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SHARP LARGE DEVIATIONS FOR THE LOG-LIKELIHOOD RATIO OF THE COX-INGERSOLL-ROSS PROCESS

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Abstract: In this paper, for the Cox-Ingersoll-Ross process in the stationary case, we investigate the sharp large deviations for the log-likelihood ratio under null hypothesis and alternative hypothesis. By using the change of measure and characteristic function techniques, we obtain the full expansion for the log-likelihood ratio.

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1 Introduction and Main Results

Consider the following Cox-Ingersoll-Ross(CIR) process

$$dX_t = (\delta + bX_t)\, dt + 2\sqrt{X_t} dB_t,\ X_0 = 0, \tag{1.1}$$

where $\delta > 2$ is a known constant, b is an unknown parameter, and $\{B_t,\ t \geq 0\}$ is a standard Brownian motion. The Cox-Ingersoll-Ross model was introduced by Cox, Ingersoll and Ross in 1985, which was mainly used to study the term structure of interest rates. If $b > 0$, the process is explosive; if $b < 0$, the process is stationary. Let $P_{\delta,b}$ denote the probability distribution of the solution of (1.1) on $C(\mathbb{R}^+, \mathbb{R})$.

Let V_T denote the log-likelihood ratio at time T, namely

$$V_T = \log \frac{dP_{\delta,b_1}}{dP_{\delta,b_0}}\bigg|_{\mathscr{F}_T},$$

by using Girsanov formula,

$$\log\left(\frac{dP_{\delta,b_1}}{dP_{\delta,b_0}}\bigg|_{\mathscr{F}_T}\right) = \frac{b_1 - b_0}{4}\left(X_T - \delta T\right) - \frac{1}{8}\left(b_1^2 - b_0^2\right)\int_0^T X_t dt, \tag{1.2}$$

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where $\mathscr{F}_T = \sigma(B_t,\ t \le T)$.

The above log-likelihood ratio process plays a crucial role in statistical inference. The maximum likelihood estimator $\hat{b}_T$ of the parameter b can be defined by maximizing the likelihood ratio. According to the value of b, the asymptotic distributions and the corresponding speeds of the maximum likelihood estimators are quite different. Overbeck [1] studied that $\hat{b}_T$ is consistent and has asymptotic normal distribution in the stationary case, while $\hat{b}_T$ has asymptotic Cauchy distribution in the explosive case. In the stationary case, Zani [2] and De Chaumaray [3] obtained the large deviations of $\hat{b}_T$, Gao and Jiang [4] obtained the moderate deviations of $\hat{b}_T$. For the parameter estimation and other issues of the Cox-Ingersoll-Ross model, see references [5–9]. In this paper, we will consider the hypothesis testing problem of this model.

Consider the following hypothesis testing problem

$$H_0 : b = b_0, \quad H_1 : b = b_1,$$

where $b_0,\ b_1 < 0$. Here, the likelihood ratio statistic $\frac{dP_{\delta,b_1}}{dP_{\delta,b_0}}\Big|_{\mathscr{F}_T}$ can be used as one of the above hypothesis testing statistics. By the Neyman-Pearson lemma, the decision region has the following form:

$$\left\{ \frac{1}{T} \log \frac{dP_{\delta,b_1}}{dP_{\delta,b_0}}\Big|_{\mathscr{F}_T} \ge c \right\},$$

where c is the constant to be solved. Large deviation principle for the log-likelihood ratio is one of the effective methods to estimate c, which has been applied by Bishwal [10], Zhao and Gao [11] to the hypothesis testing problem of the fractional Ornstein-Uhlenbeck model and Jacobi model. Since the large deviations only consider the limiting behavior, they have certain limitations in some practical statistical requirements.

The numerical approximations calculated by sharp large deviations outperform those obtained with the central limit theorem or Edgeworth expansions, so the sharp large deviations are very useful in practical situations. The sharp large deviations for the log-likelihood ratio and maximum likelihood estimator of the stationary Ornstein-Uhlenbeck process were studied by Bercu and Rouault [12]. In recent years, sharp large deviations for maximum likelihood estimators of the non-stationary Ornstein-Uhlenbeck process [13], fractional Ornstein-Uhlenbeck process [14], and Cox-Ingersoll-Ross process [15] have attracted much attention. In this paper, inspired by Bercu and Rouault [12], we investigate the sharp large deviations for the log-likelihood ratio of the Cox-Ingersoll-Ross process in the stationary case.

