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A STUDY ON THE WEIBULL AND PARETO DISTRIBUTIONS MOTIVATED BY CHVÁTAL'S THEOREM

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Abstract: Motivated by Chvátal's theorem, in this paper, we consider the infimum value problem on the probability that a random variable is at most its expectation. By employing the analytical method, the results for a random variable with the Weibull distribution or the Pareto distribution are obtained.

Keywords: Chvátal's theorem; Weibull distribution; Pareto distribution

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1 Introduction

Let $B(n,p)$ denote a binomial random variable with parameters n and p. Janson in [1] introduced the following conjecture suggested by Vašk Chvátal.

Conjecture 1 (Chvátal). For any fixed $n \geq 2$, as m ranges over $\{0, \ldots, n\}$, the probability $q_m := P(B(n, m/n) \leq m)$ is the smallest when m is closest to $\frac{2n}{3}$.

Conjecture 1 has applications in machine learning, such as the analysis of generalized boundaries of relative deviation bounds and unbounded loss functions ([2] and [3]). As to the probability of a binomial random variable exceeding its expectation, we refer to Doerr [2], Greenberg and Mohri [3], Pelekis and Ramon [4]. Janson [1] proved that Conjecture 1 holds for large n. Barabesi et al. [5] and Sun [6] gave an affirmative answer to Conjecture 1 for general $n \geq 2$. Hereafter, we call Conjecture 1 by Chvátal's theorem.

Motivated by Chvátal's theorem, Li et al. [7] considered the infimum value problem on the probability that a random variable is not more than its expectation, when its distribution is the Poisson distribution, the geometric distribution or the Pascal distribution. Sun et al. [8] investigated the corresponding infimum value problem for the Gamma distribution among

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other things. Hu et al. [9] studied the corresponding problem for some infinitely divisible distributions including the inverse Gaussian, log-normal, Gumbel and logistic distributions. In this note, we consider the infimum value problem for the Weibull distribution and the Pareto distribution in Sections 2 and 3, respectively. Before presenting the main results, we give two remarks.

Remark 1.1 (i) In virtue of Chvátal's theorem, there is a natural question as follows:

For any fixed integer $n \geq 2$, how about the minimum value of the probability $P(B(n,p) \leq np)$ for $p \in (0,1]$?

For small fixed n, we may find the solution. Up to now, we don't know the solution for general $n \geq 2$. In fact, it was posed as an open question in the first version of [7] (i.e., [10]).

(ii) Motivated by Chvátal's theorem, Li et al. [7] initiated the study on the infimum value problem on the probability that a random variable is not more than its expectation. In this topic, [8] is the second paper, this note is the third one and [9] is the fourth one.

Remark 1.2 Let X be a random variable with expectation EX. Assume that the distribution of X contains some parameters α and β. The motivation to study $\inf_{\alpha,\beta} P(X \leq EX)$ is that from it we can get $\sup_{\alpha,\beta} P(X > EX)$. Obviously, if we wish the probability $P(X > EX)$ is as large as possible, we should find $\sup_{\alpha,\beta} P(X > EX)$ or equivalently $\inf_{\alpha,\beta} P(X \leq EX)$. We wish that our work on this topic may find some applications in machine learning, statistics, finance and economics etc.

2 The Weibull Distribution

Let X be a Weibull random variable with parameters α and $\theta\,(\alpha > 0, \theta > 0)$ and the density function

$$f(x) = \frac{\alpha}{\theta} x^{\alpha-1} e^{-\frac{x^\alpha}{\theta}},\ x > 0.$$

We know that its expectation $EX = \theta^{\frac{1}{\alpha}} \Gamma\left(\frac{1}{\alpha}+1\right)$, where $\Gamma\left(\frac{1}{\alpha}+1\right)$ is the Gamma function, i.e., $\Gamma\left(\frac{1}{\alpha}+1\right) = \int_0^\infty u^{\frac{1}{\alpha}} e^{-u} du$. For any given real number $\kappa > 0$, we have

$$P\left(X \leq \kappa EX\right) = \int_0^{\kappa\theta^{\frac{1}{\alpha}}\Gamma\left(\frac{1}{\alpha}+1\right)} \frac{\alpha}{\theta} t^{\alpha-1} e^{-\frac{t^\alpha}{\theta}} dt.$$

