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TRUDINGER-MOSER INEQUALITIES ON CARTAN-HADAMARD MANIFOLDS UNDER LORENTZ NORMS

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Abstract: The aim of this paper is to study Trudinger-Moser inequalities on a Cartan-Hadamard manifold with Lorentz norm. By using the pointwise estimates of Green's function and O'Neil inequality, we obtain the sharp constants of such inequalities, which generalize the corresponding results in Euclidean spaces.

Keywords: Trudinger-Moser inequality; Lorentz space; Riemannian manifold; negative curvature; sharp constant

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1 Introduction

Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a bounded domain and $1 \leq q \leq \frac{np}{n-p}$. The Sobolev embedding theorem tells us the embedding $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ is continuous when $1 < p < n$ but $W_0^{1,n}(\Omega) \not\subseteq L^\infty(\Omega)$. Trudinger [1] established in the borderline case that $W_0^{1,n}(\Omega) \subset L_{\varphi_n}(\Omega)$, where $L_{\varphi_n}(\Omega)$ is the Orlicz space associated with the Young function $\varphi_n(t) = \exp(\beta |t|^{\frac{n}{n-1}}) - 1$ for some $\beta > 0$. In the celebrated paper[2], Moser found the optimal β. In fact, he proved that it holds

$$\frac{1}{|\Omega|} \int_\Omega \exp(\beta_n |u|^{\frac{n}{n-1}}) dx \leq C,$$

where Ω be a domain with finite measure in Euclidean n-space $\mathbb{R}^n$, $\beta_n = n \left(\frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} \right)^{\frac{1}{n-1}}$ and $u \in W_0^{1,n}(\Omega)$ with $\int_\Omega |\nabla u|^n dx \leq 1$. Furthermore, this constant β_n is sharp. Similar exponential inequalities on bounded domain have been considered firstly by A. Alvino, V. Ferone and G. Trombetti [3] in Lorentz-Sobolev spaces of $\mathbb{R}^n$.

There has also been substantial progress for Moser-Trudinger inequalities on Riemannian manifolds. In the case of compact Riemannian manifolds, the study of Trudinger-Moser inequalities can be traced back to Aubin [4], Cherrier [5,6], and Fontana [7]. In the case

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of Cartan-Hadamard manifold Kong et al [8] and Bertrand et al [9] established the sharp Trudinger-Moser inequality.

One of the aim of this paper is to establish Trudinger-Moser type inequalities on M for Lorentz-Sobolev spaces. For simplicity, we also denote by $L^{n,q}(\Omega)$ the Lorentz-Sobolev space on domain $\Omega \subset M$. One of the main results is the following:

Theorem 1.1 Let Ω be a bounded domain of M, $n \geq 2$, and let $u \in C_0^\infty(\Omega)$ be such that

$$\|\nabla_M u\|_{L^{n,q}(\Omega)} \leq 1, \quad 1 \leq q \leq \infty.$$

(1) If $1 < q < \infty$ and $Ric_g \geq -(n-1)b^2$ for some $b > 0$, then there exists a constant $C = C(n,q)$ such that

$$\frac{1}{|\Omega|}\int_\Omega e^{\beta_q |u(x)|^{q'}} dV \leq C. \tag{1.1}$$

Furthermore, this inequality is sharp in the sense that if β_q is replaced by any $\beta > \beta_q$, then inequality (1.1) can no longer hold with some C independent of u.

(2) If $q = \infty$, then for every $\beta < \beta_\infty$, there exists a constant $C = C(n,q,\beta)$ such that

$$\frac{1}{|\Omega|}\int_\Omega e^{\beta |u(x)|} dV \leq C. \tag{1.2}$$

Furthermore, this inequality is sharp in the sense that if β is replaced by any $\beta \geq \beta_\infty$, then inequality (1.2) can no longer hold with some C independent of u.