Now we state our main results.

Theorem 1.1 Under the hypothesis H_0, there exists a sequence $(d_{c,k})$ such that, for any $p > 0$ and T large enough, if $b_1 < b_0$, for all $c < \frac{\delta(b_1-b_0)^2}{8b_0}$, we have

$$P_{\delta,b_0}(V_T \le cT) = -\frac{\exp(-I(c)T + H(a_c))}{a_c\sigma_c\sqrt{2\pi T}} \left(1 + \sum_{k=1}^{p} \frac{d_{c,k}}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right),$$

if $b_1 > b_0$, for all $c > \frac{\delta(b_1-b_0)^2}{8b_0}$, we have

$$P_{\delta,b_0}(V_T \geq cT) = \frac{\exp(-I(c)T + H(a_c))}{a_c\sigma_c\sqrt{2\pi T}}\left(1 + \sum_{k=1}^{p}\frac{d_{c,k}}{T^k} + \mathcal{O}(\frac{1}{T^{p+1}})\right),$$

where

$$a_c = \frac{\delta^2(b_1^2-b_0^2)}{4(4c+\delta(b_1-b_0))^2} - \frac{b_0^2}{b_1^2-b_0^2}, \quad \sigma_c^2 = \frac{(4c+\delta(b_1-b_0))^3}{2\delta^2(b_0^2-b_1^2)},$$

$$H(a_c) = -\frac{\delta}{2}\log\left(\frac{1}{2}\left(1 + \frac{2(4c+\delta(b_1-b_0))(a_c(b_1-b_0)+b_0)}{\delta(b_1^2-b_0^2)}\right)\right),$$

and

$$I(x) = \begin{cases} \frac{\left(\delta(b_1^2-b_0^2)-2b_0(4x+\delta(b_1-b_0))\right)^2}{16(4x+\delta(b_1-b_0))(b_0^2-b_1^2)} & , \quad \frac{x}{b_0-b_1} < \frac{\delta}{4}, \\ +\infty & , \quad otherwise. \end{cases}$$

The coefficients $d_{c,1}$, $d_{c,2}$,..., $d_{c,p}$ may be explicitly given as functions of the derivatives of Λ and H (see Lemma 2.1) at point a_c. For example, the first coefficient $d_{c,1}$ is given by

$$d_{c,1} = \frac{1}{\sigma_c^2}\left(-\frac{H_2}{2} - \frac{H_1^2}{2} + \frac{\Lambda_4}{8\sigma_c^2} + \frac{\Lambda_3 H_1}{2\sigma_c^2} - \frac{5\Lambda_3^2}{24\sigma_c^4} + \frac{H_1}{a_c} - \frac{\Lambda_3}{2a_c\sigma_c^2} - \frac{1}{a_c^2}\right)$$

with $\Lambda_k = \Lambda^{(k)}(a_c)$, $H_k = H^{(k)}(a_c)$.

Theorem 1.2 Under the hypothesis H_1, there exists a sequence $(\widetilde{d}_{c,k})$ such that, for any $p > 0$ and T large enough, if $b_1 < b_0$, for all $c < -\frac{\delta(b_1-b_0)^2}{8b_1}$, we have

$$P_{\delta,b_1}(V_T \geq cT) = \frac{\exp(-\widetilde{I}(c)T + H(\widetilde{a}_c))}{\widetilde{a}_c\sigma_c\sqrt{2\pi T}}\left(1 + \sum_{k=1}^{p}\frac{\widetilde{d}_{c,k}}{T^k} + \mathcal{O}(\frac{1}{T^{p+1}})\right),$$

if $b_1 > b_0$, for all $c > -\frac{\delta(b_1-b_0)^2}{8b_1}$, we have

$$P_{\delta,b_1}(V_T \leq cT) = -\frac{\exp(-\widetilde{I}(c)T + H(\widetilde{a}_c))}{\widetilde{a}_c\sigma_c\sqrt{2\pi T}}\left(1 + \sum_{k=1}^{p}\frac{\widetilde{d}_{c,k}}{T^k} + \mathcal{O}(\frac{1}{T^{p+1}})\right),$$