By taking the change of variable $t = (\theta x)^{\frac{1}{\alpha}}$, we get

$$\begin{aligned} P\left(X \leq \kappa EX\right) &= \int_0^{\left(\kappa\Gamma\left(\frac{1}{\alpha}+1\right)\right)^\alpha} \frac{\alpha}{\theta} \left(\theta x\right)^{\frac{\alpha-1}{\alpha}} e^{-x} \frac{\theta^{\frac{1}{\alpha}}}{\alpha} x^{\frac{1}{\alpha}-1} dx \\ &= \int_0^{\left(\kappa\Gamma\left(\frac{1}{\alpha}+1\right)\right)^\alpha} e^{-x} dx \\ &= 1 - e^{-\left(\kappa\Gamma\left(\frac{1}{\alpha}+1\right)\right)^\alpha}, \end{aligned}$$

which shows that $P\left(X \leq \kappa EX\right)$ is independent of θ.

Define a function

$$g_\kappa(\alpha) := 1 - e^{-\left(\kappa\Gamma\left(\frac{1}{\alpha}+1\right)\right)^\alpha}, \quad \alpha > 0. \tag{2.1}$$

The main result of this section is

Proposition 2.1 (i) If $\kappa \le 1$, then

$$\inf_{\alpha\in(0,\infty)} g_\kappa(\alpha) = \lim_{\alpha\to\infty} g_\kappa(\alpha) = \begin{cases} 0, & \kappa < 1, \\ 1 - e^{-e^{-\gamma}}, & \kappa = 1, \end{cases}$$

where γ is the Euler's constant, i.e., $\gamma = \sum_{n=1}^{\infty}\left[\frac{1}{n} - \ln\left(1+\frac{1}{n}\right)\right]$.
(ii) If $\kappa > 1$, then

$$\min_{\alpha\in(0,\infty)} g_\kappa(\alpha) = g_\kappa(\alpha_0(\kappa)),$$

where $\alpha_0(\kappa) = \frac{1}{x_0(\kappa)-1}$, and $x_0(\kappa)$ is the unique null point of function $\varphi_\kappa(x) := (x-1)\psi(x) - \ln(\kappa\Gamma(x))$ on $(1,\infty)$, where $\psi(x)$ is the digamma function (see Definition 2.3 below).

Note that $\left(\kappa\Gamma\left(\frac{1}{\alpha}+1\right)\right)^\alpha = e^{\alpha\ln\left(\kappa\Gamma\left(\frac{1}{\alpha}+1\right)\right)}$. Let $x = \frac{1}{\alpha}+1$, and define function

$$h_\kappa(x) := \frac{\ln(\kappa\Gamma(x))}{x-1}, \quad x > 1. \tag{2.2}$$

Then

$$g_\kappa(\alpha) = 1 - e^{-e^{h_\kappa(x)}}, \tag{2.3}$$

and in order to finish the proof of Proposition 2.1, it is enough to prove the following lemma.

Lemma 2.2 (i) If $\kappa \le 1$, then

$$\inf_{x\in(1,\infty)} h_\kappa(x) = \lim_{x\to 1^+} h_\kappa(x) = \begin{cases} -\infty, & \kappa < 1, \\ -\gamma, & \kappa = 1, \end{cases}$$

where γ is the Euler's constant.
(ii) If $\kappa > 1$, then

$$\min_{x\in(1,\infty)} h_\kappa(x) = h_\kappa(x_0(\kappa)),$$

where $x_0(\kappa)$ is the unique null point of function $\varphi_\kappa(x) := (x-1)\psi(x) - \ln(\kappa\Gamma(x))$ on $(1,\infty)$, where $\psi(x)$ is the digamma function.

Before giving the proof of Lemma 2.2, we need some preliminaries on the ploygamma function.

Definition 2.3 ([11, 1.16]) Let m be any nonnegative integers. m-order ploygamma function $\psi^{(m)}$ is defined by

$$\psi^{(m)}(z) := \frac{\mathrm{d}^m}{\mathrm{d}z^m}\psi(z) = \frac{\mathrm{d}^{m+1}}{\mathrm{d}z^{m+1}}\ln\Gamma(z), \mathrm{Re}z > 0.$$

When $m = 0$, $\psi(z) := \psi^{(0)}(z) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is called digamma function.

