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FINITE RANGE SET WITH TRUNCATED MULTIPLICITY FOR MEROMORPHIC FUNCTIONS ON SOME COMPLEX DISC

WANG Yu-ting, CAO Hong-zhe
(*Mathematics Department of Nanchang University, Nanchang 330031, China*)

Abstract: In this paper, we consider the truncated multiplicity finite range set problem of meromorphic functions on some complex disc. By using the value distribution theory of meromorphic functions, we establish a second main theorem for meromorphic functions with finite growth index which share meromorphic functions (may not be small functions). As its application, we also extend the result of a finite range set with truncated multiplicity.

Keywords: meromorphic functions; finite growth index; complex disc; finite range set; truncated multiplicity

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1 Introduction

Uniqueness problem of meromorphic functions is one of the important research directions in Nevanlinna theory. In the 1920s, R. Nevanlinna [1] proved the famous five-value theorem that if two non-constant meromorphic functions f and g share five distinct values, then $f(z) \equiv g(z)$. Later on, the research on the uniqueness problem has been widely promoted, such as references [2-4]. Among them, Fujimoto [5] proposed the definition of a finite range set, and got the following result.

A finite subset S of $\mathbb{C}$ is said to be a unique range set for meromorphic functions if $f^*(S) = g^*(S)$ implies $f = g$ for arbitrary nonconstant meromorphic functions f and g on $\mathbb{C}$, where $f^*(S)$ and $g^*(S)$ denote the pull-backs of S considered as a divisor, namely, the inverse images of S counted with multiplicities by f and g respectively. A nonconstant monic polynomial $P(w)$ is called a uniqueness polynomial for meromorphic functions if for any nonconstant meromorphic functions f and g on $\mathbb{C}$, the equation $P(f) = cP(g)$ implies $f = g$, where c is a nonzero constant that possibly depends on f and g.

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Biography: Wang Yuting(2000–), female, postgraduate, major in complex analysis. E-mail:1422649627@qq.com;

Corresponding author: Cao Hongzhe. E-mail: hongzhecao@126.com.

For $S=\{a_1,\cdots,a_q\}$, we consider the polynomial

$$P_S(w)=(w-a_1)\cdots(w-a_q). \tag{1.1}$$

Obviously, if S is a unique range set of meromorphic functions, then $P_S(w)$ is a uniqueness polynomial of meromorphic functions.

Definition 1.1 [5, Definition1.2] If for any given nonconstant meromorphic function g, there exist only finitely many nonconstant meromorphic functions f such that $f^*(S)=g^*(S)$, then a finite subset S of $\mathbb{C}$ is called a finite range set for meromorphic functions.

According to the above definition, Fujimoto obtained the following result.

Theorem 1.2 [5, Theorem1.3] Take a finite set $S=\{a_1,\cdots,a_q\}$ and assume that for the polynomial $P_S(w)$ defined by (1.1), $P'_S(w)$ has exactly k distinct zeros. If $P_S(w)$ is a uniqueness polynomial for meromorphic functions and $q>k+2$, then S is a finite range set for meromorphic functions. More precisely, for an arbitrarily given nonconstant meromorphic function g, there exist at most $\frac{2q-2}{q-k-2}$ meromorphic functions f such that $f^*(S)=g^*(S)$.

It is an interesting question whether meromorphic functions on other domains also have corresponding results. Recently, Quang [6] weakened the condition $f^*(S)=g^*(S)$, and obtained the corresponding results for meromorphic functions on annulus $\mathbb{A}(R_0)$. In the light of Quang [6], we consider the meromorphic functions on the complex disc with finite growth index and obtain some results.

For $0<R\leq\infty$, we set a complex disc $\Delta(R)=\{z\in\mathbb{C};|z|<R\}$. According to Ru and Sibony [7], the growth index of meromorphic function f on $\Delta(R)$ is defined by

$$c_f=\inf\{c>0;\int_0^R\exp(cT(r,f))dr=+\infty.\}$$

Obviously, $R=+\infty$, then $c_f=0$. For convenient, if $\{c>0;\int_0^R\exp(cT(r,f))dr=+\infty.\}=\emptyset$, we will set $c_f=+\infty$. All the meromorphic functions on $\Delta(R)$ we discussed in this paper have finite growth index.

Definition 1.3 Let $S=\{a_1,\cdots,a_q\}$ be a set of distinct values in $\mathbb{C}$ and let $\ell_i(i=1,\cdots,q)$ be some positive integers (may be $+\infty$). Two meromorphic functions f and g on $\Delta(R)$ are said to share S with truncated multiplicity if

$$\sum_{i=1}^q\min\{\ell_i,v_{f-a_i}\}=\sum_{i=1}^q\min\{\ell_i,v_{g-a_i}\}.$$

If for an arbitrarily given meromorphic function g on $\Delta(R)$, there exist finitely many meromorphic functions f on $\Delta(R)$ only such that f and g share S with truncated multiplicity, then the set S is said to be a finite range set with truncated multiplicity for meromorphic functions on $\Delta(R)$.

We note that $\ell_i=+\infty(i=1,\cdots,q)$, Definition 1.3 implies Definition 1.1. Using Quang's method in [6], we obtain the following second main theorem.

Theorem 1.4 Let f be a non-constant meromorphic function on $\Delta(R)$ and let $a_1, \cdots, a_q$ be $q (q \geq 5)$ distinct meromorphic functions (may be equal to ∞). Let $\gamma(r)$ be a non-negative measurable function defined on $(0, R)$ with $\int_0^R \gamma(r)dr = +\infty$. Then, for every $\varepsilon > 0$,

$$\|_E \ \frac{2q}{5} T(r,f) \leq \sum_{i=1}^{q} \overline{N}(r, v^0_{f-a_i}) + 35 \sum_{i=1}^{q} T(r, a_i) + 17((1+\varepsilon)\log\gamma(r) + \varepsilon \log r) + o(T(r,f)).$$

Here and later on, we use the notation $\|_E \ P$ to say that the assertion P holds for all $r \in (0; R)$ outside a subset E of $(0; R)$ with $\int_0^R \gamma(r)dr < +\infty$.

