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MATRICES OVER A REDUCED RING AS SUMS OF THREE TRIPOTENTS

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Abstract: In this paper, we study reduced rings in which every element is a sum of three tripotents that commute, and determine the integral domains over which every $n \times n$ matrix is a sum of three tripotents. It is proved that for an integral domain R, every matrix in $M_n(R)$ is a sum of three tripotents if and only if $R \cong \mathbb{Z}_p$ with $p = 2, 3, 5$ or 7.

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1 Introduction

Throughout, R is an associative ring with identity. For a ring R, the set of all nilpotents is denoted by $Nil(R)$. We write $\mathbb{Z}_n$ for the ring of integral modulo n, $M_n(R)$ for the $n \times n$ matrix ring with identity matrix I_n.

Rings whose elements are sums of certain special elements have been widely studied in ring theory. Recall that an element a of a ring is a *tripotent* if $a^3 = a$. Zhou [1] investigated that, for a ring R, every element of R is a sum of a nilpotent, an idempotent and a tripotent that commute with one another, if and only if every element of R is a sum of a nilpotent and two idempotents that commute with one another. In [2], the authors determined the rings for which every element is a sum of two commuting idempotents, and characterized the rings for which every element is a sum of an idempotent and a tripotent that commute with one another. In a recent paper [3], Cui and Xia determined the rings for which every element is a sum of a nilpotent and three tripotents that commute with one another.

In [4], the authors investigated rings in which every element is a sum of two idempotents, and proved that for every $n > 1$ and any ring R, there exists a matrix $A \in M_n(R)$ such that

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A cannot be written as a sum of two idempotents. In [5], Tang, Zhou and Su proved that, for a field F and any integer $n \geq 1$, every matrix over F is a sum of three idempotents if and only if $F \cong \mathbb{Z}_2$ or $F \cong \mathbb{Z}_3$. Xia, Tang and Zhou [6] showed that, for an integral domain R and any integer $n \geq 1$, every $n \times n$ matrix over R is a sum of two tripotents if and only if $R \cong \mathbb{Z}_p$ for $p = 2, 3$ or 5. In [7], Abyzov and Tapkin determined the rings for which every matrix is a sum of two tripotents, and proved that every $n \times n$ matrix over a field F is a sum of two tripotents if and only if F is a prime field with *Char(F)* ≤ 5, where *Char(F)* is the characteristic of F.

In this paper, we go one step further investigation of this subject. In Section 2, as an extension of the work in [3], the main objective is to present the structure of reduced rings for which every element is a sum of three tripotents that commute with one another. In Section 3, we determine the integral domain R over which every $n \times n$ matrix is a sum of three tripotents, and present equivalent conditions for an $n \times n$ matrix over an integral domain R to be a sum of three tripotents.

2 Rings with property $\mathcal{P}$

In this section, we consider reduced rings for which every element is a sum of three tripotents. We call a ring R has property $\mathcal{P}$ if every element of R is a sum of three tripotents that commute. For an integer n, write $n = n \cdot 1_R \in R$ for short.

Lemma 2.1 Let R be a ring. If $4 = e + f + g$ with $e^3 = e, f^3 = f, g^3 = g$ and e, f, g all commute, then $2 \cdot 3 \cdot 5 \cdot 7 \in Nil(R)$.

Proof The proof is similar to that of [3, Lemma 2], we give a proof for a convenience. By assumption, $4^3 - 4 = 15(e + f + g) = (e + f + g)^3 - (e + f + g)$ implies

$$60 = 15(e + f + g) = 3e^2f + 3e^2g + 3ef^2 + 3f^2g + 3eg^2 + 3fg^2 + 6efg. \tag{1.1}$$