By [11, 1.7(3)] and [11, 1.9(10)], we know that

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(z+n)} = -\gamma + (z-1)\sum_{n=0}^{\infty} \frac{1}{[(n+1)(z+n)]}, \tag{2.4}$$

$$\psi^{(1)}(z) = \psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}. \tag{2.5}$$

Proof of Lemma 2.2 By (2.2) and Definition 2.3, we have

$$h'_\kappa(x) = \frac{(x-1)\psi(x) - \ln(\kappa\Gamma(x))}{(x-1)^2} = \frac{\varphi_\kappa(x)}{(x-1)^2}, \quad x > 1. \tag{2.6}$$

By (2.5), we get

$$\varphi'_\kappa(x) = (x-1)\psi^{(1)}(x) = (x-1)\sum_{n=0}^{\infty} \frac{1}{(x+n)^2} > 0, \quad \forall x > 1.$$

It follows that the function $\varphi_\kappa(x)$ is strictly increasing on the interval $(1,\infty)$.

Thus, if $\kappa \le 1$, we have $\varphi_\kappa(x) > \varphi_\kappa(1) = -\ln\kappa \ge 0, \quad \forall x > 1$. Then, by (2.6) we get $h'_\kappa(x) > 0, \quad \forall x > 1$, which implies that the function $h_\kappa(x)$ is strictly increasing on $(1,+\infty)$. Hence the function $h_\kappa(x)$ has no minimum value on $(1,\infty)$ and

$$\inf_{x\in(1,\infty)} h_\kappa(x) = \lim_{x\to1^+} h_\kappa(x) = \lim_{x\to1^+} \frac{\ln\Gamma(x)}{x-1} + \lim_{x\to1^+} \frac{\ln\kappa}{x-1}.$$

By the L'Hospital's rule and (2.4), we have

$$\lim_{x\to1^+} \frac{\ln\Gamma(x)}{x-1} = \lim_{x\to1^+} \tfrac{\Gamma'(x)}{\Gamma(x)} = \Gamma'(1) = \psi(1) = -\gamma.$$

Thus,

$$\lim_{x\in(1,\infty)} h_\kappa(x) = \begin{cases} -\infty, & \kappa < 1, \\ -\gamma, & \kappa = 1. \end{cases}$$

If $\kappa > 1$, then $\varphi_\kappa(1) = -\ln\kappa < 0$. By [11, 1.18(1)] (Stirling formula) and [11, 1.18(7)], when $z \to \infty$ we have

$$\begin{aligned} \ln\Gamma(z) &= \left(z - \tfrac{1}{2}\right)\ln z - z + \tfrac{\ln(2\pi)}{2} + o(1), \\ \psi(z) &= \ln z - \tfrac{1}{2z} + o\left(\tfrac{1}{z}\right), \quad |\arg z| < \pi. \end{aligned}$$

Then

$$\begin{aligned} \lim_{x\to\infty} \varphi_\kappa(x) &= \lim_{x\to\infty} [(x-1)\psi(x) - \ln\Gamma(x) - \ln\kappa] \\ &= \lim_{x\to\infty} \left[(x-1)\left(\ln x - \frac{1}{2x} + o\left(\frac{1}{x}\right)\right)\right. \\ &\qquad \left. - \left(\left(x - \frac{1}{2}\right)\ln x - x + \frac{\ln(2\pi)}{2} + o(1)\right) - \ln\kappa\right] \\ &= \lim_{x\to\infty} \left[x - \frac{1}{2}\ln x + \frac{1}{2x} - \frac{1}{2} - \frac{\ln(2\pi)}{2} - \ln\kappa + o(1)\right] \\ &= \infty. \end{aligned}$$

Since the function $\varphi_\kappa(x)$ is continuous, by the zero point theorem, there exists $x_0(\kappa) \in (1,\infty)$ which depends on κ satisfying that $\varphi_\kappa(x_0(\kappa)) = 0$. Moreover, combining with the monotonicity of the function $\varphi_\kappa(x)$ on the interval $(1,\infty)$, we know that $x_0(\kappa)$ is the unique null point of the function $\varphi_\kappa(x)$ and

$$\begin{aligned} \varphi_\kappa(x) &< 0, \quad \forall x \in (1, x_0(\kappa)); \\ \varphi_\kappa(x) &> 0, \quad \forall x \in (x_0(\kappa), +\infty). \end{aligned}$$

Then, by (2.6) we get

$$\begin{aligned} h'_\kappa(x) &< 0, \quad \forall x \in (1, x_0(\kappa)); \\ h'_\kappa(x) &> 0, \quad \forall x \in (x_0(\kappa), +\infty). \end{aligned}$$

Thus, the function $h_\kappa(x)$ is strictly decreasing on $(1, x_0(\kappa))$ and strictly increasing on $(x_0(\kappa), +\infty)$, which implies that

$$h_\kappa(x) \geq h_\kappa(x_0(\kappa)), \quad \forall x > 1.$$

Therefore,

$$\min_{x\in(1,\infty)} h_\kappa(x) = h_\kappa(x_0(\kappa)).$$

The proof is complete.