2 Notations and Preliminaries

We begin by quoting some preliminary facts which will be needed in the sequel and refer to [8,13] for more precise information about this subject.

Let M be an n-dimensional complete Riemannian manifold with Riemannian metric ds^2. If $\{x^i\}_{1\leq i\leq n}$ is a local coordinate system, then we can write $ds^2 = \sum g_{ij}dx^i dx^j$ so that the Laplace-Beltrami operator Δ_M in this local coordinate system is

$$\Delta_M = \sum \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}g^{ij}\frac{\partial}{\partial x^j}\right),$$

where $g = \det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$. We denote by ∇_M the corresponding gradient.

From now on, we let M be a Cartan-Hadamard manifold. That is, M is a complete, simply connected Riemannian manifold with negative curvature. Let $p \in M$ and denote by $\rho_p(x) = \mathrm{dist}(x,p)$ for all $x \in M$, where $\mathrm{dist}(\cdot,\cdot)$ denotes the geodesic distance. For simplicity, we denote by $\rho(x) = \rho_p(x)$. Then $\rho(x)$ is smooth on $M \setminus \{p\}$ and it satisfies

$$|\nabla\rho(x)| = 1, \;\; x \in M \setminus \{p\}.$$

For any $\delta > 0$, denote by $B_\delta(p) = \{x \in M : \rho(x) < \delta\}$ the geodesic ball in M. We introduce the density function $J_p(\theta,t)$ of the volume form in normal coordinates as follows (see e.g. [10], page 166-167). Choose an orthonormal basis $\{\theta, e_2, \cdots, e_n\}$ on T_pM and let $c(t) = \mathrm{Exp}_p t\theta$ be a geodesic. $\{Y_i(t)\}_{2\le i\le n}$ are Jacobi fields satisfying the initial conditions

$$Y_i(0) = 0, \;\; Y_i'(0) = e_i, \;\; 2 \le i \le n,$$

so that the density function can be given by

$$J_p(t,\theta) = t^{-n+1}\sqrt{\det(\langle Y_i(t), Y_j(t)\rangle)}, \;\; t > 0.$$

We note that $J_p(t,\theta)$ does not depend on $\{e_2, \cdots, e_n\}$ and $J_p(t,\theta) \in C^\infty(T_pM\backslash\{p\})$ by the definition of $J_p(t,\theta)$. Furthermore, if we set $J_p(\theta, 0) \equiv 1$, then $J_p(t,\theta) \in C(T_pM)$ and

$$J_p(t,\theta) = 1 + O(t^2) \;\text{ as }\; t \to 0, \tag{2.1}$$

since $Y_i(t)$ has the asymptotic expansion (see e.g. [10], page 169)

$$Y_i(t) = te_i - \frac{t^3}{6}R(c'(t), e_i)c'(t) + o(t^3),$$

where $R(\cdot,\cdot)$ is the curvature tensor on M. By the definition of $J_p(t,\theta)$, we have the following formula in polar coordinates on M:

$$\int_M f(x)dV = \int_0^\infty \int_{\mathbb{S}^{n-1}} f(\rho,\theta)\rho^{n-1}J_p(\rho,\theta)d\rho d\sigma, \;\; f \in L^1(M),$$

where $d\sigma$ denotes the canonical measure of the unit sphere of $T_p(M)$. Finally, we recall a useful fact of $J_p(t,\theta)$ which plays an important role in the study of Moser-Trudinger inequalities. If the sectional curvature K on M satisfies $K \le -b$, then (see [10], page 172, line -2, the proof of Bishop-Gunther comparison theorem)

$$\frac{1}{J_p(t,\theta)} \cdot \frac{\partial J_p(t,\theta)}{\partial t} \ge \frac{J_b'(t)}{J_b(t)}, \;\; t > 0. \tag{2.2}$$

In particular, since M is with negative curvature, we have

$$\frac{1}{J_p(t,\theta)} \cdot \frac{\partial J_p(t,\theta)}{\partial t} \ge \frac{J_0'(t)}{J_0(t)} = 0,$$

which means $J_p(t,\theta)$, as a function of t on $[0, +\infty)$, is monotonically increasing.