Multiplying both sides of (1.1) by efg gives $6e^2f^2g^2 = 36efg$, which implies $210efg = 0$. In view of (1.1), we have

$$\begin{aligned}
4^3 &= 3e^2f + 3e^2g + 3ef^2 + 3f^2g + 3eg^2 + 3fg^2 + 6efg + (e + f + g)\\
&= 3(e^2 + f^2 + g^2)e + 3(e^2 + f^2 + g^2)f + 3(e^2 + f^2 + g^2)g + 6efg - 2(e + f + g)\\
&= 3(e^2 + f^2 + g^2)(e + f + g) - 2(e + f + g) + 6efg\\
&= 12(e^2 + f^2 + g^2) - 8 + 6efg.
\end{aligned} \tag{1.2}$$

Multiplying both sides of (1.2) by 35 gives $35 \cdot 4^3 = 35 \cdot 12(e^2 + f^2 + g^2) - 35 \cdot 8$, so we have $35 \cdot 12(e^2 + f^2 + g^2) = 2^3 \cdot 3^2 \cdot 5 \cdot 7$. Note that $e^2 + f^2 + g^2 = (e + f + g)^2 - 2(ef + eg + fg) = 16 - 2(ef + eg + fg)$. Thus, $35 \cdot 12(2ef + 2eg + 2fg) = 2^3 \cdot 3 \cdot 5^2 \cdot 7$. Now multiplying both sides of (1.1) by e, f and g respectively, we obtain

$$15e + 12ef + 12eg = 3e^2f^2 + 3e^2g^2 + 3ef^2g + 3efg^2 + 6e^2fg; \tag{1.3}$$

$$15f + 12ef + 12fg = 3e^2f^2 + 3g^2f^2 + 3e^2fg + 3efg^2 + 6ef^2g; \tag{1.4}$$

$$15g + 12eg + 12fg = 3e^2g^2 + 3f^2g^2 + 3e^2fg + 3ef^2g + 6efg^2. \tag{1.5}$$

By (1.3)+(1.4)+(1.5), we have

$$\begin{aligned}15(e+f+g) &= 12efg(e+f+g)+6(e^2f^2+e^2g^2+f^2g^2)-24(ef+eg+fg)\\ &= 12efg(e+f+g)+6(ef+eg+fg)^2-12efg(e+f+g)-24(ef+eg+fg)\\ &= 6(ef+eg+fg)^2-24(ef+eg+fg).\end{aligned}$$

Hence $2\cdot 60 = 3(2ef+2eg+2fg)^2-24(2ef+2eg+2fg)$, multiplying both sides by $35^2\cdot 48$. It follows that $2^6\cdot 3^2\cdot 5^3\cdot 7^2 \in Nil(R)$, so $2\cdot 3\cdot 5\cdot 7\in Nil(R)$.

Recall that a ring R is reduced if it contains no nonzero nilpotents (i.e., $Nil(R)=\{0\}$).

Lemma 2.2 Let R be a reduced ring. Then R has property $\mathcal{P}$ if and only if $R\cong R_1\times R_2\times R_3\times R_4$ where each R_i has property $\mathcal{P}$ with $2=0$ in R_1, $3=0$ in R_2, $5=0$ in R_3 and $7=0$ in R_4.

Proof It is clear that the direct product of rings with property $\mathcal{P}$ is also with property $\mathcal{P}$, so the sufficiency follows. We now prove the necessity. In view of Lemma 2.1, we get $2\cdot 3\cdot 5\cdot 7=0\in R$ since R is reduced. Thus, $2R\cap 3R\cap 5R\cap 7R=0$. By the Chinese Remainder Theorem,

$$R\cong R/2R\times R/3R\times R/5R\times R/7R.$$

Write $R_1=R/2R$, $R_2=R/3R$, $R_3=R/5R$ and $R_4=R/7R$. Then every R_1,R_2,R_3,R_4 has property $\mathcal{P}$ for $i=1,2,3,4$. Clearly, $2=0$ in R_1, $3=0$ in R_2, $5=0$ in R_3 and $7=0$ in R_4, and $R\cong R_1\times R_2\times R_3\times R_4$.

We use $J(R)$ to denote the Jacobson radical of a ring R.