3 The Pareto Distribution

Let X be a Pareto random variable with parameters a and $\theta\,(a > 0, \theta > 0)$ and the density function

$$f(x) = \theta a^\theta x^{-(\theta+1)} I_{(a,\infty)}(x).$$

When $\theta > 1$, the expectation of X is $EX = \frac{\theta a}{\theta-1}$. Then, for any given real number $\kappa > 0$, we have

$$P(X \leq \kappa EX) = \int_a^{\frac{\kappa\theta a}{\theta-1}} \theta a^\theta t^{-(\theta+1)} dt = -\left(\frac{a}{t}\right)^\theta \Big|_a^{\frac{\kappa\theta a}{\theta-1}} = 1 - \left(\frac{\theta-1}{\kappa\theta}\right)^\theta,$$

which shows that $P(X \leq \kappa EX)$ is independent of a. Note that, in order to make sense of the above equality, if $\kappa < 1$, the parameter θ should satisfy that $1 < \theta \leq \frac{1}{1-\kappa}$; and if $\kappa \geq 1$, the parameter θ should satisfy that $\theta > 1$.

Define a function

$$g_\kappa(\theta) := 1 - \left(\frac{\theta-1}{\kappa\theta}\right)^\theta, \ 1 < \theta \leq \frac{1}{1-\kappa}, \kappa < 1 \text{ or } \theta > 1, \kappa \geq 1.$$

The main result of this section is

Proposition 3.1 (i) If $\kappa < 1$, then

$$\min_{\theta\in\left(1,\frac{1}{1-\kappa}\right]} g_\kappa(\theta) = g_\kappa\left(\frac{1}{1-\kappa}\right) = 0.$$

(ii) If $\kappa = 1$, then $\inf\limits_{\theta \in (1,\infty)} g_1(\theta) = \lim\limits_{\theta \to \infty} g_1(\theta) = 1 - e^{-1}$.
(iii) If $\kappa > 1$, then $\min\limits_{\theta \in (1,\infty)} g_\kappa(\theta) = g_\kappa(\theta_0(\kappa))$, where $\theta_0(\kappa) = \frac{1}{1 - x_0(\kappa)}$, and $x_0(\kappa)$ is the unique null point of function $\varphi_\kappa(x) := 1 - \frac{1}{x} - \ln \frac{x}{\kappa}$ on the interval $(0,1)$.

Note that $\left(\frac{\theta-1}{\kappa\theta}\right)^\theta = e^{\theta \ln \frac{\theta-1}{\kappa\theta}}$. Let $x = 1 - \frac{1}{\theta}$ and define function

$$h_\kappa(x) := \frac{\ln \frac{x}{\kappa}}{x-1}, \ 0 < x \le \kappa, \kappa < 1, \text{ or } 0 < x < 1, \kappa \ge 1. \tag{3.1}$$

Then

$$g_\kappa(\theta) = 1 - e^{-h_\kappa(x)}, \tag{3.2}$$

and in order to finish the proof of Proposition 3.1, it is enough to prove the following lemma.

Lemma 3.2
(i) If $\kappa < 1$, then $\min\limits_{x \in (0,\kappa]} h_\kappa(x) = h_\kappa(\kappa) = 0$.
(ii) If $\kappa = 1$, then $\inf\limits_{x \in (0,1)} h_1(x) = \lim\limits_{x \to 1^-} h_1(x) = 1$.
(iii) If $\kappa > 1$, then $\min\limits_{x \in (0,1)} h_\kappa(x) = h_\kappa(x_0(\kappa))$, where $x_0(\kappa)$ is the unique null point of function $\varphi_\kappa(x) := 1 - \frac{1}{x} - \ln \frac{x}{\kappa}$ on the interval $(0,1)$.

