Vol. 44 (2024)
No.5

数 学 杂 志
J. of Math. (PRC)

The UNIFORM C^0 ESTIMATE AND WEIGHTED ESTIMATE OF GENERALIZED CHRISTOFFEL-MINKOWSKI PROBLEMS

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Abstract: In this paper, we consider generalized Christoffel-Minkowski problems as follows

$$\frac{\sigma_k(u_{ij}+u\delta_{ij})}{\sigma_l(u_{ij}+u\delta_{ij})}=u^{p-1}f(x),\quad x\in\mathbb{S}^n,$$

where $0\le l<k\le n$, $p-1>0$ and f is positive, and we establish the weighted gradient estimate and uniform C^0 estimate for the positive convex even solutions, which is a generalization of Guan-Xia [1] and Guan [2].

Keywords: weighted gradient estimate; convex solution; minkowski type problem
2010 MR Subject Classification: 35J60; 35B45
Document code: A **Article ID:** 0255-7797(2024)05-0397-09

1 Introduction

In this paper, we consider the following form of Hessian quotient type equation

$$\frac{\sigma_k(u_{ij}+u\delta_{ij})}{\sigma_l(u_{ij}+u\delta_{ij})}=u^{p-1}f(x),\quad x\in\mathbb{S}^n,\tag{1.1}$$

where σ_k is the k-th elementary symmetric function and u_{ij} is the second order covariant derivative of u with respect to an orthonormal frame on $\mathbb{S}^n$, and a function $u\in C^2(\mathbb{S}^n)$ is called convex if

$$(u_{ij}+u\delta_{ij})>0,\quad on\quad \mathbb{S}^n.\tag{1.2}$$

In fact, the equation (1.1) corresponds to a class of Lp Minkowski type problem. The Lp Minkowski problem introduced by Lutwak[3] is a generalisation of the classical Minkowski problem.

Given a Borel measure μ on the unit sephere $\mathbb{S}^n$, the Lp Minkowski problem concerns with the existence of a unique convex body $\mathbb{K}$ in $\mathbb{R}^{n+1}$ so that μ is the Lp surface area measure of $\mathbb{K}$,

$$d\mu=u^{1-p}dS_k,\tag{1.3}$$

* **Received date:** 2024-03-19 **Accepted date:** 2024-05-21
Foundation item: Supported by National Natural Science Foundation of China(12171260).
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where S_k is the ordinary surface area measure of $\mathbb{K}$ and $u : \mathbb{S}^n \mapsto \mathbb{R}$ is the support function of $\mathbb{K}$. In the case of $p = 1$, the Lp Minkowski problem reduces to the classical Minkowski problem. The classical Minkowski problem was considered by Minkowski in [4], which is to find the necessary and sufficient conditions on a given measure so that it is exactly the surface area measure of a convex body. The classical Minkowski problem corresponds to solve a Monge-Ampère type equation

$$\det(u_{ij} + u\delta_{ij}) = f(x), \quad x \in \mathbb{S}^n. \tag{1.4}$$

Many important contributions to Minkowski problems were done by Minkowski [5, 6], Alexandrov [7], Nirenberg [8], and Cheng-Yau [9], et al. Since the classical Minkowski problem, many Minkowski type problems have been introduced and extensively studied.

The L_p Minkowski problem ($p \geq 1$) is the problem of prescribing L_p surface area measure which was introduced by Lutwak [3], and is to solve a Hessian type geometric PDE

$$\det(u_{ij} + u\delta_{ij}) = u^{p-1} f(x), \quad x \in \mathbb{S}^n, \tag{1.5}$$

and many important contributions to L_p Minkowski problems were done by Lutwak[3], Chou-Wang [10], Guan-Lin [11], Böröczky-Lutwak-Yang-Zhang[12], Lutwak-Oliker[13] and Lutwak-Yang-Zhang[14] et al.