Lemma 2.3 Let R be a reduced ring, and $p\in\{2,3,5,7\}$. The following are equivalent:

(1) R has property $\mathcal{P}$ with $p\in J(R)$.

(2) R satisfies the identity $x^p=x$ with $p=0$ in R.

(3) R is a subdirect product of $\mathbb{Z}_p$'s.

Proof (1) $\Rightarrow$ (2) By Lemma 2.1, $2\cdot 3\cdot 5\cdot 7=0\in R$. Since $p\in\{2,3,5,7\}$ and $p\in J(R)$, the rest three elements of $\{2,3,5,7\}$ are all units. So $p=0\in R$.

We first show that for any $h^3=h\in R$, $h^p-h=0\in R$. Indeed, if $p\in\{3,5,7\}$ then $h^p=h$; if $p=2$ then $(e^2-e)^2=0$ since $2=0\in R$, which yields $e^2-e=0$. Now, let $x\in R$. Then $x=e+f+g$ with $e^3=e, f^3=f, g^3=g$ and e,f,g all commute. From $p=0\in R$, we get $x^p-x=(e+f+g)^p-(e+f+g)=(e^p-e)+(f^p-f)+(g^p-g)=0$, and therefore, $x^p=x$.

(2) $\Leftrightarrow$ (3) follows from [8, Ex. 12.11].

(3) $\Rightarrow$ (1) By [3, Theorem 10], the result follows.

Combining Lemma 2.2 and Lemma 2.3, we have the following result.

Theorem 2.4 Let R be a reduced ring. The following are equivalent:

(1) A ring R has property $\mathcal{P}$.

(2) $R \cong R_1 \times R_2 \times R_3 \times R_4$, where R_1 is zero or a subdirect product of $\mathbb{Z}_2$'s, R_2 is zero or a subdirect product of $\mathbb{Z}_3$'s, R_3 is zero or a subdirect product of $\mathbb{Z}_5$'s and R_4 is zero or a subdirect product of $\mathbb{Z}_7$'s.

3 Matrix Rings

Recall that if K is a field, then every $n \times n$ matrix over K with $n \geq 2$ is similar to its rational canonical form. Let R be a ring and $a_0, a_1, ..., a_{n-1}$ be elements in R. The matrix

$$C = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_{n-2} \\ 0 & 0 & \cdots & 1 & a_{n-1} \end{pmatrix}$$

is called the companion matrix associated with $a_0, a_1, ..., a_{n-1}$. We will set $tr(C) = a_{n-1}$.

Lemma 3.1 [6, Lemma 4.3]. Let R be a ring and $n \geq 2$. For any idempotent $(n-1) \times (n-1)$ block E and for any $(n-1) \times 1$ block X over R, the $n \times n$ matrices

$$\begin{pmatrix} E & X \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} -E & X \\ 0 & 1 \end{pmatrix}$$

are tripotent.

Lemma 3.2 Let R be a ring and C be an $n \times n$ companion matrix associated with $a_0, a_1, ..., a_{n-1} \in R$. If $tr(C) = a_{n-1} \in \{3, 2, 1, 0\}$, then C is a sum of three tripotents in $M_n(R)$.

Proof It is clear if $n = 1$, so we are assuming $n \geq 2$. If $tr(C) = a_{n-1} \in \{2, 1, 0\}$, then by [7, Proposition 3], C is a sum of an idempotent and a tripotent, and so the result follows. Consequently, we can assume $tr(C) = 3$, and decompose $C = E + F + G$ as follows, where $E^3 = E$ is a consequence of Lemma 3.1, and F, G are tripotents.
If n is even, then

$$C = \begin{pmatrix} -1 & & & & & & a_0 \\ & -1 & 0 & & & & a_1 \\ & 1 & 0 & & & & a_2 \\ & & & \ddots & & & \vdots \\ & & & & -1 & 0 & a_{n-3} \\ & & & & 1 & 0 & a_{n-2} \\ & & & & & & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & & & \\ 1 & 1 & & & \\ & & \ddots & & \\ & & & 0 & 0 \\ & & & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & 1 \end{pmatrix}.$$