Proof (i) If $\kappa < 1$, by (3.1) we have

$$h'_\kappa(x) = \frac{1 - \frac{1}{x} - \ln \frac{x}{\kappa}}{(x-1)^2} = \frac{\varphi_\kappa(x)}{(x-1)^2}, \quad 0 < x \le \kappa. \tag{3.3}$$

By the definition of $\varphi_\kappa(x)$, we get that

$$\varphi'_\kappa(x) = \frac{1-x}{x^2} > 0, \quad \forall 0 < x \le \kappa.$$

It follows that function $\varphi_\kappa(x)$ is strictly increasing on $(0,\kappa]$ and thus

$$\varphi_\kappa(x) \le \varphi_\kappa(\kappa) = \frac{\kappa - 1}{\kappa} < 0, \quad \forall 0 < x \le \kappa. \tag{3.4}$$

Then, by (3.3) and (3.4) we get $h'_\kappa(x) < 0, \quad \forall 0 < x \le \kappa$, which implies that function $h_\kappa(x)$ is strictly decreasing on $(0,\kappa]$. Thus $\min\limits_{x \in (0,\kappa]} h_\kappa(x) = h_\kappa(\kappa) = 0$.

If $\kappa \ge 1$, by (3.1) and the definition of $\varphi_\kappa(x)$ again, we also have

$$h'_\kappa(x) = \frac{\varphi_\kappa(x)}{(x-1)^2}, \quad 0 < x < 1, \tag{3.5}$$

and

$$\varphi'_\kappa(x) = \frac{1-x}{x^2} > 0, \quad \forall 0 < x < 1.$$

It follows that function $\varphi_\kappa(x)$ is strictly increasing on $(0,1)$.

(ii) If $\kappa = 1$, then

$$\varphi_\kappa(x) < \varphi_\kappa(1) = -\ln\kappa = 0, \quad \forall 0 < x < 1.$$

By (3.5), we get that $h'_\kappa(x) < 0, \quad \forall 0 < x < 1$, which implies that function $h_\kappa(x)$ is strictly decreasing on $(0,1)$. Thus,

$$\inf_{x\in(0,1)} h_\kappa(x) = \lim_{x\to 1^-} h_\kappa(x) = \lim_{x\to 1^-} \frac{\ln x}{x-1} = 1.$$

(iii) If $\kappa > 1$, then $\varphi_\kappa(1) = \ln\kappa > 0$. Moreover,

$$\lim_{x\to 0^+} \varphi_\kappa(x) = \lim_{x\to 0^+} \left(1 - \frac{1}{x} - \ln x + \ln\kappa\right) = \lim_{x\to 0^+} \left(1 + \ln\kappa - \frac{x\ln x + 1}{x}\right) = -\infty.$$

Since the function $\varphi_\kappa(x)$ is continuous on $(0,1)$, by the zero point theorem, there exists $x_0(\kappa) \in (0,1)$ depending on parameter κ fulfills that $\varphi_\kappa(x_0(\kappa)) = 0$. By the monotonicity of function $\varphi_\kappa(x)$ on $(0,1)$, we know that $x_0(\kappa)$ is the unique null point of $\varphi_\kappa(x)$ and

$$\begin{aligned} \varphi_\kappa(x) &< 0, \quad \forall x \in (0, x_0(\kappa)); \\ \varphi_\kappa(x) &> 0, \quad \forall x \in (x_0(\kappa), 1). \end{aligned}$$

Then, by (3.5) we have

$$\begin{aligned} h'_\kappa(x) &< 0, \quad \forall x \in (0, x_0(\kappa)); \\ h'_\kappa(x) &> 0, \quad \forall x \in (x_0(\kappa), 1). \end{aligned}$$

Therefore, the function $h_\kappa(x)$ is strictly decreasing on $(0, x_0(\kappa))$ and is strictly increasing on $(x_0(\kappa), 1)$. Thus

$$\min_{x\in(0,1)} h_\kappa(x) = h_\kappa(x_0(\kappa)).$$

The proof is complete.

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由Chvátal定理引出的关于Weibull分布和Pareto分布的研究

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摘要：受Chvátal 定理的启发，本文研究了随机变量不超过其期望的概率的下确界问题. 利用分析的方法，我们得到了当随机变量的分布为Weibull 分布或Pareto分布时该随机变量不超过其期望的概率的下确界.

关键词：Chvátal定理；Weibull分布；Pareto分布

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