As its application, we obtain the results as follows.

Theorem 1.5 Take a finite set $S = \{a_1, \cdots, a_q\}$ of q distinct values in $\mathbb{C}$ and assume that for the polynomial $P_S(w)$ defined by (1.1), $P'_S(w)$ has exactly k distinct zeros. Let $\ell_i (1 \leq i \leq q)$ be some positive integers (may be $\ell_i = +\infty$). If $P_S(w)$ is a uniqueness polynomial for meromorphic functions on $\Delta(R)$, then any given nonconstant meromorphic function g on $\Delta(R)$ with finite growth index, if $\sum_{f^*(S)=g^*(S)} c_f < \frac{1}{372}\{2(q-1) - 5(k+1+\sum_{i=1}^{q} \frac{35}{\ell_i})\}$, then the number of elements in set $A = \{f | f^*(S) = g^*(S)\}$ does not exceed four, namely, S is a finite range with truncated multiplicity.

Remark 1.6 If $R = +\infty$, we have $c_{f_i} = 0 (1 \leq i \leq q)$, let $\ell_i = \ell (1 \leq i \leq q)$, then the condition of the above theorem becomes $q > \frac{(5k+7)\ell}{2\ell - 175}$. Furthermore, $\ell = +\infty$, we get $q > \frac{(5k+7)}{2}$, which is much greater than the number $k + 2$ in Theorem 1.2. Then, the most difficult part comes from the fact that “how to get a better number in our situation”.

2 Some Definitions and Results

In this section, we introduce the preliminaries of Nevanlinna theory for meromorphic functions on $\Delta(R)$, detailed reference [8-10].

Let v be a divior on $\Delta(R)$,which is regarded as a function on $\Delta(R)$ with values in $\mathbb{Z}$ such that $\mathrm{Supp}(v) = \{z; v(z) \neq 0\}$ is a discrete subset of $\Delta(R)$. We define the counting function of v to be:

$$n(t) = \sum_{|z| \leq t} v(z) (0 \leq t \leq R_0), N(r, v) = \int_0^r \frac{n(t) - n(0)}{t} dt.$$

Let f be a non-constant meromorphic function on $\Delta(R)$, we define

(1) v^0_f (resp.v^∞_f) is the divisor of zeros (resp. divisor of poles) of f.

(2) $v^0_{f,\geq k} = \max\{k, v^0_f\}$.

For a meromorphic function a, we then define $N(r, a, f) = N(r, v^0_{f-a})$ and the truncated counting function is defined by

$$\overline{N}(r, a, f) = \overline{N}(r, v^0_{f-a}) = N(r, \min\{1, v^0_{f-a}\}).$$

The proximity function of f is defined by

$$m(r, f) = \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi}.$$

For a meromorphic function a, we define $m(r,a,f)=m(r,\frac{1}{f-a})$, (we regard $\frac{1}{f-\infty}$ as f), where $\log^+ x=\max\{0,\log x\}$.

The characteristic function of f is defined by $T(r,f)=m(r,f)+N(r,v_f^\infty)$. The first main theorem states that

$$m(r,\frac{1}{f-a})+N(r,v^0_{f-a})=T(r,f)+O(1)$$

for arbitrarily meromorphic function a.

Lemma 2.1 [Lemma on logarithmic derivative, [7, Theorem 5.1]] Let $f(z)$ be a meromorphic function on $\Delta(R)(0<R\leq\infty)$, and let $\gamma(r)$ be a non-negative measurable function defined on $(0,R)$ with $\int_0^R\gamma(r)dr=+\infty$. Then, for $\varepsilon>0$, we have

$$\|_E\ m(r,\frac{f'}{f})\leq(1+\varepsilon)\log\gamma(r)+\varepsilon\log r+o(\log T(r,f)).$$

Theorem 2.2 [Corollary 1.8 in [7], second main theorem, Theorem2.4 in [11]] Let f be a non-constant meromorphic function on $\Delta(R)(0<R\leq\infty)$, let $\gamma(r)$ be a non-negative measurable function defined on $(0,R)$ with $\int_0^R\gamma(r)dr=+\infty$ and let $a_1,\cdots,a_q$ be q distinct values in $\mathbb{C}\cup\{\infty\}$. Then, for $\varepsilon>0$, we have

$$\|_E\ (q-2)T(r,f)\leq\sum_{i=1}^q\overline{N}(r,v^0_{f-a_i})+(1+\varepsilon)\log\gamma(r)+\varepsilon\log r+o(\log T(r,f)).$$

3 Proof of Theorem 1.4

In order to prove our main theorem, we need the following lemmas.

Lemma 3.1 [6, Lemma 3.1] Let f_1 and f_2 be two meromorphic functions on $\Delta(R)$, and let a_1,a_2, and a_3 be three distinct meromorphic functions on $\Delta(R)$ (being not equal to ∞) such that $f_2=\frac{(f_1-a_1)}{(f_1-a_2)}$. Then we have
(a) $T(r,f_2)\geq T(r,f_1)-\sum_{i=1}^2T(r,a_i)+O(1)$,
(b) $\overline{N}(r,v^0_{f_2})+\overline{N}(r,v^0_{f_2-1})+\overline{N}(r,v^\infty_{f_2})\leq\overline{N}(r,v^\infty_{f_1})+\sum_{i=1}^2(\overline{N}(r,v^0_{f_1-a_i})+2T(r,a_i))+O(1)$.
And that, if we set $b=\frac{a_3-a_1}{a_3-a_2}$, then
(c) $\overline{N}(r,v^0_{f_2-b})\leq\overline{N}(r,v^0_{f_1-a_3})+T(r,a_1)+2T(r,a_2)+T(r,a_3)+O(1)$.

