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THE PERFECT INTEGER k-MATCHINGS AND k-FACTOR-CRITICAL GRAPHS

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Abstract: This article investigates the existence of perfect integer k-matchings and k-factor critical graphs. The extension constant represents the connectivity strength of a graph. For regular graphs, a sufficient condition for the existence of perfect integer k-matching is given using the extension constant, which extends the results of Hamers et al. and Cioabă et al. In addition, for regular graphs, a sufficient condition for the existence of k-factor-critical graphs based on extension constant is also given.

Keywords: perfect integer k-matching; k-factor-critical graph; connectivity; expansion constant

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1 Introduction

All graphs considered here are undirected, connected and simple. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The order of G is $|V(G)|$. For a vertex $v \in V(G)$, define $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and $\Gamma(v) = \{e \in E(G) : e \text{ is incident with } v \text{ in } G\}$. An integer k-matching of a graph $G = (V, E)$ is a function h that assigns to each edge an integer in $\{0, \cdots, k\}$ such that $\sum\limits_{e\in\Gamma(v)} f(e) \le k$ for each $v \in V$. A vertex v of G is saturated by an integer k-matching h or v is h-saturated if $\sum\limits_{e\in\Gamma(v)} f(e) = k$, otherwise, v is h-unsaturated. An integer k-matching is perfect if $\sum\limits_{e\in\Gamma(v)} f(e) = k$ for every vertex $v \in V(G)$.

Clearly, an integer k-matching is perfect if and only if its size is $\frac{k|V(G)|}{2}$. Note that when k is odd, a graph with a perfect integer k-matching has an even number of vertices. For $k = 1$, the integer (or perfect) 1-matching is just the matching (or perfect matching) in usual sense. If $k = 2$, then the perfect 2-matching coincides with the one defined by Tutte ([1]).

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Integer k-matchings have attracted many researchers' attention, such as Lu et al. ([2]) studied the perfect k-matchings of general graph and gave a sufficient and necessary condition for its existence, which is the Tutte's Theorem for perfect integer k-matching of a graph. Liu et al. ([3]) proved that when k is even, the integer k-matching number of G equals k times its fractional matching number. Hence, by the Berge-Tutte formula of fractional matching in ([4]), they can easily obtained the integer k-matching analogue of the Berge-Tutte Formula when k is even.

In addition, Gallai (1963, [5]) introduced the concepts of factor-critical graphs, i.e. a graph G is factor-critical if $G-v$ has a perfect matching for every vertex v of G. Very recently, Liu et al. ([6]) extended this definition and defined k-factor-critical graph. A connected graph G with at least three vertices is said to be k-factor-critical if for any $v \in V(G)$, there exists an integer k-matching h such that $\sum\limits_{e\in\Gamma(v)} f(e) = k-1$ and other vertices are h-saturated. We note that this definition is different from the one defined by Favaron and Odile ([7]). Moreover, Liu et al. ([6]) proved the integer k-matching analogue of the Berge-Tutte Formula when k is odd, they also gave sufficient and necessary conditions for the existence of k-factor-critical.

For two subsets $S, T \subset V(G)$, let $e(S,T)$ denotes the number of edges of G joining S to T. For a set X, we denote the cardinality of X by $|X|$. Given a subset $S \subset V(G)$, we let $e(S,S^c)$ denote the number of edges with exactly one endpoint in S, where S^c denotes the complement of S in $K_{|V(G)|}$. The expansion constant of a graph G is $h(G) = \min\frac{e(S,S^c)}{|S|}$, where the minimum is taken over all $S \subset V(G)$ with $|S| \le \frac{|V(G)|}{2}$, the expansion constant of a graph represents a strength of connection of the graph ([8]).

In [9], Cioabă proved a sufficient condition for the existence of a perfect matching in a regular graph in terms of expansion constant and λ_3, which improved the result of Brouwer and Haemers ([10]). For regular graphs, Lu et al. ([2]) provided a sufficient condition for the existence of perfect k-matching in terms of the edge connectivity. Based on this, we will consider the existence of perfect integer k-matching and k-factor-critical graph in terms of the expansion constant. We present a sufficient condition for the existence of a perfect integer k-matching in a regular graph in terms of its expansion constant, which generalizes the results of Haemers et al. ([10]) and Cioabă et al. ([9]). Furthermore, a sufficient condition for the existence of k-factor-critical graphs for regular graphs in terms of its expansion constant is obtained.

2 Expansion and Perfect Integer k-Matching

In this section, we determine a lower bound on the expansion constant of a regular graph, which implies the existence of a perfect integer k-matching.

Given a subset $S \subseteq V(G)$, we say $G-S$ is the subgraph obtained from G by deleting all vertices of S and their incident edges. In 2014, Lu et al. ([2]) obtained the Tutte's Theorem for perfect integer k-matching of a graph.

