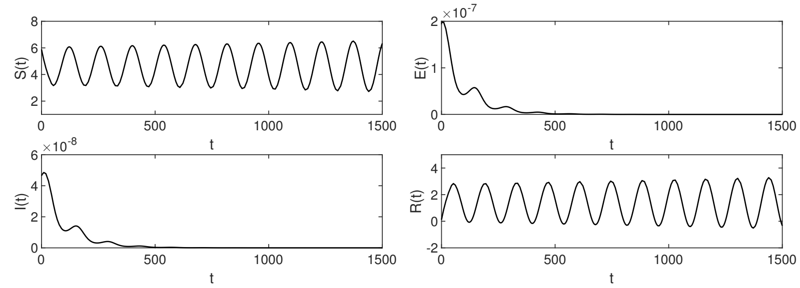


Fig.4. $\tau = 44 > \tau_1$, a Hopf bifurcation occurs in Eq.(2.1).

Fig.4 shows the case of $\tau = 44 > \tau_1$. It can be seen that when τ passes through the critical value τ_1, the Hopf bifurcation occurs and a cluster of periodic solutions bifurcate from P_0. In addition, according to the definition of $\widetilde{R}_0$, when $\widetilde{R}_0 < 1$, the maximum number of individuals that can be infected by an infectious individual is less than 1, so when $\widetilde{R}_0 < 1$ the risk can always be gradually eliminated.

4.2 Simulation results of P^*

According to Lyapunov's first method, we use the linearized model Eq.(3.13) to study the local stability of Eq.(2.1) near P^*.

We conclude that $P^*(S^*, E^*, I^*, R^*)$ exists only when $\widetilde{R}_0 > 1$ in Theorem 3.1. Therefore, we take $\mu = 0.02$, $\varepsilon = 0.6$, $\omega = 0.5$, $T = 500$, $\beta_1 = 0.07$, $\beta_2 = 0.08$, $\alpha = 0.1$, $b = 0.3$, $\sigma = 0.02$, $m = 0.6$. The calculation shows that $\widetilde{R}_0 = 322.8327 > 1$. At this time, the system has a risky equilibrium $P^*(1.5238, 370.2347, 40.4944, 47.2527)$. In addition, we can get $l_{14} = -0.0733 < 0$ and $h'(v_{20}) = 282.5254 \neq 0$, so the conditions $(H_{10}) - (H_{12})$ in Theorem 3.3 hold. From the above parameters, we can get $\theta_{10} = 0.6277$, so the corresponding time delay critical value $\tau_2 = 2.6425$.

We take $\tau = 2.5 < 2.6425$ in Fig.5, it can be seen that P^* is locally asymptotically stable. But when τ passes through the critical value 2.6425, P^* will lose stability and a cluster of periodic solutions will bifurcate from P^*. Therefore, we take $\tau = 2.65 > 2.6425$, and the corresponding result is shown in Fig.6. Furthermore, the simulation results also confirmed the similarity of local stability between Eq.(2.1) and Eq.(3.13).

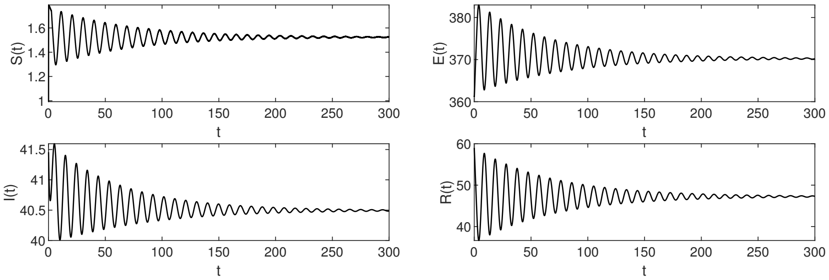


Fig.5. $\tau = 2.5 < \tau_2$, P^* is locally asymptotically stable.