Lemma 3.2 [6, Lemma 3.2] Let f_1 and f_2 be two meromorphic functions on $\Delta(R)$, and let a_1, a_2, a_3 and a_4 be four distinct meromorphic functions on $\Delta(R)$ such that

$$f_2=\frac{f_1-a_1}{f_1-a_2}\cdot\frac{a_3-a_2}{a_3-a_1}.$$

Then we have
(a) $T(r,f_2)\geq T(r,f_1)-\sum_{i=1}^3T(r,a_i)+O(1)$,
(b) $\overline{N}(r,v^0_{f_2})+\overline{N}(r,v^0_{f_2-1})+\overline{N}(r,v^\infty_{f_2})\leq\sum_{i=1}^3(\overline{N}(r,v^0_{f_1-a_i})+3T(r,a_i))+O(1)$.
If we set $b=\frac{a_4-a_1}{a_4-a_2}\cdot\frac{a_3-a_2}{a_3-a_1}$, then
(c) $\overline{N}(r,v^0_{f_2-b})\leq\overline{N}(r,v^0_{f_1-a_4})+2T(r,a_1)+3T(r,a_2)+2T(r,a_3)+T(r,a_4)$.

In fact, Quang proved the above two lemmas for meromorphic function on $\mathbb{A}(R_0)$. But their proof process still holds for meromorphic functions on $\Delta(R)$. To prove Theorem 1.4, we just only prove the following lemma.

Lemma 3.3 Let g be a nonconstant meromorphic function on $\Delta(R)$. Let a_1, a_2, a_3, a_4 and a_5 be five distinct meromorphic functions on $\Delta(R)$ (may be equal to ∞). We have

$$\|_E \ 2T(r,g) \le \sum_{i=1}^{5} \overline{N}(r, v^0_{g-a_i}) + 35\sum_{i=1}^{5} T(r,a_i) + 17S(r) + o(T(r,g)),$$

where $S(r) = (1+\varepsilon)\log\gamma(r) + \varepsilon\log r$.

Proof The proof of the lemma 3.3 is divided into two parts.

Part 1. We first consider the case where $a_i \not\equiv \infty$ for all $i = 1, \cdots, 5$. Put

$$\begin{aligned} f =& \frac{g-a_1}{g-a_2}\cdot\frac{a_3-a_2}{a_3-a_1}, b_1 = \frac{a_4-a_1}{a_4-a_2}\cdot\frac{a_3-a_2}{a_3-a_1}, \\ b_2 =& \frac{a_5-a_1}{a_5-a_2}\cdot\frac{a_3-a_2}{a_3-a_1}, b_3 = 0, b_4 = 1. \end{aligned}$$

By Lemma 3.2, we have

$$\begin{aligned} T(r,g) &\le T(r,f) + \sum_{i=1}^{3} T(r,a_i) + O(1), \\ T(r,b_1) &\le \sum_{i=1}^{4} T(r,a_i) + O(1), \\ T(r,b_2) &\le \sum_{i=1}^{3} T(r,a_i) + T(r,a_5) + O(1), \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \overline{N}(r, v^\infty_f) + \sum_{i=1}^{4} \overline{N}(r, v^0_{f-b_i}) \le \sum_{i=1}^{5} \overline{N}(r, v^0_{g-a_i}) + 7T(r,a_1) + 9T(r,a_2) \\ + 7T(r,a_3) + T(r,a_4) + T(r,a_5) + O(1). \end{aligned} \tag{3.2}$$

We need to prove the following proposition.

proposition 3.4

$$\|_E \ 2T(r,f) \le \overline{N}(r, v^\infty_f) + \sum_{i=1}^{4} \overline{N}(r, v^0_{f-b_i}) + \sum_{i=1}^{2} 18T(r,b_i) + 17S(r) + o(T(r,f)).$$

Actually, if b_1 or b_2 is constant, then the proposition directly follows from Theorem 2.2. Therefore, we may assume that both b_1 and b_2 are not constant. We define

$$F = \begin{vmatrix} ff' & f' & f^2 - f \\ b_1 b_1' & b_1' & b_1^2 - b_1 \\ b_2 b_2' & b_2' & b_2^2 - b_2 \end{vmatrix}. \tag{3.3}$$

Consider the following two cases.

Case 1: $F(z)\equiv 0$. For (3.3), by elementary transformation of determinants, we have

$$F=f(f-1)b_1(b_1-1)b_2(b_2-1)\begin{vmatrix}\frac{f'}{f-1}-\frac{b_2'}{b_2-1} & \frac{f'}{f}-\frac{b_2'}{b_2} & 0\\ \frac{b_1'}{b_1-1}-\frac{b_2'}{b_2-1} & \frac{b_1'}{b_1}-\frac{b_2'}{b_2} & 0\\ \frac{b_2'}{b_2-1} & \frac{b_2'}{b_2} & 1\end{vmatrix}.$$

Therefore

$$(\frac{b_1'}{b_1}-\frac{b_2'}{b_2})(\frac{f'}{f-1}-\frac{b_2'}{b_2-1})\equiv(\frac{b_1'}{b_1-1}-\frac{b_2'}{b_2-1})(\frac{f'}{f}-\frac{b_2'}{b_2}). \tag{3.4}$$

We discuss the following four subcases.

