Vol. 44 (2024)
No.5

数 学 杂 志
J. of Math. (PRC)

NORMALITY AND THE NUMBER OF ZEROS

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Abstract: In this paper, we study normal families of meromorphic functions. By using the idea in [11], we obtain some normality criteria for families of meromorphic functions that concern the number of zeros of the differential polynomial, which extends the related result of Li, and Chen et al.. An example is given to show that the hypothesis on the zeros of $a(z)$ is necessary.

Keywords: Meromorphic function; normal family; differential polynomial; zero

2010 MR Subject Classification: 30D45

Document code: **A** **Article ID:** **0255-7797(2024)05-0377-06**

1 Introduction

Let D be a domain in the complex plane $\mathbb{C}$, and $\mathcal{F}$ be a family of meromorphic functions in D. The family $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if every sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence $\{f_{n_j}\}$ such that $\{f_{n_j}\}$ converges spherically locally uniformly on D to a meromorphic function or ∞ [1, 2, 3].

In 2009, Xu, Wu and Liao[4](cf. [5]) proved the following result.

Theorem A Let $a(\neq 0)$ and b be finite complex numbers, n, k be two positive integers with $n \geq k+1$, and let $\mathcal{F}$ be a family of meromorphic functions in a domain D. If, for each $f \in \mathcal{F}$, f has only zeros of multiplicity at least $k+1$, and $f + a(f^{(k)})^n \neq b$ in D, then $\mathcal{F}$ is normal in D.

In 2013, Lei, Fang and Zeng[6] showed that Theorem A is still valid for $n \geq 2$. Under the condition that $f \neq 0$ for each $f \in \mathcal{F}$, Li[7] obtained the following result.

Theorem B Let $a(\neq 0)$ and b be finite complex numbers, $n(\geq 2), k$ be two positive integers, and let $\mathcal{F}$ be a family of zero-free meromorphic functions in a domain D. If, for each $f \in \mathcal{F}$, $f + a(f^{(k)})^n - b$ has at most nk distinct zeros in D, ignoring multiplicities, then $\mathcal{F}$ is normal in D.

Chen[8] extended $f + a(f^{(k)})^n - b$ in Theorem B to $f^m + a(f^{(k)})^n - b$, as follows.

Theorem C Let $\mathcal{F}$ be a family of zero-free meromorphic functions in a domain D, and let n, k and m be positive integers with $n \geq m+1$, and $a(\neq 0)$ and b be finite complex

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* **Received date:** 2024-05-19 **Accepted date:** 2024-06-24

Foundation item: Supported by NNSF of China(11471163).

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numbers. If, for each $f \in \mathcal{F}$, $f^m + a(f^{(k)})^n - b$ has at most nk distinct zeros in D, ignoring multiplicities, then $\mathcal{F}$ is normal in D.

Let $a_1(z), a_2(z), \cdots, a_k(z)$ be holomorphic functions. Set

$$L[f](z) = f^{(k)}(z) + a_1(z)f^{(k-1)}(z) + \cdots + a_k(z)f(z), \tag{1.1}$$

we call $L[f]$ a linearly differential polynomial of f.

In this paper, we replace '$f^{(k)}$' by '$L[f]$' in Theorem C, and prove the following result.

Theorem 1 Let $\mathcal{F}$ be a family of zero-free meromorphic functions in a domain D, let $a(z)(\neq 0)$ be a holomorphic function in D, and let n, k and m be positive integers with $n \geq m+1$, and b a finite complex number. If, for each $f \in \mathcal{F}$, $f^m(z) + a(z)(L[f](z))^n - b$ has at most nk distinct zeros (ignoring multiplicities) in D, then $\mathcal{F}$ is normal in D, where $L[f](z)$ is defined as in (1.1).

A natural problem arises: does Theorem C still hold if the constant $a(\neq 0)$ is replaced by the function $a(z)(\not\equiv 0)$? In this paper, we shall prove the following result.

Theorem 2 Let $\mathcal{F}$ be a family of zero-free meromorphic functions in a domain D, let $a(z)(\not\equiv 0)$ be a holomorphic function in D, whose zeros have multiplicities at most $n-1$, and let n, k and m be positive integers with $n \geq m+1$, and b a finite complex number. If, for each $f \in \mathcal{F}$, $f^m(z) + a(z)(L[f](z))^n - b$ has at most nk zeros (counting multiplicities) in D, then $\mathcal{F}$ is normal in D, where $L[f](z)$ is defined as in (1.1).

Remark 1 Clearly, Theorems 1 and 2 generalize Theorems B and C.

Remark 2 The condition that all zeros of $a(z)$ have multiplicities at most $n-1$ in Theorem 2 is necessary, as is shown by the following example.