3 Rearrangement

We firstly recall the rearrangement of functions on M. Suppose f is a nonnegative function on M. The non-increasing rearrangement of f is defined by

$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \le t\}, \tag{3.1}$$

where $\lambda_f(s) = |\{x \in M : f(x) > s\}|$. Here we use the notation $|\Sigma|$ for the measure of a measurable set $\Sigma \subset M$. Set

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds.$$

Denote by$L^{p,q}(\Omega)$ the Lorentz space of those function f on a domain $\Omega\subset M$ satisfying

$$\|f\|_{L^{p,q}(\Omega)}=\begin{cases}\left\|t^{\frac{1}{p}-\frac{1}{q}}f^*(t)\right\|_{L^q(0,|\Omega|)}, & 1\le q<\infty;\\ \sup\limits_{t>0}f^*(t)t^{1/p}, & q=\infty\end{cases}$$

is finite.Similarly, denote by

$$\|f\|^*_{L^{p,q}(\Omega)}=\begin{cases}\left\|t^{\frac{1}{p}-\frac{1}{q}}f^{**}(t)\right\|_{L^q(0,|\Omega|)}, & 1\le q<\infty;\\ \sup\limits_{t>0}f^{**}(t)t^{1/p}, & q=\infty.\end{cases}$$

We remark that for $1<q<+\infty$ and $0<r\le+\infty$, there holds (see [11], Theorem 3.4)

$$\|f\|_{L^{q,r}}\le\|f\|^*_{L^{q,r}}\le C(p,q)\|f\|_{L^{q,r}},\tag{3.2}$$

where $C(p,q)$ is a constant depending on p and r. Moreover, we have the following generalization of Young's inequality for convolution (see [12]):

Proposition 3.1 Let $1<r,p_1,p_2<+\infty$ and $1\le s,q_1,q_2\le\infty$. If

$$\frac{1}{p_1}+\frac{1}{p_2}-1=\frac{1}{r}\quad\text{and}\quad\frac{1}{q_1}+\frac{1}{q_2}\ge\frac{1}{s},$$

then there exists $C>0$ such that

$$\|f*g\|_{L^{r,s}}\le C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}},\quad f\in L(p_1,q_1),\quad g\in L(p_2,q_2).\tag{3.3}$$

We also need the following inequality (see [12], Corollary 1.8)

$$\|f*g\|_\infty\le\int_0^\infty f^*(t)g^*(t)dt.\tag{3.4}$$

Lemma 3.2 Let $0<\alpha<n$ and set $\phi_\alpha=\frac{1}{\rho^{n-\alpha}J_p(\rho,\theta)}$. Then

$$\phi^*_\alpha(t)\le\left(\frac{t}{\omega_n}\right)^{-(n-\alpha)/n},\quad t>0,$$

where $\omega_n=\pi^{n/2}/\Gamma(1+n/2)$ is the volume of $\mathbb{S}^n$.

Proof As in the proof of [8], Lemma 3.2, we set, for any $s>0$,

$$\lambda_{\phi_\alpha}(s)=\int_{\{x\in M:\phi_\alpha(x)>s\}}dV=\int_{\{(\rho,\theta)\in M:\rho^{n-\alpha}J_p(\rho,\theta)<s^{-1}\}}dV.\tag{3.5}$$

We denote by $\rho_\theta(s)$ the solution of $\rho^{n-\alpha}J_p(\rho,\theta)=s^{-1}$. Then $\rho_\theta(s)^{n-\alpha}J_p(\rho_\theta(s),\theta)=s^{-1}$ and

$$\lambda_{\phi_\alpha}(s)=\int_{\{(\rho,\theta)\in M:\rho^{n-1}J_p(\rho,\theta)<s^{-1}\}}dV=\int_{\mathbb{S}^{n-1}}\int_0^{\rho_\theta(s)}\rho^{n-1}J_p(\rho,\theta)d\sigma d\rho.$$