Subcase 1: $\frac{b_1'}{b_1}\equiv\frac{b_2'}{b_2}$. Then we have $\frac{b_1'}{b_1-1}\equiv\frac{b_2'}{b_2-1}$ or $\frac{f'}{f}\equiv\frac{b_2'}{b_2}$, if $\frac{b_1'}{b_1-1}\equiv\frac{b_2'}{b_2-1}$, then b_1 and b_2 are constant, and get a contradiction. If $\frac{f'}{f}\equiv\frac{b_2'}{b_2}$, then there is $f=cb_2$ with a constant c. This implies that $T(r,f)=T(r,b_2)$, and the proposition is proved in this subcase.

Subcase 2: $\frac{b_1'}{b_1-1}\equiv\frac{b_2'}{b_2-1}$. Using the same arguments as in subcase 1, we obtain the inequality of the proposition in this subcase.

Subcase 3: $\frac{b_1'}{b_1}\not\equiv\frac{b_2'}{b_2}$, $\frac{b_1'}{b_1-1}\not\equiv\frac{b_2'}{b_2-1}$, $\frac{b_1'}{b_1}-\frac{b_2'}{b_2}\equiv\frac{b_1'}{b_1-1}-\frac{b_2'}{b_2-1}$. The identity (3.4) implies that

$$\frac{f'}{f-1}-\frac{f'}{f}\equiv\frac{b_2'}{b_2-1}-\frac{b_2'}{b_2}.$$

Then we get

$$\frac{f-1}{f}\equiv c\cdot\frac{b_2-1}{b_2},$$

where c is a constant. Therefore $f=\frac{b_2}{(c-1)b_2-c}$, and $T(r,f)=T(r,b_2)$. Again, we obtain the inequality of the proposition in this subcase.

Subcase 4: $\frac{b_1'}{b_1}\not\equiv\frac{b_2'}{b_2}$, $\frac{b_1'}{b_1-1}\not\equiv\frac{b_2'}{b_2-1}$, $\frac{b_1'}{b_1}-\frac{b_2'}{b_2}\not\equiv\frac{b_1'}{b_1-1}-\frac{b_2'}{b_2-1}$. The identity (3.4) may be rewritten as

$$(\frac{b_1'}{b_1}-\frac{b_2'}{b_2})\frac{f'}{f-1}-(\frac{b_1'}{b_1-1}-\frac{b_2'}{b_2-1})\frac{f'}{f}\equiv\frac{b_1'b_2'}{b_1b_2-1}-\frac{b_2'b_1'}{b_2b_1-1}. \tag{3.5}$$

We see that each zero of f must be a zero or a 1-point or a pole of $b_j(j=1,2)$ or a zero of $\frac{b_1'}{b_1-1}-\frac{b_2'}{b_2-1}$. Therefore,

$$\min\{1,v_f^0\}\le\sum_{i=1,2}\sum_{a=0,1,\infty}\min\{1,v_{b_i-a}^0\}+\min\{1,v^0_{\frac{b_1'}{b_1-1}-\frac{b_2'}{b_2-1}}\}. \tag{3.6}$$

Similarly, we have

$$\min\{1,v_{f-1}^0\}\le\sum_{i=1,2}\sum_{a=0,1,\infty}\min\{1,v_{b_i-a}^0\}+\min\{1,v^0_{\frac{b_1'}{b_1}-\frac{b_2'}{b_2}}\}. \tag{3.7}$$

From (3.5), we also see that each pole of f must be a zero or a 1-point or a pole of $b_j(j=1,2)$ or a zero of $(\frac{b_1'}{b_1}-\frac{b_2'}{b_2})-(\frac{b_1'}{b_1-1}-\frac{b_2'}{b_2-1})$. By the same arguments, we have

$$\min\{1,v_f^\infty\}\le\sum_{i=1,2}\sum_{a=0,1,\infty}\min\{1,v_{b_i-a}^0\}+\min\{1,v^0_{(\frac{b_1'}{b_1}-\frac{b_2'}{b_2})-(\frac{b_1'}{b_1-1}-\frac{b_2'}{b_2-1})}\}. \tag{3.8}$$

Combining (3.6)-(3.8), we have

$$\begin{aligned}\sum_{a=0,1,\infty} \min\{1, v^0_{f-a}\} \leq \sum_{i=1,2}\sum_{a=0,1,\infty} &\min\{1, v^0_{b_i-a}\} + \min\{1, {v^0}_{\frac{b_1'}{b_1-1}-\frac{b_2'}{b_2-1}}\} \\ &+ \min\{1, v^0_{\frac{b_1'}{b_1}-\frac{b_2'}{b_2}}\} + \min\{1, {v^0}_{(\frac{b_1'}{b_1}-\frac{b_2'}{b_2})-(\frac{b_1'}{b_1-1}-\frac{b_2'}{b_2-1})}\}.\end{aligned} \tag{3.9}$$

By Theorem 2.2, we get

$$\begin{aligned}\|_E\ T(r,f) \leq& \overline{N}(r, v^0_f) + \overline{N}(r, v^0_{f-1}) + \overline{N}(r, v^\infty_f) + S(r) \\ \leq& 2\sum_{i=1,2}(T(r, \frac{b_i'}{b_i}) + T(r, \frac{b_i'}{b_i-1})) + S(r) \\ \leq& 2\sum_{i=1,2}(N(r, v^\infty_{\frac{b_i'}{b_i}}) + N(r, v^\infty_{\frac{b_i'}{b_i-1}})) + 2\sum_{i=1,2}(m(r, \frac{b_i'}{b_i}) + m(r, \frac{b_i'}{b_i-1})) + S(r) \\ \leq& 2\sum_{i=1,2}(N(r, v^0_{b_i}) + 2N(r, v^\infty_{b_i}) + N(r, v^0_{b_i-1})) + 9S(r) \\ \leq& 8T(r, b_1) + 8T(r, b_2) + 9S(r).\end{aligned}$$

Then we have the desired inequality of the proposition in this subcase.