Example 1 Let $m = k = 1, n = 2$, $D = \{z : |z| < 1\}$, $a(z) = z^2, b = -1$, and $\mathcal{F} = \{f_j\}$, where $f_j(z) = \{1/jz\}(j = 1, 2, \cdots)$.

We know that

$$f_j(z) + a(z)(f_j'(z))^2 - b = \frac{jz^2 + jz + 1}{j^2z^2}$$

has at most 2 zeros. But $\mathcal{F}$ is not normal at 0.

Remark 3 Since normality is a local property, the condition that for each $f \in \mathcal{F}$, $f^m(z)+a(z)(L[f](z))^n-b$ has at most nk zeros (ignoring multiplicities in Theorem 1, counting multiplicities in Theorem 2) in D can be replaced by that for each $z_0 \in D$, there exists some $r > 0$ such that $f^m(z) + a(z)(L[f](z))^n - b$ has at most nk zeros (ignoring multiplicities in Theorem 1, counting multiplicities in Theorem 2) in $\Delta_r(z_0) = \{z : |z - z_0| < r\}$.

Throughout this paper, we denote by $\mathbb{C}$ the complex plane, by $\mathbb{C}^*$ the punctured complex plane $\mathbb{C} \setminus \{0\}$. For $r > 0$, we shall write $\Delta_r = \{z : |z| < r\}$ and $\Delta_r' = \{z : 0 < |z| < r\}$. When $r = 1$, we drop the subscript.

2 Some Lemmas

To prove our results, we need some lemmas.

Lemma 1 [9, 10] Let $\mathcal{F}$ be a family of zero-free functions meromorphic in a domain D. If $\mathcal{F}$ is not normal at $z_0 \in D$, then for each $-1 < \alpha < \infty$, there exist a sequence of complex numbers $z_n \in D$ with $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0$ and a sequence of functions $f_n \in \mathcal{F}$ such that

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^{\alpha}} \to g(\zeta)$$

converges spherically and locally uniformly on $\mathbb{C}$, where g is a nonconstant zero-free meromorphic function of order at most 2. In particular, g is of order at most 1 if g is entire on $\mathbb{C}$.

Lemma 2 [11] Let f be a nonconstant zero-free rational function, and k a positive integer. Then $f^{(k)} - 1$ has at least $k+1$ distinct zeros.

Lemma 3 [8] Let $n \geq 2, m, k$ be positive integers, and $a \in \mathbb{C}^*$. If f is a zero-free transcendental meromorphic function in $\mathbb{C}$, then $f^m + a(f^{(k)})^n$ has infinitely many zeros.

Lemma 4 [8] Let n, m, k be positive integers with $n \geq m+1$, let $a(\neq 0), b$ be two finite complex number. If f is a nonconstant zero-free rational function, then $f^m + a(f^{(k)})^n - b$ has at least $nk+1$ distinct zeros in $\mathbb{C}$.

3 Proof of Theorems

Proof of Theorem 1 Suppose, on the contrary, that $\mathcal{F}$ is not normal at $z_0 \in D$. We consider two cases.

Case 1. $b = 0$.

by Lemma 1, there exist points $z_j \to z_0$, positive numbers $\rho_j \to 0$ and functions f_j such that

$$g_j(\zeta) = \rho_j^{-\frac{nk}{n-m}} f_j(z_j + \rho_j \zeta) \to g(\zeta) \tag{3.1}$$

spherically uniformly on compact subsets of $\mathbb{C}$, where g is a nonconstant zero-free meromorphic function of order at most 2. In particular, g is of order at most 1 if g is entire on $\mathbb{C}$.

By Lemmas 3 and 4, $(g(\zeta))^m + a(z_0)(g^{(k)}(\zeta))^n$ has at least $nk+1$ distinct zeros in $\mathbb{C}$. Let $\zeta_1, \zeta_2, \cdots, \zeta_{nk+1}$ are distinct zeros of $g^m(\zeta) + a(z_0)(g^{(k)}(\zeta))^n$.

We claim that $g^m(\zeta) + a(z_0)(g^{(k)}(\zeta))^n \not\equiv 0$. Otherwise, $g^m(\zeta) + a(z_0)(g^{(k)}(\zeta))^n \equiv 0$, then g has no poles since $n \geq m+1$, that is, g is entire. Hence g is of order at most 1. Noting that g is nonconstant zero-free, g has the form $g(\zeta) = e^{c\zeta + d}$, with $c \in \mathbb{C}^*$ and $d \in \mathbb{C}$. This gives

$$g^m(\zeta) + a(z_0)(g^{(k)}(\zeta))^n = e^{m(c\zeta+d)}[1 + a(z_0)c^{nk}e^{(n-m)(c\zeta+d)}] \not\equiv 0$$

since $n \geq m+1$, a contradiction. The claim is proved.