Therefore, since $\phi_\alpha^*(t) = \inf\{s > 0 : \lambda_{\phi_\alpha}(s) \le t\}$, we have

$$t = \lambda_{\phi_\alpha}(\phi_\alpha^*(t)) = \int_{\mathbb{S}^{n-1}} \int_0^{\rho_\theta(\phi_\alpha^*(t))} \rho^{n-1} J_p(\rho,\theta) d\sigma d\rho, \tag{3.6}$$

where $\rho_\theta(\phi_\alpha^*(t))$ satisfies

$$\rho_\theta(\phi_\alpha^*(t))^{n-\alpha} J_p(\rho_\theta(\phi_\alpha^*(t)),\theta) = \frac{1}{\phi_\alpha^*(t)}. \tag{3.7}$$

For simplicity, we set $\rho_\theta(t) = \rho_\theta(\phi_\alpha^*(t))$ in the rest of proof. Then,

$$t = \lambda_{\phi_\alpha}(\phi_\alpha^*(t)) = \int_{\mathbb{S}^{n-1}} \int_0^{\rho_\theta(t)} \rho^{n-1} J_p(\rho,\theta) d\sigma d\rho$$

and $\rho_\theta(t)$ satisfies

$$\rho_\theta(t)^{n-\alpha} J_p(\theta, \rho_\theta(t)) = \frac{1}{\phi_\alpha^*(t)}.$$

Thus, since $J_p(\rho,\theta)$, as a function of ρ on $[0,+\infty)$, is monotonically increasing and $J_p(\rho,\theta) \ge J_p(0,\theta) = 1$, we have

$$\begin{aligned}
t &= \int_{\mathbb{S}^{n-1}} \int_0^{\rho_\theta(t)} \rho^{n-1} J_p(\rho,\theta) d\sigma d\rho \\
&\le \int_{\mathbb{S}^{n-1}} \int_0^{\rho_\theta(t)} \rho^{n-1} J_p(\theta,\rho_\theta(t)) d\sigma d\rho \\
&= \int_{\mathbb{S}^{n-1}} J_p(\theta,\rho_\theta(t)) \left(\int_0^{\rho_\theta(t)} \rho^{n-1} d\rho \right) d\sigma \\
&= \frac{1}{n} \int_{\mathbb{S}^{n-1}} J_p(\theta,\rho_\theta(t)) \rho_\theta^n(t) d\sigma \\
&\le \frac{1}{n} \int_{\mathbb{S}^{n-1}} J_p^{n/(n-\alpha)}(\theta,\rho_\theta(t)) \rho_\theta^n(t) d\sigma \\
&= \frac{1}{n} \int_{\mathbb{S}^{n-1}} \left[J_p(\theta,\rho_\theta(t)) \rho_\theta^{n-\alpha}(t) \right]^{n/(n-\alpha)} d\sigma \\
&= \frac{1}{n} [\phi_\alpha^*(t)]^{-n/(n-\alpha)} |\mathbb{S}^{n-1}| = [\phi_\alpha^*(t)]^{-n/(n-\alpha)} \omega_n.
\end{aligned}$$

Therefore, $\phi_\alpha^*(t) \le \left(\frac{t}{\omega_n}\right)^{-(n-\alpha)/n}, \ \ t > 0$. This completes the proof of Lemma 3.2.