Case 2: $F(z) \not\equiv 0$. we set

$$\begin{aligned}&\delta(z) = min\{1, \mid b_1(z) \mid, \mid b_2(z) \mid, \mid b_1(z) - 1 \mid, \mid b_2(z) - 1 \mid, \mid b_1(z) - b_2(z) \mid\}, \\ &\theta_j(r) = \{\theta : \mid f(re^{i\theta}) - b_j(re^{i\theta}) \mid \leq \delta(re^{i\theta})\} (j = 1, 2), \\ &\theta_3(r) = \{\theta : \mid f(re^{i\theta}) \mid \leq \delta(re^{i\theta})\}, \\ &\theta_4(r) = \{\theta : \mid f(re^{i\theta}) - 1 \mid \leq \delta(re^{i\theta})\}.\end{aligned}$$

It is clear that the sets $\theta_i(r) \cap \theta_j(r) (i \neq j, i, j = 1, 2, 3, 4)$ have at most many finite points. Therefore we easily get that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{\delta(re^{i\theta})} d\theta \leq& m(r, \frac{1}{b_1}) + m(r, \frac{1}{b_2}) + m(r, \frac{1}{b_1-1}) \\ &+ m(r, \frac{1}{b_2-1}) + m(r, \frac{1}{b_1-b_2}) \\ \leq& T(r) + O(1),\end{aligned} \tag{3.10}$$

where $T(r) = 3T(r, b_1) + 3T(r, b_2)$. We also get that

$$\begin{aligned}ff' &= (f - b_1)(f' - b_1') + b_1'(f - b_1) + b_1(f' - b_1') + b_1 b_1', \\ f' &= (f' - b_1') + b_1', \\ f^2 - f &= (f - b_1)^2 + (2b_1 - 1)(f - b_1) + b_1^2 - b_1.\end{aligned}$$

Substituting these functions (on the right-hand side) into (3.3), we have

$$F = \begin{vmatrix} \varphi & f' - b_1' & \psi \\ b_1 b_1' & b_1' & b_1^2 - b_1 \\ b_2 b_2' & b_2' & b_2^2 - b_2 \end{vmatrix}, \tag{3.11}$$

where

$$\begin{aligned}
&\varphi=(f-b_1)(f'-b_1')+b_1'(f-b_1)+b_1(f'-b_1'),\\
&\psi=(f-b_1)^2+(2b_1-1)(f-b_1).
\end{aligned}$$

From (3.11), we see that each zero with multiplicity $p(p>1)$ of $(f-b_1)$ is neither a pole of b_1 nor a pole of b_2, which must be a zero of F with multiplicity at least $p-1$. Similarly, each zero with multiplicity $p(p>1)$ of $(f-b_2)$ is neither a pole of b_1 nor a pole of b_2, which must be a zero of F with multiplicity at least $p-1$. Furthermore, from (3.3), we see that each zero of multiplicity $p(p>1)$ of f or $f-1$ is neither a pole of b_1 nor a pole of b_2, which must be a zero of F with multiplicity at least $p-1$. This implies that

$$\sum_{i=1}^{4}(N(r,v^0_{f-b_i})-\overline{N}(r,v^0_{f-b_i}))\le N(r,v^0_F). \tag{3.12}$$

We now estimate this quantities $m(r,\frac{1}{f-b_j})(1\le j\le 4)$.

We first analyze $j=3$, since $\mid f(re^{i\theta})-b_3\mid=\mid f(re^{i\theta})\mid\le 1$ for each $\theta\in\theta_3(r)$, we have

$$F=fb_1b_2\begin{vmatrix} f' & \frac{f'}{f} & f-1\\ b_1' & \frac{b_1'}{b_1} & b_1-1\\ b_2' & \frac{b_2'}{b_2} & b_2-1\end{vmatrix}=fb_1b_2G. \tag{3.13}$$

For G,

$$\log^+G\le(\log^+f+\log^+\frac{f'}{f})+\sum_{i=1}^{2}(\log^+b_i+\log^+\frac{b_i'}{b_i}). \tag{3.14}$$

Thus

$$\log^+F\le\log^+f+\sum_{i=1}^{2}\log^+b_i+\log^+G. \tag{3.15}$$

Hence

$$\begin{aligned}
\|_E\ \frac{1}{2\pi}\int_{\theta_3(r)}\log^+|\frac{F}{f-b_3}|d\theta\le{}&\frac{1}{2\pi}\int_{\theta_3(r)}\log^+\frac{\mid f'\mid}{\mid f\mid}d\theta\\
&+\sum_{i=1}^{2}\frac{1}{2\pi}\int_{\theta_3(r)}(2\log^+\mid b_i\mid+\log^+\frac{\mid b_i'\mid}{\mid b_i'\mid})+O(1)\\
\le{}&m(r,\frac{f'}{f})+m(r,\frac{b_1'}{b_1})+m(r,\frac{b_2'}{b_2})\\
&+\frac{1}{2\pi}\int_{\theta_3(r)}2\log^+|b_1|d\theta+\frac{1}{2\pi}\int_{\theta_3(r)}2\log^+|b_2|d\theta+O(1)\\
\le{}&\frac{1}{2\pi}\int_{\theta_3(r)}2\log^+|b_1|d\theta+\frac{1}{2\pi}\int_{\theta_3(r)}2\log^+|b_2|d\theta+3S(r)+O(1).
\end{aligned} \tag{3.16}$$

Here the above inequality comes from the fact that

$$\log^+ |\det(x_{ij}; 1 \leq i, j \leq 3)| \leq \sum_{i=1}^{3} \log^+ \max\{|x_{ij}|; 1 \leq j \leq 3\} + O(1) \tag{3.17}$$

for every $3 * 3$ matrix of complex numbers $(x_{ij})_{1 \leq i,j \leq 3}$.