By (3.1), we have

$$g_j^{(t)}(\zeta) = \rho_j^{t-\frac{nk}{n-m}} f_j^{(t)}(z_j + \rho_j \zeta) \to g^{(t)}(\zeta), (t = 1, 2, \cdots.)$$

locally uniformly on $\mathbb{C}\setminus g^{-1}(\infty)$. Since $a_1(z), a_2(z), \cdots, a_k(z)$ are holomorphic functions, we obtain that

$$\begin{aligned} & \rho_j^{-\frac{mnk}{n-m}}\left[f_j^m(z_j+\rho_j\zeta)+a(z_j+\rho_j\zeta)\left(L[f_j](z_j+\rho_j\zeta)\right)^n\right] \\ = \ & g_j^m(\zeta)+a(z_j+\rho_j\zeta)\left(g_j^{(k)}(\zeta)+\rho_j a_1(z_j+\rho_j\zeta)g_j^{(k-1)}(\zeta)+\cdots+\rho_j^k a_k(z_j+\rho_j\zeta)g_j(\zeta)\right)^n \\ \to \ & g^m(\zeta)+a(z_0)(g^{(k)}(\zeta))^n \end{aligned}$$

locally uniformly on $\mathbb{C}\setminus g^{-1}(\infty)$.

This and Hurwitz's theorem imply that, for $i=1,2,\cdots,nk+1$, there exist $\zeta_{j,i}$ such that $\zeta_{j,i}\to\zeta_i$ $(j\to\infty)$, and for sufficiently large j,

$$f_j^m(z_j+\rho_j\zeta_{j,i})+a(z_j+\rho_j\zeta_{j,i})\left(L[f_j](z_j+\rho_j\zeta_{j,i})\right)^n=0.$$

On the other hand, we know that $f_j^m+a(L[f_j])^n$ has at most nk distinct zeros. It follows that there exists $1\le i\ne i'\le nk+1$ such that

$$z_j+\rho_j\zeta_{j,i}=z_j+\rho_j\zeta_{j,i'},$$

so that $\zeta_{j,i}=\zeta_{j,i'}$. Letting $j\to\infty$, $\zeta_i=\zeta_{i'}$, a contradiction.

Case 2. $b\ne 0$.

By Lemma 1, there exist points $z_j\to z_0$, positive numbers $\rho_j\to 0$ and functions f_j such that

$$g_j(\zeta)=\rho_j^{-k}f_j(z_j+\rho_j\zeta)\to g(\zeta) \tag{3.2}$$

spherically uniformly on compact subsets of $\mathbb{C}$, where g is a nonconstant zero-free meromorphic function of order at most 2. By (3.2), we have

$$\begin{aligned} & f_j^m(z_j+\rho_j\zeta)+a(z_j+\rho_j\zeta)\left(L[f_j](z_j+\rho_j\zeta)\right)^n-b \\ = \ & \rho_j^{mk}g_j^m(\zeta)+a(z_j+\rho_j\zeta)\left(g_j^{(k)}(\zeta)+\rho_j a_1(z_j+\rho_j\zeta)g_j^{(k-1)}(\zeta)+\cdots+\rho_j^k a_k(z_j+\rho_j\zeta)g_j(\zeta)\right)^n-b \\ \to \ & a(z_0)(g^{(k)}(\zeta))^n-b \end{aligned}$$

locally uniformly on $\mathbb{C}\setminus g^{-1}(\infty)$.

Obviously, $a(z_0)(g^{(k)}(\zeta))^n-b\not\equiv 0$. Otherwise, $a(z_0)(g^{(k)}(\zeta))^n-b\equiv 0$, then $(g^{(k)}(\zeta))^n=b/a(z_0)$, and thus g is a polynomial of degree k, which is impossible since $g\ne 0$.

Then, arguing as in Case 1, $a(z_0)(g^{(k)}(\zeta))^n-b$ has at most nk distinct zeros in $\mathbb{C}$ since $f_j^m+a(L[f_j])^n-b$ has at most nk distinct zeros in D.