Lemma 3.3 Let $0 < \alpha < n$ and set $\widetilde{\phi}_\alpha = \frac{1}{\rho^{n-\alpha} \fint_{\mathbb{S}^{n-1}} J_p(\rho,\theta) d\theta}$, where $\fint_{\mathbb{S}^{n-1}} J_p(\rho,\theta) d\theta = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} J_p(\rho,\theta) d\theta$. Then

$$\widetilde{\phi}_\alpha^*(t) \le \left(\frac{t}{\omega_n}\right)^{-(n-\alpha)/n}, \ \ t > 0,$$

Proof The proof is similar to that given in Lemma 3.2. Set

$$\lambda_{\widetilde{\phi}_\alpha}(s) = \int_{\{x \in M : \widetilde{\phi}_\alpha(x) > s\}} dV = \int_{\{(\rho,\theta) \in M : \rho^{n-\alpha} \int_{\mathbb{S}^{n-1}} J_p(\rho,\theta) d\theta < s^{-1}\}} dV.$$

Denote by $\rho(s)$ the solution of $\rho^{n-\alpha}\int_{\mathbb{S}^{n-1}} J_p(\rho,\theta)d\theta = s^{-1}$. Then $\rho(s)$ satisfies

$$\rho(s)^{n-\alpha}\int_{\mathbb{S}^{n-1}} J_p(\rho(s),\theta)d\theta = s^{-1}$$

and

$$\lambda_{\widetilde{\phi}_\alpha}(s) = \int_{\{(\rho,\theta)\in M:\rho^{n-1}J_p(\rho,\theta)<s^{-1}\}} dV = \int_{\mathbb{S}^{n-1}}\int_0^{\rho(s)} \rho^{n-1}J_p(\rho,\theta)d\sigma d\rho.$$

With the same argument as that in the proof of Lemma 3.2, we have

$$\begin{aligned} t =&\lambda_{\widetilde{\phi}_\alpha}(\widetilde{\phi}^*_\alpha(t)) = \int_{\mathbb{S}^{n-1}}\int_0^{\rho(\widetilde{\phi}^*_\alpha(t))} \rho^{n-1}J_p(\rho,\theta)d\sigma d\rho \\ \le& \int_{\mathbb{S}^{n-1}} J_p(\theta,\rho(\widetilde{\phi}^*_\alpha(t)))d\sigma \int_0^{\rho(\widetilde{\phi}^*_\alpha(t))} \rho^{n-1}d\rho \\ =&\frac{|\mathbb{S}^{n-1}|}{n}\rho^n(\widetilde{\phi}^*_\alpha(t)) \fint_{\mathbb{S}^{n-1}} J_p(\theta,\rho(\widetilde{\phi}^*_\alpha(t))d\sigma \\ \le&\omega_n\left[\rho^{n-\alpha}(\widetilde{\phi}^*_\alpha(t)) \fint_{\mathbb{S}^{n-1}} J_p(\theta,\rho(\widetilde{\phi}^*_\alpha(t))d\sigma\right]^{\frac{n}{n-\alpha}} = \omega_n[\widetilde{\phi}^*_\alpha(t)]^{-n/(n-\alpha)}. \end{aligned}$$

The rest of proof is completely to that given in Lemma 3.2 and we omit it.

For simplicity, we set $\phi = \phi_1 = \frac{1}{\rho^{n-\alpha}J_p(\rho,\theta)}$. Then by by Lemma 3.2,

$$\phi^*(t) \le \left(\frac{t}{\omega_n}\right)^{-(n-1)/n}, \quad t>0. \tag{3.8}$$

4 Proofs of Theorems 1.1

Now we can give the proof of the main result.

Proof of Theorem 1.1

(1) Without loss of generality, we assume $\Omega \subset B(x_0,R) = \{x\in M : d(x,x_0) < R\}$ for some $x_0 \in M$ and $R>0$. Then by (3.1) in [8] we have

$$|u(p)| \le \frac{1}{n\omega_n}\int_M |\nabla_M u|\frac{1}{\rho^{n-1}J_p(\rho,\theta)}dV \le \frac{1}{n\omega_n}\int_M |\nabla_M u|\frac{1}{\rho^{n-1}}dV.$$