Therefore, for $j = 3$, we get

$$\begin{aligned} \|_E \; m(r, \frac{1}{f - b_3}) \leq & \frac{1}{2\pi} \int_{\theta_3(r)} \log^+ |\frac{1}{f - b_3}| d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{\delta(re^{i\theta})} d\theta \\ \leq & \frac{1}{2\pi} \int_{\theta_3(r)} \log^+ |\frac{F}{f - b_3}| d\theta \\ & + \frac{1}{2\pi} \int_{\theta_3(r)} \log^+ |\frac{1}{F}| d\theta + T(r) + O(1) \\ \leq & \frac{1}{2\pi} \int_{\theta_3(r)} (\log^+ \frac{1}{| F |} + 2\log^+ | b_1 | + 2\log^+ | b_2 |) d\theta \\ & + 3S(r) + T(r) + O(1). \end{aligned} \tag{3.18}$$

Similarly, we get

$$\begin{aligned} \|_E \; m(r, \frac{1}{f - b_4}) \leq & \frac{1}{2\pi} \int_{\theta_4(r)} (\log^+ \frac{1}{| F |} + 2\log^+ | b_1 | + 2\log^+ | b_2 |) d\theta \\ & + 3S(r) + T(r) + O(1). \end{aligned} \tag{3.19}$$

On the other hand, Since $|f(re^{i\theta}) - b_1(re^{i\theta})| \leq \delta(re^{i\theta}) \leq 1$ for every $\theta \in \theta_1(r)$, by (3.11),

$$\begin{aligned} \log^+ |\frac{F(re^{i\theta})}{f(re^{i\theta}) - b_1}| \leq & \log^+ \frac{| f'(re^{i\theta}) - b_1'(re^{i\theta}) |}{| f(re^{i\theta}) - b_1(re^{i\theta}) |} d\theta + 2\log^+ | b_1(re^{i\theta}) | \\ & + \log^+ \frac{| b_1'(re^{i\theta}) |}{| b_1(re^{i\theta}) |} + \sum_{i=1}^{2} (2\log^+ | b_i(re^{i\theta}) | + \log^+ \frac{| b_i'(re^{i\theta}) |}{| b_i(re^{i\theta}) |}) + O(1). \end{aligned}$$

Similarly, for $j = 1$, we get

$$\begin{aligned} \|_E \; m(r, \frac{1}{f - b_1}) \leq & \frac{1}{2\pi} \int_{\theta_1(r)} (\log^+ \frac{1}{| F |} + 4\log^+ | b_1 | + 2\log^+ | b_2 |) d\theta \\ & + 4S(r) + T(r) + O(1). \end{aligned} \tag{3.20}$$

$$\begin{aligned} \|_E \; m(r, \frac{1}{f - b_2}) \leq & \frac{1}{2\pi} \int_{\theta_2(r)} (\log^+ \frac{1}{| F |} + 2\log^+ | b_1 | + 4\log^+ | b_2 |) d\theta \\ & + 4S(r) + T(r) + O(1). \end{aligned} \tag{3.21}$$

Combining (3.18) – (3.21), we obtain

$$\|_E \; \sum_{i=1}^{4} m(r, \frac{1}{f - b_i}) \leq m(r, \frac{1}{F}) + 4m(r, b_1) + 4m(r, b_2) + 4T(r) + 14S(r).$$

Therefore,

$$\|_E 4T(r,f) \le N(r,v_f^0) + N(r,v_{f-1}^0) + N(r,v_{f-b_1}^0) + N(r,v_{f-b_2}^0) - N(r,v_F^0) \\ + T(r,F) + 4m(r,b_1) + 4m(r,b_2) + 4T(r) + 14S(r) + o(T(r,f)).$$

Combining with (3.12), we have

$$\|_E\ 4T(r,f) \le \overline{N}(r,v_f^0) + \overline{N}(r,v_{f-1}^0) + \overline{N}(r,v_{f-b_1}^0) + \overline{N}(r,v_{f-b_2}^0) + T(r,F) \\ + 4m(r,b_1) + 4m(r,b_2) + 4T(r) + 14S(r) + o(T(r,f)). \tag{3.22}$$

Moreover, from (3.3) and (3.17),

$$m(r,F) \le 2m(r,f) + m(r,\frac{f'}{f}) + \sum_{i=1}^{2}(2m(r,b_i) + m(r,\frac{b_i'}{b_i})) \\ \le 2m(r,f) + 2m(r,b_1) + 2m(r,b_2) + 3S(r)$$

and

$$N(r,v_F^\infty) \le 2N(r,v_f^\infty) + \overline{N}(r,v_f^\infty) + 3\sum_{i=1}^{2} N(r,v_{b_i}^\infty).$$

These inequalities imply that

$$\|_E\ T(r,F) \le 2T(r,f) + \overline{N}(r,v_f^\infty) + 2\sum_{i=1}^{2} N(r,v_{b_i}^\infty) + \frac{2}{3}T(r) + 3S(r). \tag{3.23}$$

From (3.22) and (3.23), we get

$$\|_E\ 2T(r,f) \le \overline{N}(r,v_f^0) + \overline{N}(r,v_{f-1}^0) + \overline{N}(r,v_{f-b_1}^0) + \overline{N}(r,v_{f-b_2}^0) \\ + \overline{N}(r,v_f^\infty) + 18T(r,b_1) + 18T(r,b_2) + 17S(r) + o(T(r,f)).$$

Therefore, this proposition is proved.