If n is odd, then

$$C = \begin{pmatrix} -1 & 0 & & & & a_0 \\ 1 & 0 & & & & a_1 \\ & & \ddots & & & \vdots \\ & & & -1 & 0 & a_{n-3} \\ & & & 1 & 0 & a_{n-2} \\ & & & & & 1 \end{pmatrix} + \begin{pmatrix} 1 & & & & & \\ & 0 & 0 & & & \\ & 1 & 1 & & & \\ & & & \ddots & & \\ & & & & 0 & 0 \\ & & & & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & 1 \end{pmatrix}.$$

According to Lemma 3.2, we come to the following result. We use $U(R)$ to denote the set of all units of a ring R.

Theorem 3.3 Let R be an integral domain. The following are equivalent:

(1) Every matrix in $M_n(R)$ is a sum of three tripotents for any integer $n \geq 1$.

(2) Every matrix in $M_2(R)$ is a sum of three tripotents.

(3) $R \cong \mathbb{Z}_p$ for $p = 2, 3, 5$ or 7.

Proof (1) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (3) Note that the integral domain R can be embed into its field of fraction. For any $E^3 = E \in M_2(R)$. In view of [6, Lemma 3.1], $tr(E) \in \{-2, -1, 0, 1, 2\} \subseteq \mathbb{Z} \cdot 1_R$. Let $a \in R$, and let $A = aE_{11}$, where E_{11} is a matrix with (1, 1)-entry 1 and zeros elsewhere. By hypothesis, there exist tripotents $A_1, A_2, A_3 \in M_2(R)$ such that

$$A = aE_{11} = A_1 + A_2 + A_3.$$

Then $a = tr(A) = tr(A_1) + tr(A_2) + tr(A_3) \in \mathbb{Z} \cdot 1_R$. So R is a finite ring since each $tr(A_i)$ is of the form $k \cdot 1_R$ and $k \in \{-2, -1, 0, 1, 2\}$, $i = 1, 2, 3$. Hence, R is a field.

Now, let $4I_2 = E + F + G$ where $E^3 = E$, $F^3 = F$ and $G^3 = G$. Then

$$8 = tr(4I_2) = tr(E) + tr(F) + tr(G) \in 2\mathbb{Z} \cdot 1_R.$$

Note that $tr(E) + tr(F) + tr(G) \in \{-6, -4, -2, 0, 2, 4, 6\}$. So we have $2m = 0$ and $m \in \{1, 2, \ldots, 7\}$. If $2 \in U(R)$, then R is isomorphic to one of the fields $\mathbb{Z}_3, \mathbb{Z}_5$ or $\mathbb{Z}_7$. If $2 \notin U(R)$, then $R \cong \mathbb{Z}_2$. It is clear that every matrix over $\mathbb{Z}_2$ is a sum of three tripotents. Therefore, $R \cong \mathbb{Z}_p$ where $p = 2, 3, 5$ or 7.

(3) $\Rightarrow$ (1) By Lemma 3.2, if $tr(A) \in \{3, 2, 1, 0\}$ then A is a sum of three tripotents. Notice that a matrix A is a sum of three tripotents if and only if $-A$ is a sum of three tripotents. So the case for $tr(A) \in \{-3, -2, -1\}$ follows immediately. Thus, every matrix in $M_n(R)$ is a sum of three tripotents.

Now, combining Theorem 2.4 and Theorem 3.3, we have the following result.

Proposition 3.4 Let R be a reduced ring with property $\mathcal{P}$. Then every $n \times n$ matrix over R is a sum of three tripotents.

Proof In view of Theorem 2.4, R is isomorphic to the product of R_1, R_2, R_3, R_4, where R_1 is a subdirect product of $\mathbb{Z}_2$'s, R_2 is a subdirect product of $\mathbb{Z}_3$'s, R_3 is a subdirect product of $\mathbb{Z}_5$'s and R_4 is a subdirect product of $\mathbb{Z}_7$'s.