With the same arguments as in [9], we have, for some $A>0$ and $\epsilon>0$,

$$(\frac{1}{\rho^{n-1}})^*(t) \le \left(\frac{t}{\omega_n}\right)^{-(n-1)/n} + At^{-\frac{n-1-\epsilon}{n}}, \quad t\to 0+.$$

Applying O'Neil's lemma (see [12], Lemma 1.5), we have, by (3.1) in [8],

$$\begin{aligned} u^*(t) \le&\frac{1}{n\omega_n}\left(\frac{1}{t}\int_0^t |\nabla_M u|^*(s)ds \int_0^t \left[\left(\frac{s}{\omega_n}\right)^{-(n-1)/n} + As^{-\frac{n-1-\epsilon}{n}}\right]ds+\right. \\ &\left.\int_t^{|\Omega|} |\nabla_M u|^*(s)\left[\left(\frac{s}{\omega_n}\right)^{-(n-1)/n} + As^{-\frac{n-1-\epsilon}{n}}\right]ds\right), \end{aligned}$$

i.e.

$$\begin{aligned}u^*(t) \leq &\frac{1}{n\omega_n^{1/n}}\left((nt^{-\frac{n-1}{n}}+A\frac{n}{1+\epsilon}t^{-\frac{n-1-\epsilon}{n}})\int_0^t|\nabla_M u|^*(s)ds+\right.\\ &\left.\int_t^{|\Omega|}|\nabla_M u|^*(s)\left[s^{-\frac{n-1}{n}}+A\left(\omega_n\right)^{-(n-1)/n}s^{-\frac{n-1-\epsilon}{n}}\right]ds\right).\end{aligned} \tag{4.1}$$

With the same arguments as in [13], there exists a constant $C>0$ independent of u such that

$$\frac{1}{|\Omega|}\int_\Omega e^{\beta_q|u|^{q'}}dV<C.$$

Now we prove the sharpness of β_q. Denote by $B_r=\{x\in M:\rho(x)<r\}$. Set, for each $\varepsilon\in(0,1)$,

$$f_\varepsilon(\rho)=\begin{cases}\int_\rho^1\left[t\fint_{\mathbb{S}^{n-1}}J_p(t,\theta)d\theta\right]^{-1}dt\cdot\left\{\int_\varepsilon^1\left[t\fint_{\mathbb{S}^{n-1}}J_p(t,\theta)d\theta\right]^{-1}dt\right\}^{-1}, & \varepsilon\leq\varrho\leq 1;\\ 1, & 0\leq\rho\leq\varepsilon.\end{cases}$$

We compute

$$|\nabla_M f_\varepsilon(\rho)|=\begin{cases}\left[\rho\fint_{\mathbb{S}^{n-1}}J_p(\rho,\theta)d\theta\right]^{-1}\left\{\int_\varepsilon^1\left[t\fint_{\mathbb{S}^{n-1}}J_p(t,\theta)d\theta\right]^{-1}dt\right\}^{-1}, & \varepsilon<\varrho\leq 1;\\ 0, & 0\leq\rho<\varepsilon.\end{cases}$$

Following the proof of Lemma 3.3, we have

$$|\nabla_M f_\varepsilon|^*(t)\leq\left\{\int_\varepsilon^1\left[t\fint_{\mathbb{S}^{n-1}}J_p(t,\theta)d\theta\right]^{-1}dt\right\}^{-1}\left(\frac{\omega_n}{t+|B_\varepsilon|}\right)^{\frac{1}{n}},\ \ 0\leq t<|B_1|-|B_\varepsilon|$$