The Christoffel-Minkowski problem concerns with the existence of convex bodies with prescribed k-th surface area measure, which corresponds to finding convex solutions of the following geometric PDE

$$\sigma_k(u_{ij} + u\delta_{ij}) = f(x), \quad x \in \mathbb{S}^n. \tag{1.6}$$

Important contributions to Christoffel-Minkowski problems were done by Guan-Ma [15] and Guan-Ma-Zhou [16] et al. The key tool is the constant rank theorem for fully nonlinear partial differential equations.

The L_p-Christoffel-Minkowski problem corresponds to finding convex solutions of the following geometric PDE

$$\sigma_k(u_{ij} + u\delta_{ij}) = u^{p-1} f(x), \quad x \in \mathbb{S}^n. \tag{1.7}$$

Equation (1.7) has been studied by Hu-Ma-Shen [17] in the case $p - 1 \geq k$, and Guan-Xia[1] for $1 < p < k + 1$ and even prescribed data, by using the constant rank theorem.

This article is organized as follows. In Section 2, we present some properties of $\sigma_k(\lambda)$ in Gårding's cone Γ_k , which are important to the a priori estimates. In Section 3, we prove Theorem 3.1. At last we prove Theorem 4.2 in Section 4.

2 Preliminaries

In this section, we recall the definition and some basic properties of elementary symmetric functions, which could be found in [18].

Definition 2.1 For any $k = 1, 2, \dots, n$, we set

$$\sigma_k(\lambda) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}, \qquad \text{for any} \quad \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n. \tag{2.1}$$

For convenience, let $\sigma_0 = 1$ and $\sigma_k = 0$ for $k > n$.

Denote by $\sigma_k(\lambda\,|i)$ the symmetric function with $\lambda_i = 0$ and $\sigma_k(\lambda\,|ij)$ the symmetric function with $\lambda_i = \lambda_j = 0$.

The following standard formulas of elementary symmetric functions are needed.

Proposition 2.2 Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ and $k = 0, 1, \dots, n$. Then

$$\begin{aligned} &\sigma_k(\lambda) = \sigma_k(\lambda|i) + \lambda_i\sigma_{k-1}(\lambda|i), \quad \forall\, 1 \le i \le n, \\ &\sum_{i=1}^{n} \lambda_i\sigma_{k-1}(\lambda|i) = k\sigma_k(\lambda), \\ &\sum_{i=1}^{n} \sigma_k(\lambda|i) = (n-k)\sigma_k(\lambda). \end{aligned}$$

Proposition 2.3 Let $W = \{W_{ij}\}$ be an $n \times n$ symmetric matrix and $\lambda(W) = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be the eigenvalues of W. If $W = \{W_{ij}\}$ is diagonal and $\lambda_i = W_{ii}$, then

$$\begin{aligned} &\frac{\partial \lambda_i}{\partial W_{ii}} = 1, \quad \frac{\partial \lambda_k}{\partial W_{ij}} = 0, \text{otherwise}, \\ &\frac{\partial^2 \lambda_i}{\partial W_{ij}\partial W_{ji}} = \frac{1}{\lambda_i - \lambda_j}, \quad i \ne j \text{ and } \lambda_i \ne \lambda_j, \\ &\frac{\partial^2 \lambda_i}{\partial W_{kl}\partial W_{pq}} = 0, \text{otherwise}. \end{aligned}$$

Definition 2.1 can be extended to symmetric matrices by letting $\sigma_k(W) = \sigma_k(\lambda(W))$, where $\lambda(W) = (\lambda_1(W), \lambda_2(W), \cdots, \lambda_n(W))$ are the eigenvalues of the symmetric matrix W. We also denote by $\sigma_k(W\,|i\,)$ the symmetric function with W deleting the i-row and i-column and $\sigma_k(W\,|ij\,)$ the symmetric function with W deleting the i, j-rows and i, j-columns. Then we have the following properties.