We now prove the lemma of part 1. From (3.1) , (3.2) and proposition 3.4, we get

$$\|_E\ 2T(r,g) \le \sum_{i=1}^{5} \overline{N}(r,v_{g-a_i}^0) + 45\sum_{j=1,3} T(r,a_{i_j}) + 47T(r,a_{i_2}) \\ + 19\sum_{j=4,5} T(r,a_{i_j}) + 17S(r) + o(T(r,g)),$$

for any permutation$(i_1,\cdots,i_5)$ of $\{1,\cdots,5\}$.Summing up both sides of the above inequalities over all such permutations, we have

$$\|_E\ 2T(r,g) \le \sum_{i=1}^{5} \overline{N}(r,v_{g-a_i}^0) + 35\sum_{i=1}^{5} T(r,a_i) + 17S(r) + o(T(r,g)). \tag{3.24}$$

Therefore, the part 1 is proved.

Part 2. We now prove the part 2, where there is a function among $\{a_1, \cdots, a_5\}$ equal to ∞; for example, $a_1 \equiv \infty$. We set $h = \frac{g-a_2}{g-a_3}, c_1 = \frac{a_4-a_2}{a_4-a_3}, c_2 = \frac{a_5-a_2}{a_5-a_3}, c_3 = 0, c_4 = 1$. By Lemma 3.1, we have

$$\begin{aligned} T(r,g) &\le T(r,h) + \sum_{i=2}^{3} T(r,a_i) + O(1), \\ T(r,c_1) &\le \sum_{i=2}^{4} T(r,a_i) + O(1), \\ T(r,c_2) &\le \sum_{i=2}^{3} T(r,a_i) + T(r,a_5) + O(1), \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} &\overline{N}(r,v_h^0) + \overline{N}(r,v_h^\infty) + \overline{N}(r,v_{h-1}^0) + \overline{N}(r,v_{h-c_1}^0) + \overline{N}(r,v_{h-c_2}^0) \\ \le &\sum_{i=1}^{5} \overline{N}(r,v_{g-a_i}^0) + 4T(r,a_2) + 8T(r,a_3) + T(r,a_4) + T(r,a_5) + O(1). \end{aligned} \tag{3.26}$$

Applying Proposition 3.4 for functions h, c_1, c_2, c_3, and c_4, we get

$$\|_E \; 2T(r,h) \le \overline{N}(r,v_h^\infty) + \sum_{i=1}^{4} \overline{N}(r,v_{h-c_i}^0) + \sum_{i=1}^{2} 18T(r,c_i) + 17S(r) + O(1).$$

Combining (3.25) , (3.26) and the above inequality, we get

$$\begin{aligned} \|_E \; 2T(r,g) \le &\sum_{i=1}^{5} \overline{N}(r,v_{g-a_i}^0) + 42T(r,a_{i_2}) + 46T(r,a_{i_3}) \\ &+ 19 \sum_{j=4,5} T(r,a_{i_j}) + 17S(r) + o(T(r,g)), \end{aligned}$$

for any permutation $(i_2, \cdots, i_5)$ of $\{2, \cdots, 5\}$. Summing up both sides of the above inequalities over all such permutations, we get

$$\|_E \; 2T(r,g) \le \sum_{i=1}^{5} \overline{N}(r,v_{g-a_i}^0) + \frac{63}{2} \sum_{i=2}^{5} T(r,a_i) + 17S(r) + o(T(r,g)). \tag{3.27}$$

Therefore, the part 2 is proved. We complete the proof of the lemma.

4 Proof of Theorem 1.5

Assuming there exists a meromorphic function g on $\Delta(R)$ and mutually distinct meromorphic functions $f_i (1 < i < 5)$ on $\Delta(R)$ such that f_i and g share the set S with truncated multiplicity, where $g = f_1$.

Put $P_S(w) = (w-a_1)\cdots(w-a_q), \psi_i = P_S(f_i)/P_S(g) (1 < i < 5)$, $\varphi_j = P_S'(f_j)f_j'/P_S(f_j)$ and $\varphi = \varphi_1$. We have

$$\varphi = \alpha_j + \varphi_j, T(r, P_S(f_j)) = qT(r,f_j) + O(1)$$

where $\alpha_j = -\frac{\psi_j'}{\psi_j}(1 < j < 5)$. Let $T(r) = \sum_{i=1}^{5} T(r, f_i)$, by lemma 2.1, we have

$$m(r, \varphi_j) = S_{P_S(f_j)}(r) \leq S(r) + o(T(r)) \text{ for all } 1 < j < 5.$$

Since each pole of φ_j is a simple pole, which is either zero or a pole of $P_S(f_j)$, we get

$$N(r, v_{\varphi_j}^{\infty}) = \overline{N}(r, v_{P_S(f_j)}^{0}) + \overline{N}(r, v_{P_S(f_j)}^{\infty}) \leq 2qT(r, f_j).$$

This implies

$$\|_E\ T(r, \varphi_j) = m(r, \varphi_j) + N(r, v_{\varphi_j}^{\infty}) \leq 2qT_0(r, f_j) + S(r) + o(T(r)).$$

In particular, $T(r, \varphi_j) = O(T(r, g))$. In addition, by Theorem 2.2, we get

$$\begin{aligned}\|_E\ N(r, v_{\varphi}^{\infty}) &= \overline{N}(r, v_{P_S(g)}^{0}) + \overline{N}(r, v_{P_S(g)}^{\infty}) \\ &= \overline{N}(r, v_{P_S(f_j)}^{0}) + \overline{N}(r, v_{P_S(f_j)}^{\infty}) \\ &= \sum_{i=1}^{q} \overline{N}(r, v_{f_j - a_i}^{0}) + q\overline{N}(r, v_{f_j}^{\infty}) \\ &\geq (q-1)(T(r, f_j) + \overline{N}(r, v_{f_j}^{\infty})) - S_{f_j}(r).\end{aligned}$$