and

$$|\nabla_M f_\varepsilon|^*(t)=0,\ \ |B_1|-|B_\varepsilon|\leq t\leq|B_1|.$$

Therefore,

$$\begin{aligned}\|\nabla_M f_\varepsilon\|_{L^{n,q}(B_1)}^{q'}&=\left[\int_0^{|B_1|}\left(t^{\frac{1}{n}}|\nabla_M f|^*(t)\right)^q\frac{dt}{t}\right]^{\frac{1}{q-1}}\\ &\leq\frac{\omega_n^{q'/n}}{\left\{\int_\varepsilon^1\left[t\fint_{\mathbb{S}^{n-1}}J_p(t,\theta)d\theta\right]^{-1}dt\right\}^{q'}}\left[\int_0^{|B_1|-|B_\varepsilon|}\left(\frac{t}{t+|B_\varepsilon|}\right)^{\frac{q}{n}}\frac{dt}{t}\right]^{\frac{1}{q-1}}.\end{aligned}$$

Substituting $s=1+\frac{t}{|B_\varepsilon|}$ in the integral we have

$$\|\nabla_M f_\varepsilon\|_{L^{n,q}(B_1)}^{q'}\leq\frac{\omega_n^{q'/n}}{\left\{\int_\varepsilon^1\left[t\fint_{\mathbb{S}^{n-1}}J_p(t,\theta)d\theta\right]^{-1}dt\right\}^{q'}}\left[\int_1^{\frac{|B_1|}{|B\varepsilon|}}\left(1-\frac{1}{s}\right)^{\frac{q}{n}}\frac{ds}{s-1}\right]^{\frac{1}{q-1}}. \tag{4.2}$$

Now assume that

$$\frac{1}{|B_1|}\int_{B_1}\exp\left[\beta\left(\frac{|f_\varepsilon|}{\|\nabla_M f_\varepsilon\|_{L^{n,q}(B_1)}}\right)^{q'}\right]dV\le C$$

for some $\beta>0$. Using the fact $f_\varepsilon\equiv 1$ on B_ε, we have

$$\frac{|B_\varepsilon|}{|B_1|}\exp\left(\beta\frac{1}{\|\nabla_M f_\varepsilon\|^{q'}_{L^{n,q}(B_1)}}\right)\le C.$$

By (4.2), we have

$$\begin{aligned}\beta\le&\left(\ln C+\ln|B|+\ln|B_\varepsilon|^{-1}\right)\|\nabla_M f_\varepsilon\|^{q'}_{L^{n,q}(B_1)}\\ \le&\omega_n^{q'/n}\frac{\ln C+\ln|B|+\ln|B_\varepsilon|^{-1}}{\left\{\int_\varepsilon^1\left[t\,⨍_{\mathbb{S}^{n-1}}J_p(t,\theta)d\theta\right]^{-1}dt\right\}^{q'}}\left[\int_1^{\frac{|B_1|}{|B\varepsilon|}}\left(1-\frac{1}{s}\right)^{\frac{q}{n}}\frac{ds}{s-1}\right]^{\frac{1}{q-1}}.\end{aligned}\tag{4.3}$$

By (2.1), we have

$$|B_\varepsilon|=\int_0^\varepsilon\int_{\mathbb{S}^{n-1}}\rho^{n-1}J(\rho,\theta)d\rho d\theta=\omega_n\varepsilon^n(1+O(\varepsilon^2)),\ \ \varepsilon\to 0+.$$

Also by (2.1), one can easily check

$$\lim_{\varepsilon\to 0+}\frac{\ln C+\ln|B|+\ln|B_\varepsilon|^{-1}}{\int_\varepsilon^1\left[t\,⨍_{\mathbb{S}^{n-1}}J_p(t,\theta)d\theta\right]^{-1}dt}=n$$

and

$$\lim_{\varepsilon\to 0+}\frac{1}{\int_\varepsilon^1\left[t\,⨍_{\mathbb{S}^{n-1}}J_p(t,\theta)d\theta\right]^{-1}dt}\int_1^{\frac{|B_1|}{|B\varepsilon|}}\left(1-\frac{1}{s}\right)^{\frac{q}{n}}\frac{ds}{s-1}=n.$$