Take $M \in M_n(R_1)$. Let S be a subring of R_1 generated by the elements of the matrix M. So the ring S is finite, and whence S is a finite direct product of $\mathbb{Z}_2$'s. By Theorem 3.3, every matrix over S is a sum of three tripotents. Hence, $M \in M_n(R_1)$ is a sum of three tripotents. A similar argument as the above reveals that if $M \in M_n(R_i)$, then M is also a sum of three tripotents for $i = 2, 3, 4$.

Let $A \in M_n(R)$. Write $A = (A_1, A_2, A_3, A_4) \in M_n(R_1) \times M_n(R_2) \times M_n(R_3) \times M_n(R_4)$, where $A_i \in M_n(R_i), i = 1, 2, 3, 4$. Clearly, A is a sum of three tripotents in $M_n(R)$.

The following example is useful to reflect conclusions.

Example 3.5 Every $n \times n$ matrix over $\mathbb{Z}_{210}$ is a sum of three tripotents.

Proof By the Chinese Reminder Theorem, $\mathbb{Z}_{210} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7$. Then $\mathbb{Z}_{210}$ is a reduced ring. By Theorem 2.4, $\mathbb{Z}_{210}$ has property $\mathcal{P}$. In view of Proposition 3.4, every $n \times n$ matrix over $\mathbb{Z}_{210}$ is a sum of three tripotents.

Proposition 3.6 Let R be a reduced ring. The following are equivalent:

(1) Every matrix in $M_n(R)$ is a sum of three tripotents.

(2) $R \cong S \times T$ where $x^5 = x$ for all $x \in S$ and $x^7 = x$ for all $x \in T$ with $7 = 0$.

Proof (1) $\Rightarrow$ (2) Suppose that every matrix in $M_n(R)$ is a sum of three tripotents. As R is a reduced ring, R is a subdirect product of integral domain $\{R_\alpha\}$. Since $M_n(R_\alpha)$ is a homomorphic image of $M_n(R)$, it follows that every matrix in $M_n(R_\alpha)$ is a sum of three tripotents. Thus, by Theorem 3.3 $R_\alpha \cong \mathbb{Z}_p$ for $p \in \{2, 3, 5, 7\}$. Let $S = R_1 \times R_2 \times R_3$ where R_1 is a subdirect product of $\mathbb{Z}_2$'s, R_2 is a subdirect product of $\mathbb{Z}_3$'s, R_3 is a subdirect product of $\mathbb{Z}_5$'s and T is a subdirect product of $\mathbb{Z}_7$'s. So $R \cong S \times T$ where $x^5 = x$ for all $x \in S$ and $x^7 = x$ for all $x \in T$ with $7 = 0$.

(2) $\Rightarrow$ (1) Notice that $M_n(R) \cong M_n(S) \times M_n(T)$. By [6, Theorem 4.7], every matrix in $M_n(S)$ is a sum of two tripotents. In view of [8, Ex. 12.11], T is a subdirect product of $\mathbb{Z}_7$'s. So by the proof of Proposition 3.4, every matrix in $M_n(T)$ is a sum of three tripotents. Therefore every matrix in $M_n(R)$ is a sum of three tripotents.

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约化环上可分解为三个三幂等矩阵之和的矩阵

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摘要: 本文研究了每个元素均可表示为三个互相交换的三幂等元之和的约化环, 给出了整环上任意 n 阶矩阵均可表示为三个三幂等矩阵之和的判定条件, 证明了对于整环 R, $M_n(R)$ 中任意矩阵可分解为三个三幂等矩阵之和当且仅当 $R \cong \mathbb{Z}_p$, 其中 $p = 2, 3, 5$ 或 7.

关键词: 三幂等元; 友矩阵; 约化环; 矩阵环

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