Lemma 2.1 (see [2]) Let $k \ge 2$ be even. A graph G has a perfect integer k-matching if and only if

$$i(G-S) \le |S| \ for\ all\ S \subseteq V(G),$$

where $i(G-S)$ is the number of isolated vertices of $G-S$.

Let $odd(G)$ denote the number of odd components with order at least three in G.

Lemma 2.2 (see [2]) Let $k \ge 1$ be odd. A graph G has a perfect integer k-matching if and only if

$$odd(G-S) + k \cdot i(G-S) \le k|S| \ for\ all\ S \subseteq V(G),$$

where $odd(G-S)$ is the number of odd components with order at least three of $G-S$ and $i(G-S)$ is the number of isolated vertices of $G-S$.

Theorem 2.1 Let G be a d-regular graph with n vertices.

(1) If k is even, and

$$h(G) \ge \begin{cases} \frac{d-2}{d+1}, & if\ d\ is\ even, \\ \frac{d-1}{d+1}, & otherwise, \end{cases}$$

then G has a perfect integer k-matching;

(2) For even n. If k is odd, and

$$h(G) \ge \begin{cases} \frac{d-2}{d+1}, & if\ d\ is\ even, \\ \frac{d-2}{d+2}, & otherwise, \end{cases}$$

then G has a perfect integer k-matching.

Proof (1) Assume G has no perfect integer k-matching. By Lemma 2.1, there is a vertex set S such that $i(G-S) > |S|$. Let $|S| = s$ and $i(G-S) = q$. We easily observe $q \ge s+1$. Let $V' = V(G) \backslash S$. Thus $|V'| = n - s$. We denote the q isolated vertices of $G-S$ by $v_1, v_2, \cdots, v_q$. Let $G_1, G_2, \cdots, G_q$ be the subgraphs induced by v_i and $N_G(v_i)$, where $N_G(v_i)$ is the set of neighbours of v_i and $1 \le i \le q$. Denote by n_i and e_i the order and the size of G_i, respectively.

For $i \in [q]$, let t_i denote the number of edges in G between G_i and $S - N_G(v_i)$. Since G is connected, it follows that $t_i \ge 1$ for each $i \in [q]$. Because v_i is adjacent only to vertices in S, we deduce that $2e_i = d(d+1) - t_i$. Clearly t_i is even.

The sum of the degrees of the vertices in S is at least the number of edges between S and $\cup_{i=1}^{q} v_i$. Thus, $ds \ge \sum\limits_{i=1}^{q} t_i$. Since $q \ge s+1$, it follows that there are at least two t_i's such that $t_i < d$. This implies there are at least two t_i's satisfying $t_i \le d-2$ if d is even and $t_i \le d-1$ if d is odd.

The fact $n_1 + n_2 < n$ implies that there is at least one i such that $n_i < \frac{n}{2}$. Without losing generality, assume $n_1 < \frac{n}{2}$. Then we obtain $\frac{t_1}{n_1} \le \frac{d-2}{d+1}$ if d is even and $\frac{t_1}{n_1} \le \frac{d-1}{d+1}$ if d is odd. This contradicts the assumption made on the expansion constant, which finishes the proof.

(2) By the definition of perfect integer k-matching, we know that a graph with a perfect integer k-matching has an even number of vertices when k is odd. So we only discuss the case n is even.

Assume G has no perfect integer k-matching. By Lemma 2.2, there is a vertex set S such that $odd(G-S)+k\cdot i(G-S)>k|S|$. Let $|S|=s$ and $odd(G-S)+i(G-S)=q$. By easily process, we can obtain $q>s$. But since n is even, $s+q$ is even, hence $q\geq s+2$. Let $V'=V(G)\backslash S$. Thus $|V'|=n-s$. We denote the q components by $G_1,G_2,\cdots,G_q$. Denote by n_i and e_i the order and the size of G_i respectively.

For $i\in[q]$, let t_i denotes the number of edges with one endpoint in G_i and another in S. Since G is connected, it follows that $t_i\geq 1$ for each $i\in[q]$. Because vertices in G_i are adjacent only to vertices in G_i or S, we deduce that $2e_i=dn_i-t_i=d(n_i-1)+d-t_i$. Because n_i is odd, it follows that $d-t_i$ is even. Hence, t_i and d have the same parity for each i.

The sum of the degrees of the vertices in S is at least the number of edges between S and $\cup_{i=1}^{q}G_i$. Thus, $ds\geq\sum\limits_{i=1}^{q}t_i$. Since $q\geq s+2$, it follows that there are at least three t_i's such that $t_i<d$. This implies there are at least three t_i's satisfying $t_i\leq d-2$. If $t_i\leq d-2$, then $n_i>1$. Assume $t_i\leq d-2$ for $i\in[3]$. If $t_i\leq d-2$, then $n_i(n_i-1)\geq 2e_i=dn_i-t_i\geq dn_i-d+2$. Thus, $n_i\geq d+\frac{2}{n_i-1}$. Hence, $n_i\geq d+1$. Now, since d and n_i are both odd, we obtain $n_i\geq d+2$.