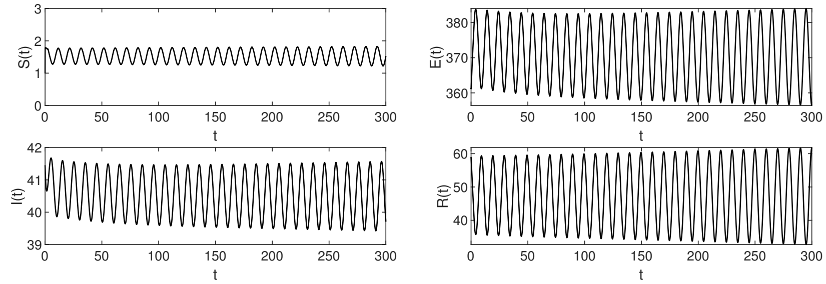
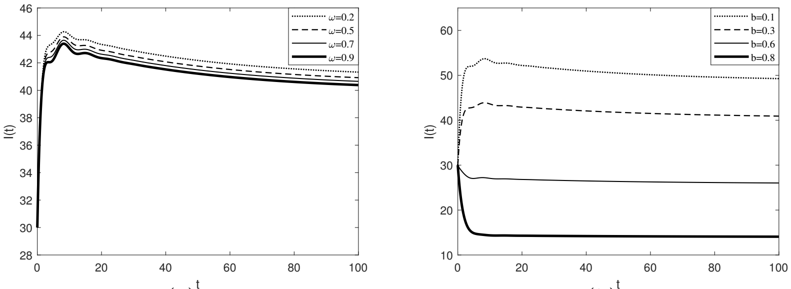


Fig.6. $\tau = 2.65 > \tau_2$, a Hopf bifurcation occurs in Eq.(2.1).

4.3 The Effect of ω and b on $I(t)$

In this part we will test the effect of preventive intensity (ω) and self-rescue capabilities (b) on the number of $I(t)$. From Eq.(3.1) we can conclude that both ω and b will effect the value of R_0 and the speed of risk propagation.

In addition, we can see from Fig.7 that when risks spread in the system, the improvement



(a) (b)

Fig.7. Trend chart of $I(t)$ changing with (a) ω, (b) b.

of self-rescue capabilities can significantly reduce the number of $I(t)$, and even cause a negative growth in $I(t)$. In contrast, the prior prevention of regulatory agencies has little effect in reducing the number of $I(t)$, which shows the restriction of relevant policies cannot fundamentally solve risk infection in the operation of financial systems.

5 Conclusion

In this paper, we comprehensively consider the prior prevention of regulatory agencies, the self-rescue of individuals, and the temporary immunity period in financial risk contagion, build a dynamic model to describe the contagion process of financial risks. It has been shown that the temporary immunity time delay has a significant impact on the stability of risk contagion model. When relevant parameters satisfy certain conditions, the system will exist a critical value τ_i for immunity period. If the immunity period is shorter than τ_i, the model will be stable, but if the immunity period is longer than τ_i, the model will undergo Hopf bifurcation. This illustrates that when risks are transmitted among individuals, recovered groups do not gain permanent immunity, but rather there exists a critical value that allows the system to operate smoothly.

Furthermore, we derived the conditions for the disappearance and persistence of risks, demonstrated that the system was stable and risks would eventually disappear when $\widetilde{R}_0 < 1$. If $\widetilde{R}_0 > 1$, risks will persist. This provides theoretical support for government and enterprises to effectively monitor risks. Finally, it was found that when risks spread in a financial system, the prior prevention of regulatory agencies can only serve as an auxiliary role. However, the self-rescue capabilities of individuals have a significant impact on the number of infections. Therefore, individuals should actively utilize their own resources to counteract risks when infected with risks.

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含免疫期与救助的一类金融风险传染模型的动力学分析

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摘要: 本文研究了一类具有时滞的易感-潜伏-感染-免疫(SEIR)金融风险传染模型的动力学性质. 利用稳定性理论和Hopf分岔理论分析了平衡点的稳定性与Hopf分岔产生的解析条件. 在已有传染病模型的基础上, 本文综合考虑了监管机构的事先预防和个体自身的救助来改进模型, 得出两者均能影响感染个体的数量. 同时获得了系统产生Hopf分岔的解析条件, 并用数值模拟加以验证.

关键词: 金融风险传染; 自身救助; 暂时免疫; 时滞

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