Proposition 2.4 If $W = \{W_{ij}\}$ is diagonal and m is a positive integer, then

$$\frac{\partial \sigma_m(W)}{\partial W_{ij}} = \begin{cases} \sigma_{m-1}(W\,|i\,), & \text{if } i = j, \\ 0, & \text{if } i \ne j, \end{cases}$$

and

$$\frac{\partial^2 \sigma_m(W)}{\partial W_{ij}\partial W_{kl}} = \begin{cases} \sigma_{m-2}(W\,|ik\,), & \text{if } i = j, k = l, i \ne k, \\ -\sigma_{m-2}(W\,|ik\,), & \text{if } i = l, j = k, i \ne j, \\ 0, & \text{otherwise}. \end{cases}$$

Recall that the classic Gårding's cone is defined as

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_i(\lambda) > 0, \forall 1 \le i \le k\},$$

and the following properties are well known.

Proposition 2.5 For $\lambda \in \Gamma_k$ and $k > l \ge 0$, $r > s \ge 0$, $k \ge r$, $l \ge s$, we have

$$\left[\frac{\sigma_k(\lambda)/C_n^k}{\sigma_l(\lambda)/C_n^l}\right]^{\frac{1}{k-l}} \le \left[\frac{\sigma_r(\lambda)/C_n^r}{\sigma_s(\lambda)/C_n^s}\right]^{\frac{1}{r-s}},$$

with equality if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n > 0$.

Proposition 2.6 (1) Γ_k are convex cones, and $\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n$.

(2) If $\lambda = (\lambda_1, \dots, \lambda_n) \in \Gamma_k$ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, then $\lambda_k > 0$,

$$\sigma_{k-1}(\lambda|n) \ge \sigma_{k-1}(\lambda|n-1) \ge \cdots \ge \sigma_{k-1}(\lambda|1) > 0.$$

(3) If $\lambda = (\lambda_1, \dots, \lambda_n) \in \Gamma_k$, then $\sigma_k(\lambda)^{\frac{1}{k}}$ and $\left[\frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}\right]^{\frac{1}{k-l}}$ $(0 \le l < k \le n)$ are concave with respect to λ. Equivalently, for any $(\xi_1, \dots, \xi_n)$, we have

$$\sum_{i,j=1}^n \frac{\partial^2 \left[\frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}\right]}{\partial\lambda_i \partial\lambda_j} \xi_i\xi_j \le (1 - \frac{1}{k-l}) \frac{[\sum_{i=1}^n \frac{\partial \left[\frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}\right]}{\partial\lambda_i} \xi_i]^2}{\frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}}.$$

3 Weighted Gradient Estimate

Theorem 3.1 Suppose $u \in C^3(\mathbb{S}^n)$ is a positive convex solution of the equation (1.1), where $0 \le l < k \le n$, $p - 1 > 0$, and $f \in C^1(\mathbb{S}^n)$ is a smooth, positive function. Then we have the weighted gradient estimate

$$\frac{|\nabla u|^2}{u^\gamma} \le A(\max_{\mathbb{S}^n} u)^{2-\gamma}, \quad \forall\, x \in \mathbb{S}^n, \tag{3.1}$$

where $\gamma = \min\{1, \frac{p-1}{k-l}\}$ and A is a positive constant depending on n, k, l, p, $min_{\mathbb{S}^n} f$ and $\|f\|_{C^1}$.

Proof Following the idea of [1, 2], we prove Theorem 3.1 by a contradiction argument. Let

$$\Phi = \frac{|\nabla u|^2}{u^\gamma},$$

where $\gamma = \min\{1, \frac{p-1}{k-l}\} \in (0, 1]$. Denote $M_u = \max_{\mathbb{S}^n} u$. Assume Φ attains maximum at x_0. We may choose an orthonormal frame on $\mathbb{S}^n$ such that

$$u_1 = |\nabla u|, \quad \text{and } \{u_{ij}\}_{2 \le i,j \le n} \text{ is diagonal.}$$

In the following, we compute at x_0. Then we have at x_0,

$$0 = (\log \Phi)_i = \frac{2u_k u_{ki}}{|\nabla u|^2} - \gamma\frac{u_i}{u}, \tag{3.2}$$

hence we have

$$\begin{aligned} u_{11} &= \frac{\gamma}{2}\frac{u_1^2}{u}, \\ u_{1i} &= 0, \quad i = 2, \cdots, n. \end{aligned} \tag{3.3}$$