The fact $n_1+n_2+n_3<n$ implies that there is at least one i such that $n_i<\frac{n}{2}$. Without losing generality, assume $n_1<\frac{n}{2}$. Then we obtain $\frac{t_1}{n_1}\leq\frac{d-2}{d+1}$ if d is even and $\frac{t_1}{n_1}\leq\frac{d-2}{d+2}$ if d is odd. This contradicts the assumption made on the expansion constant, which finishes the proof.

From the above Theorem 2.1, if n is even and let $k=1$, then we may deduce the following results obtained by Cioabă [9], which provides an expansion constant condition to guarantee that there exists a perfect matching in a regular graph G.

Corollary 2.1 (see [9, 10]) Let G be a d-regular graph. If n is even and

$$h(G)\geq\begin{cases}\frac{d-2}{d+1}, & if\ d\ is\ even,\\ \frac{d-2}{d+2}, & if\ d\ is\ odd,\end{cases}$$

then G has a perfect matching.

3 Expansion and k-Factor-Critical Graph

In this section, for a positive integer k and a regular graph G, we obtain a condition of k-factor-critical according to a lower bound of the expansion constant $h(G)$.

In 2021, Liu et al. ([6]) gave a sufficient and necessary condition for the existence of k-factor-critical graph.

Lemma 2.3 (see [6]) A connected graph G with at least three vertices is k-factor-critical if and only if G has odd number of vertices and $odd(G-S)+k\cdot i(G-S)\leq k|S|-1$ for any $\emptyset\neq S\subseteq V(G)$.

Theorem 2.2 Let G be a d-regular graph with n vertices. If n is odd and

$$h(G) \geq \begin{cases} \frac{d-2}{d+1}, & if\ d\ is\ even, \\ \frac{d-2}{d+2}, & if\ d\ is\ odd, \end{cases}$$

then G is k-factor-critical.

Proof Suppose G satisfies the conditions of this theorem, but it is not k-factor-critical. By Lemma 2.3, there is a vertex set S such that $odd(G-S)+k\cdot i(G-S) \geq k|S|$. Let $|S| = s$ and $odd(G-S)+i(G-S) = q$, we can get $q \geq s$. And since n is odd, $s+q$ is odd, hence $q \geq s+1$.

Let $G_1, G_2, \cdots, G_q$ be the q odd components with odd vertices of $G-S$. We denote the order and size of G_i by n_i and e_i, respectively. For $i \in [q]$, let t_i denote the number of edges with one endpoint in G_i and another in S. We know $t_i \geq 1$ for $i \in [q]$ because G is connected. Since vertices in G_i are adjacent only to vertices in G_i or S, we can get that $2e_i = dn_i - t_i = d(n_i-1)+d-t_i$. This implies that $d-t_i$ is even since n_i is odd. Therefore, t_i and d have the same parity for each i.

The sum of degrees of the vertices of S is at least the number of edges between S and the odd component G_i for $i \in [q]$. Hence, $ds \geq \sum\limits_{i=1}^{q} t_i$. Because $q \geq s+1$, it follows that there are at least two t_i's such that $t_i < d$. Hence, there are at least two t_i's such that $t_i < d-2$. If $t_i \leq d-2$, then $n_i > 1$. Without loss of generality, we assume $t_i \leq d-2$ for $i \in [2]$, then $n_i(n_i-1) \geq 2e_i = dn_i - t_i$. Hence, $n_i \geq d + \frac{2}{n_i-1}$. Then $n_i \geq d+1$. If d is odd, we get $n_i \geq d+2$ since n_i is odd.

There is at least one i such that $n_i < \frac{n}{2}$ because of $n_1+n_2 < n$. We assume $n_1 < \frac{n}{2}$. Therefore, we get $\frac{t_1}{n_1} \leq \frac{d-2}{d+1}$ if d is even and $\frac{t_1}{n_1} \leq \frac{d-2}{d+2}$ if d is odd. This contradicts the condition made on the expansion constant, which completes the proof.

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完美整数 k- 匹配和 k- 因子临界图

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摘要: 本文研究完美整数 k- 匹配和 k- 因子临界图的存在性. 扩张常数表示图的连通强度, 对于正则图, 利用扩张常数给出了完美整数 k-匹配存在的一个充分条件, 这推广了 Hamers 等人和 Cioabă 等人的结果. 此外, 对于正则图, 基于扩张常数还给出了 k-因子临界图存在的一个充分条件.

关键词: 完美整数 k-匹配; k-因子临界图; 连通性; 扩张常数

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