It yields that

$$\|_E\ (q-1)T(r, f_j) \leq T(r, \varphi) - (q-1)\overline{N}(r, v_{f_j}^{\infty}) + o(T(r)). \tag{4.1}$$

Hence

$$\begin{aligned}T(r, \alpha_j) &= m(r, \alpha_j) + N(r, v_{\alpha_j}^{\infty}) \\ &\leq m(r, \varphi) + m(r, \varphi_j) + \overline{N}(r, v_{\psi_j}^{0}) + \overline{N}_0(r, v_{\psi_j}^{\infty}) + O(1) \\ &\leq \overline{N}(r, v_{P_S(g)/P_S(f_j)}^{0}) + \overline{N}(r, v_{P_S(g)/P_S(f_j)}^{\infty}) + 2S(r) + o(T(r)) \\ &\leq \sum_{i=1}^{q} \overline{N}(r, v_{f_j - a_i, \geq \ell_i}^{0}) + 2S(r) + o(T(r)) \\ &\leq \sum_{i=1}^{q} \frac{1}{\ell_i} T(r, f_j) + 2S(r) + o(T(r)).\end{aligned}$$

This also implies $T(r, \alpha_j) = O(T(r, g))$.

We now show that $\alpha_1, \cdots, \alpha_5$ are mutually distinct. Indeed, supposing that $\alpha_i = \alpha_j$ for some $i \neq j$. Then there exists a nonzero constant c_0 with $\psi_i = c_0\psi_j$, and hence $c_0 P_S(f_i) = P_S(f_j)$. Since $P_S(w)$ is a uniqueness polynomial for meromorphic functions on $\Delta(R_0)$, we have $f_i = f_j$, it is a contradiction.

Applying Theorem 1.4, we have

$$\|_E\ 2T(r, \varphi) \leq \sum_{j=1}^{5} \overline{N}(r, v_{\varphi - \alpha_j}^{0}) + 35 \sum_{j=1}^{5} T(r, \alpha_j) + 17S(r) + o(T(r, \varphi)).$$

On the other hand,

$$\overline{N}(r, v^0_{\varphi-\alpha_j}) = \overline{N}(r, v^0_{\varphi_j}) \le \overline{N}(r, v^0_{f'_j}) + \sum_{t=1}^{k} \overline{N}(r, v^0_{f_j-e_t}),$$

where $e_1, \cdots, e_k$ are all of distinct zeros of $P'_S(w)$. Since $\overline{N}(r, v^0_{f_j-e_t}) \le T(r, f_j) + O(1)$ and

$$\begin{aligned}\overline{N}(r, v^0_{f'_j}) &\le T(r, f'_j) + O(1) = m(r, f'_j) + N(r, v^\infty_{f'_j}) + O(1)\\ &\le m(r, f_j) + m(r, \frac{f'_j}{f_j}) + N(r, v^\infty_{f_j}) + \overline{N}(r, v^\infty_{f_j}) + O(1)\\ &\le T(r, f_j) + \overline{N}(r, v^\infty_{f_j}) + S(r) + o(T(r)),\end{aligned}$$

we have

$$\sum_{j=1}^{5} \overline{N}(r, v^0_{\varphi-\alpha_j}) \le (k+1)\sum_{j=1}^{5} T(r, f_j) + \sum_{j=1}^{5} \overline{N}(r, v^\infty_{f_j}) + 5S(r) + o(T(r)). \tag{4.2}$$

Combining (4.1) and (4.2), we obtain

$$\begin{aligned}\|_E\ 2(q-1)T(r, f_i) \le& 2T(r, \varphi) - 2(q-1)\overline{N}(r, v^\infty_{f_i})\\ \le& (k+1+\sum_{i=1}^{q}\frac{35}{\ell_i})\sum_{j=1}^{5} T(r, f_j) + \sum_{j=1}^{5} \overline{N}(r, v^\infty_{f_j})\\ &- 2(q-1)\overline{N}(r, v^\infty_{f_i}) + 372S(r) + o(T(r)),\end{aligned}$$

for all $i = 1, \cdots, 5$.

Summing up this inequality over all $i = 1, \cdots, 5$,

$$\|_E\ 2(q-1)\sum_{i=1}^{5} T(r, f_i) \le 5(k+1+\sum_{i=1}^{q}\frac{35}{\ell_i})\sum_{j=1}^{5} T(r, f_j) + 372 \times 5S(r) + o(T(r)).$$

Set $\gamma(r) = \exp\{\min_{1\le j\le 5}\{c_{f_j}\}+\varepsilon\}T(r)$. Then $S(r) = (1+\varepsilon)(\min_{1\le j\le 5}\{c_{f_j}\}+\varepsilon)T(r)+\varepsilon\log r$. Letting $\varepsilon \to 0$,and $r \to R_0$ $(r \notin E)$, we get

$$372\sum_{i=1}^{5} c_{f_i} \ge 372 \times 5 \min_{1\le j\le 5}\{c_{f_j}\} \ge \{2(q-1) - 5(k+1+\sum_{i=1}^{q}\frac{35}{\ell_i})\}.$$

It is a contradiction. We complete the proof of the theorem.

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复圆盘上亚纯函数的截断型有限象集

王玉婷，曹红哲

(南昌大学数学系, 江西 南昌 330031)

摘要: 本文研究了复圆盘上亚纯函数的截断型有限象集问题,利用亚纯函数值分布理论,建立了具有有限增长指标的亚纯函数分担亚纯函数(可能不是小函数)的第二基本定理; 并应用此结果推广了亚纯函数的截断型有限象集结果.

关键词: 亚纯函数; 有限增长指标; 复圆盘; 有限象集; 截断重数

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