Hence $\{u_{ij}\}_{1\leq i,j\leq n}$ is diagonal, $\{b_{ij}\}_{1\leq i,j\leq n}$ is diagonal with $b_{ij} := u_{ij}+u\delta_{ij}$, and $\{F^{ij}\}_{1\leq i,j\leq n}$ is diagonal, where

$$F^{ij} = \frac{\partial\left[\frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}\right]}{\partial b_{ij}}. \tag{3.4}$$

Also, at x_0, we have

$$\begin{aligned} 0 &\geq F^{ii}(\log\Phi)_{ii} \\ &= F^{ii}\frac{2u_{ii}^2 + 2u_l u_{lii}}{|\nabla u|^2} - \gamma\frac{F^{ii}u_{ii}}{u} + \gamma(1-\gamma)\frac{F^{ii}u_i^2}{u^2} \\ &= \frac{2F^{ii}u_{ii}^2}{u_1^2} + \frac{2F^{ii}u_1(b_{ii,1} - u_i\delta_{1i})}{u_1^2} - \gamma\frac{F^{ii}u_{ii}}{u} + \gamma(1-\gamma)\frac{F^{ii}u_i^2}{u^2} \\ &\geq \frac{2F^{ii}u_{ii}^2}{u_1^2} + 2(p-1)u^{p-2}f + \frac{2u^{p-1}f_1}{u_1} - 2F^{11} - \gamma\frac{F^{ii}b_{ii}}{u} + 2(1-\gamma)\frac{F^{11}u_{11}}{u} \\ &\geq 2F^{11}(\frac{u_{11}^2}{u_1^2} - 1) + 2(p-1)\frac{F}{u} + \frac{2u^{p-1}f_1}{u_1} - \gamma(k-l)\frac{F}{u}. \end{aligned} \tag{3.5}$$

By (3.3), if $A \geq \frac{4}{\gamma^2}$,

$$\frac{u_{11}^2}{u_1^2} - 1 \geq \frac{\gamma^2}{4}A\frac{M_u}{u} - 1 \geq 0. \tag{3.6}$$

By the definition of Φ, we have

$$\begin{aligned} \frac{2u^{p-1}f_1}{u_1} &\geq -Cu^{p-1}\Phi^{-\frac{1}{2}}u^{-\frac{\gamma}{2}} \\ &\geq -\frac{C}{\sqrt{A}}M_u^{-1+\frac{\gamma}{2}}u^{p-1}u^{-\frac{\gamma}{2}} \\ &\geq -\frac{C}{\sqrt{A}}u^{p-1}u^{-1} \\ &\geq -\frac{C}{\sqrt{A}}\frac{F}{u}. \end{aligned} \tag{3.7}$$

Combining (3.5), (3.6) and (3.7), we have

$$\begin{aligned} 0 &\geq F^{ii}(\log\Phi)_{ii} \\ &\geq 2F^{11}(\frac{u_{11}^2}{u_1^2} - 1) + 2(p-1)\frac{F}{u} - \frac{C}{\sqrt{A}}\frac{F}{u} - \gamma(k-l)\frac{F}{u} \\ &\geq [2(p-1) - \frac{C}{\sqrt{A}} - \gamma(k-l)]\frac{F}{u} \\ &> 0, \end{aligned} \tag{3.8}$$

if we choose $A > \frac{C^2}{p-1}$. This is a contradiction. Hence Theorem 3.1 holds.

Remark 3.2 If we choose

$$\Phi = \frac{|\nabla u|^2}{(u - \min u)^\gamma}, \tag{3.9}$$

following the idea of [1, 2], we can prove

$$\frac{|\nabla u(x)|^2}{(u(x) - \min u)^\gamma} \le A \max_{\mathbb{S}^n} u^{2-\gamma}, \quad \forall\, x \in \mathbb{S}^n. \tag{3.10}$$

4 The Uniform C^0 Estimate

Following Lemma 3.1 in [2], we can get the positive lower bound and upper bound of u. In fact, we can prove the following lemma (see [19]).