We write

$$a(z_0)(g^{(k)}(\zeta))^n-b=a(z_0)\prod_{i=1}^{n}(g^{(k)}(\zeta)-c_i), \tag{3.3}$$

where $c_1,c_2,\cdots,c_n$ are distinct zeros of $z^n-b/a(z_0)$. If g is a rational function, it follows from Lemma 2 that $g^{(k)}(\zeta)-c_i$ has at least $k+1$ distinct zeros for $1\le i\le n$. By (3.3),

$a(z_0)(g^{(k)}(\zeta))^n - b$ has at least $n(k+1) > nk+1$ distinct zeros, a contradiction. Then g is transcendental. By (3.3), and Nevanlinna's first and second fundamental theorems, we have

$$\begin{aligned} T(r, g^{(k)}) &\leq \bar{N}(r, g^{(k)}) + \sum_{i=1}^{n} \bar{N}\left(r, \frac{1}{g^{(k)} - c_i}\right) + S(r, g^{(k)}) \\ &\leq \frac{1}{k+1} N(r, g^{(k)}) + \bar{N}\left(r, \frac{1}{a(z_0)(g^{(k)}(\zeta))^n - b}\right) + S(r, g^{(k)}) \\ &\leq \frac{1}{k+1} T(r, g^{(k)}) + S(r, g^{(k)}), \end{aligned}$$

a contradiction. Theorem 1 is thus proved.

Proof of Theorem 2 By Theorem 1, it is enough to prove that $\mathcal{F}$ is normal at the zeros of $a(z)$ in D. Without loss of generality, we assume $D = \Delta$, $a(z) = z^l \varphi(z)$, where $l(\leq n-1)$ is a positive integer, $\varphi(z)$ is holomorphic and zero-free in Δ, and $\varphi(0) = 1$. Now $\mathcal{F}$ is normal in Δ' by Theorem 1, and we need to prove that $\mathcal{F}$ is normal at $z = 0$.

Suppose, on the contrary, that $\mathcal{F}$ is not normal at $z = 0$. Since for each $f \in \mathcal{F}$, $f(z) \neq 0$ in Δ, we know that the family $\mathcal{F}_1 = \{1/f : f \in \mathcal{F}\}$ is holomorphic in Δ and normal on Δ', but not normal at $z = 0$. Then there exists a sequence $\{1/f_j\} \subset \mathcal{F}_1$ which converges locally uniformly in Δ', but not on Δ. The maximum modulus principle implies that $1/f_j \to \infty$ in Δ'. Thus $f_j \to 0$ converges locally uniformly in Δ', and hence $f_j^{(i)} \to 0$ converges locally uniformly in Δ' for $1 \leq i \leq k$ by Weierstrass theorem.

Clearly, there exists $0 < \delta < 1$ such that

$$f_j^m(z) + a(z)(L[f_j(z)])^n - b \neq 0$$

on $|z| = \delta$. If $\{f_j\}$ is holomorphic in Δ_δ, again by the maximum modulus, $f_j \to \infty$ converges locally uniformly in Δ'_δ, which contradicts $f_j \to 0$ in Δ'. So $\{f_j\}$ is not holomorphic in Δ_δ.

By the argument principle, we have

$$\begin{aligned} &n\left(\delta, \frac{1}{f_j^m(z) + a(z)(L[f_j(z)])^n - b}\right) - n\left(\delta, f_j^m(z) + a(z)(L[f_j(z)])^n - b\right) \\ &\qquad = \frac{1}{2\pi i} \int_{|z|=\delta} \frac{\left(f_j^m(z) + a(z)(L[f_j(z)])^n - b\right)'}{f_j^m(z) + a(z)(L[f_j(z)])^n - b} dz \\ &\qquad \to 0, \end{aligned}$$

where $n(r, 1/f)$ is the number of zeros of f in Δ_r and $n(r, f)$ is the number of poles of f in Δ_r, counting multiplicity. Then for sufficiently large j,

$$n\left(\delta, \frac{1}{f_j^m(z) + a(z)(L[f_j(z)])^n - b}\right) = n\left(\delta, f_j^m(z) + a(z)(L[f_j(z)])^n - b\right). \tag{3.4}$$

In view of $n \geq m+1$ and $l \leq n-1$, we see that

$$n\left(\delta, f_j^m(z) + a(z)(L[f_j(z)])^n - b\right) \geq n(k+1) - l > nk.$$

On the other hand, by the assumption of Theorem ,

$$n\left(\delta, \frac{1}{f_j^m(z)+a(z)(L[f_j(z)])^n-b}\right) \leq nk.$$

These, together with (3.4), yield that $nk > nk$, a contradiction. This completes the proof of Theorem 2.

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正规定则与零点个数

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摘要: 本文研究了亚纯函数族正规定则, 利用文献[11]中的思想, 得到了关于微分多项式零点个数的亚纯函数正规定则, 结果推广了Li, Chen等人的相关结果. 另外举例说明了定理中微分多项式的系数$a(z)$零点条件的必要.

关键词: 亚纯函数; 正规族; 微分多项式; 零点

MR(2010)**主题分类号:** 30D45　　**中图分类号:** O174.52