Therefore, passing the limit $\varepsilon\to 0+$ in (4.3) yields

$$\beta\le\omega_n^{q'/n}n^{1+\frac{1}{q-1}}=\beta_q.$$

(2) Since $\|\nabla_M u\|_{L^{n,\infty}(\Omega)}\le 1$, we have

$$|\nabla_M u|^*\le t^{-\frac{1}{n}},\ \ 0\le t\le|\Omega|.\tag{4.4}$$

Therefore, by (4.1), we have for $0<t<|\Omega|$,

$$\begin{aligned}u^*(t)\le&\frac{1}{n\omega_n^{1/n}}\left((nt^{-\frac{n-1}{n}}+A\frac{n}{1+\epsilon}t^{-\frac{n-1-\epsilon}{n}})\int_0^t|\nabla_M u|^*(s)ds+\right.\\ &\left.\int_t^{|\Omega|}|\nabla_M u|^*(s)\left[s^{-\frac{n-1}{n}}+A\left(\omega_n\right)^{-(n-1)/n}s^{-\frac{n-1-\epsilon}{n}}\right]ds\right)\\ \le&\frac{1}{\beta_\infty}\left(\frac{n^2}{n-1}+\frac{n^2A}{(n-1)(1+\epsilon)}t^{\frac{\epsilon}{n}}+\frac{n}{\epsilon}(|\Omega|^{\frac{\epsilon}{n}}-t^{\frac{\epsilon}{n}})+\ln\frac{|\Omega|}{t}\right).\end{aligned}$$

Thus, for every $\beta < \beta_\infty$, we have

$$\begin{aligned}\frac{1}{|\Omega|}\int_\Omega e^{\beta|u(x)|}dV =& \frac{1}{|\Omega|}\int_0^{|\Omega|} e^{\beta u^*(t)}dt \\ \leq& \frac{1}{|\Omega|}\int_0^{|\Omega|} \exp\frac{\beta}{\beta_\infty}\left(\frac{n^2}{n-1}+\frac{n^2A}{(n-1)(1+\epsilon)}t^{\frac{\epsilon}{n}}+\frac{n}{\epsilon}(|\Omega|^{\frac{\epsilon}{n}}-t^{\frac{\epsilon}{n}})+\ln\frac{|\Omega|}{t}\right)dt < \infty.\end{aligned}$$

To see the constant β_∞ is sharp, we consider the function

$$f(\rho) = \frac{1}{\beta_\infty}n\int_\rho^1\left[t\fint_{\mathbb{S}^{n-1}}J_p(t,\theta)d\theta\right]^{-1}dt, \quad 0<\rho\leq 1.$$

By (2.1), we have

$$f(\rho) = \frac{1}{\beta_\infty}n\ln\frac{1}{\rho}+O(\rho^2), \quad \rho\to 0+.$$

Therefore,

$$\int_{B_1} e^{\beta_\infty|f(x)|}dV = \infty.$$

On the other hand, since $|\nabla_M f| = \frac{n}{\beta_\infty\rho\fint_{\mathbb{S}^{n-1}}J_p(\rho,\theta)d\theta} = \frac{1}{\omega_n^{1/n}\rho\fint_{\mathbb{S}^{n-1}}J_p(\rho,\theta)d\theta}$, we have, by Lemma 3.3,

$$|\nabla_M f|^*(t) \leq t^{-\frac{1}{n}}$$

and thus

$$\|\nabla_M f\|_{L^{n,\infty}(B_1)} \leq 1.$$

The proof is thereby completed.

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Cartan-Hadamard流形上关于Lorentz范数的Trudinger-Moser不等式

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摘要: 本文研究了Cartan-Hadamard流形上带Lorentz范数的Trudinger-Moser不等式. 利用了相关格林函数的逐点估计以及O'Neil不等式, 我们得到了该不等式的最佳常数, 推广了相应欧氏空间上的结果.

关键词: Trudinger-Moser不等式；Lorentz空间；Riemannian流形；负曲率；最佳常数

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