Lemma 4.1 Assume u is a positive even convex function on $\mathbb{S}^n$ satisfying condition

$$\frac{|\nabla u|^2}{u^\gamma}(x) \le A \max_{\mathbb{S}^n} u^{2-\gamma}, \quad \forall\, x \in \mathbb{S}^n, \tag{4.1}$$

for some $\gamma \in (0, 1)$ and $A > 0$. Then the following non-collapsing estimate holds,

$$\frac{\max u}{\min u} \le C, \tag{4.2}$$

where C depends only on n, γ and A.

Proof The proof is similar with [2]. For complete, we give the proof here.

Let Ω be the convex body with support function u. Since u is even, the center of mass of Ω is the origin. From the John Lemma, there is an ellipsoid E centered at the origin, such that

$$E \subset \Omega \subset (n+1)^{\frac{3}{2}} E.$$

Write E

$$\frac{x_1^2}{b_1^2} + \cdots + \frac{x_{n+1}^2}{b_{n+1}^2} \le 1,$$

with longest axis b_1, and the shortest axis b_{n+1}. We have

$$b_1 \le \max u \le (n+1)^{\frac{3}{2}} b_1, \quad b_{n+1} \le \min u \le (n+1)^{\frac{3}{2}} b_{n+1}.$$

Recall that the support function of E is

$$u_E(x) = \sqrt{b_1^2 x_1^2 + \cdots + b_{n+1}^2 x_{n+1}^2}, \quad x \in \mathbb{S}^n,$$

and then

$$u_E(x) \le u(x) \le (n+1)^{\frac{3}{2}} u_E(x), \quad x \in \mathbb{S}^n.$$

Restrict the support function u_E to the slice $S := \{x \in \mathbb{S}^n | x = (x_1, 0, \cdots, 0, x_{n+1})\}$. Set

$$v(s) = u_E(s, 0, \cdots, 0, \sqrt{1-s^2}) = \sqrt{b_1^2 s^2 + b_{n+1}^2(1-s^2)} \geq b_1 s, \quad s \in [0,1].$$

Hence

$$v(t(\frac{b_{n+1}}{b_1})^{\frac{2-\gamma}{2}}) \geq t b_1^{\frac{\gamma}{2}} b_{n+1}^{\frac{2-\gamma}{2}},$$

for $t \in [0, (\frac{b_1}{b_{n+1}})^{\frac{2-\gamma}{2}}]$. On the other hand, set $q(s) = u(s, 0, \cdots, 0, \sqrt{1-s^2})^{\frac{2-\gamma}{2}}$. By the weighted gradient estimate

$$|\frac{d}{ds} q(s)| \leq \frac{|\nabla u(s, 0, \cdots, 0, \sqrt{1-s^2})|}{u^{\frac{\gamma}{2}}} \leq A^{\frac{1}{2}} (\max u)^{\frac{2-\gamma}{2}} \leq A^{\frac{1}{2}} (n+1)^{\frac{3(2-\gamma)}{4}} b_1^{\frac{2-\gamma}{2}}.$$

Hence

$$\begin{aligned} q(t(\frac{b_{n+1}}{b_1})^{\frac{2-\gamma}{2}}) \leq & q(0) + A^{\frac{1}{2}} (n+1)^{\frac{3(2-\gamma)}{4}} b_1^{\frac{2-\gamma}{2}} \cdot t(\frac{b_{n+1}}{b_1})^{\frac{2-\gamma}{2}} \\ \leq & [(n+1)^{\frac{3}{2}} b_{n+1}]^{\frac{2-\gamma}{2}} + t A^{\frac{1}{2}} (n+1)^{\frac{3(2-\gamma)}{4}} b_{n+1}^{\frac{2-\gamma}{2}} \\ = & (n+1)^{\frac{3(2-\gamma)}{4}} [1 + t A^{\frac{1}{2}}] b_{n+1}^{\frac{2-\gamma}{2}}. \end{aligned}$$

Thus

$$u(t(\frac{b_{n+1}}{b_1})^{\frac{2-\gamma}{2}}, 0, \cdots, 0, \sqrt{1 - t^2(\frac{b_{n+1}}{b_1})^{2-\gamma}}) \leq (n+1)^{\frac{3}{2}} [1 + t A^{\frac{1}{2}}]^{\frac{2}{2-\gamma}} b_{n+1}.$$

Since $u_E(x) \leq u(x)$, we obtain

$$t b_1^{\frac{\gamma}{2}} b_{n+1}^{\frac{2-\gamma}{2}} \leq (n+1)^{\frac{3}{2}} [1 + t A^{\frac{1}{2}}]^{\frac{2}{2-\gamma}} b_{n+1}.$$

Let $t = A^{-\frac{1}{2}}$, then we have

$$\frac{b_1}{b_{n+1}} \leq (n+1)^{\frac{3}{\gamma}} 2^{\frac{4}{(2-\gamma)\gamma}} A^{\frac{1}{\gamma}}.$$

Hence

$$\frac{\max u}{\min u} \leq (n+1)^{\frac{3}{2}} \frac{b_1}{b_{n+1}} \leq (n+1)^{(\frac{3}{2} + \frac{3}{\gamma})} 2^{\frac{4}{(2-\gamma)\gamma}} A^{\frac{1}{\gamma}}.$$

Now we start to prove Theorem 4.2.

Theorem 4.2 Suppose $u \in C^3(\mathbb{S}^n)$ is a positive convex even solution of the equation (1.1), where $0 \leq l < k \leq n$, $p - 1 > 0$, and $f \in C^1(\mathbb{S}^n)$ is a positive even function. Then we have the following uniform C^0 estimate

$$0 < c_0 \leq u \leq C_0, \quad \text{if } p - 1 \neq k - l, \tag{4.3}$$

$$0 < 1 \leq \frac{u}{\min u} \leq C_0, \quad \text{if } p - 1 = k - l, \tag{4.4}$$

where c_0 and C_0 are two positive constants depending only on n, k, l, p, $\min_{\mathbb{S}^n} f$ and $||f||_{C^1}$.

Proof When $p-1>k-l$, we can get directly from the equation (1.1)

$$\min u \geq \Big[\frac{C_n^k}{C_n^l}\frac{1}{\min f}\Big]^{\frac{1}{p-1-(k-l)}}, \quad \max u \geq \Big[\frac{C_n^k}{C_n^l}\frac{1}{\max f}\Big]^{\frac{1}{p-1-(k-l)}}.$$

When $p-1=k-l$, Theorem 4.2 holds from Theorem 3.1 and Lemma 4.1. When $0<p-1<k-l$, we know from the equation (1.1)

$$\min u \leq \Big[\frac{C_n^k}{C_n^l}\frac{1}{\max f}\Big]^{\frac{1}{p-1-(k-l)}}, \quad \max u \geq \Big[\frac{C_n^k}{C_n^l}\frac{1}{\min f}\Big]^{\frac{1}{p-1-(k-l)}}.$$

Remark 4.3 From [20], the constant rank theorem holds if $f^{-\frac{1}{p-1+k-l}}$ is spherical convex and $p-1\geq 0$. From [16], the existence theorem of the positive convex even solutions of (1.1) holds by the method of degree theory.

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广义Christoffel-Minkowski问题的一致C^0估计和加权估计

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摘 要: 本文考虑了广义Christoffel-Minkowski问题

$$\frac{\sigma_k(u_{ij}+u\delta_{ij})}{\sigma_l(u_{ij}+u\delta_{ij})} = u^{p-1}f(x), \quad x \in \mathbb{S}^n,$$

其中$0 \le l < k \le n$是整数, $p-1>0$, f是一个正函数. 对于上述方程的的正凸偶解, 本文建立了解的加权梯度估计和一致C^0估计. 这是对Guan-Xia[1]和Guan[2]中结果的一般化.

关 键 词: 加权梯度估计; 凸解; Minkowski型问题

MR(2010)主题分类号: 35J60; 35B45 中图分类号: O175.25