where

$$\widetilde{a}_c = \frac{\delta^2(b_1^2-b_0^2)}{4(4c+\delta(b_1-b_0))^2} - \frac{b_1^2}{b_1^2-b_0^2}, \quad \sigma_c^2 = \frac{(4c+\delta(b_1-b_0))^3}{2\delta^2(b_0^2-b_1^2)},$$

$$H(\widetilde{a}_c) = -\frac{\delta}{2}\log\left(\frac{1}{2}\left(1 + \frac{2(4c+\delta(b_1-b_0))(\widetilde{a}_c(b_1-b_0)+b_1)}{\delta(b_1^2-b_0^2)}\right)\right),$$

and

$$\widetilde{I}(x) = \begin{cases} \frac{\left(\delta(b_1^2-b_0^2)-2b_1(4x+\delta(b_1-b_0))\right)^2}{16(4x+\delta(b_1-b_0))(b_0^2-b_1^2)} & , \quad \frac{x}{b_0-b_1} < \frac{\delta}{4}, \\ +\infty & , \quad otherwise. \end{cases}$$

Similarly, the coefficients $\widetilde{d}_{c,1}$, $\widetilde{d}_{c,2}$,..., $\widetilde{d}_{c,p}$ can be calculated explicitly.

By Theorems 1.1 and 1.2, we get

Corollary 1.1 For any closed subset $F \subset \mathbb{R}$,

$$\limsup_{T\to\infty} \frac{1}{T} \log P_{\delta,b_0}\left(\frac{1}{T}\log \frac{dP_{\delta,b_1}}{dP_{\delta,b_0}}\bigg|_{\mathscr{F}_T} \in F\right) \le -\inf_{x\in F} I(x),$$

and for any open subset $G \subset \mathbb{R}$,

$$\liminf_{T\to\infty} \frac{1}{T} \log P_{\delta,b_0}\left(\frac{1}{T}\log \frac{dP_{\delta,b_1}}{dP_{\delta,b_0}}\bigg|_{\mathscr{F}_T} \in G\right) \ge -\inf_{x\in G} I(x),$$

where $I(x)$ is defined in Theorem 1.1.

Corollary 1.2 For any closed subset $F \subset \mathbb{R}$,

$$\limsup_{T\to\infty} \frac{1}{T} \log P_{\delta,b_1}\left(\frac{1}{T}\log \frac{dP_{\delta,b_1}}{dP_{\delta,b_0}}\bigg|_{\mathscr{F}_T} \in F\right) \le -\inf_{x\in F} \widetilde{I}(x),$$

and for any open subset $G \subset \mathbb{R}$,

$$\liminf_{T\to\infty} \frac{1}{T} \log P_{\delta,b_1}\left(\frac{1}{T}\log \frac{dP_{\delta,b_1}}{dP_{\delta,b_0}}\bigg|_{\mathscr{F}_T} \in G\right) \ge -\inf_{x\in G} \widetilde{I}(x),$$

where $\widetilde{I}(x)$ is defined in Theorem 1.2.

2 Preparatory Lemmas

In this section, we propose several lemmas that play an important role in the proof of Theorem 1.1.

In order to study the sharp large deviations for the log-likelihood ratio, we consider the logarithmic moment generating function under P_{δ,b_0}, i.e.,

$$\Lambda_T(\lambda) = \log E_{\delta,b_0} \exp\left\{\lambda \log \frac{dP_{\delta,b_1}}{dP_{\delta,b_0}}\right\}, \quad \forall\, \lambda \in \mathbb{R}.$$

Let

$$\mathcal{D}_{\Lambda_T} = \{\lambda \in \mathbb{R}, \Lambda_T(\lambda) < +\infty\}$$

be the domain of Λ_T.

Lemma 2.1 Set $\varphi(\lambda) = -\sqrt{b_0^2 + \lambda(b_1^2 - b_0^2)}$, $h(\lambda) = \frac{\lambda(b_1-b_0)+b_0}{\varphi(\lambda)}$.

(a) For all $\lambda \in \mathcal{D}_\Lambda$, we have

$$\Lambda_T(\lambda) = T\Lambda(\lambda) + H(\lambda) + R_T(\lambda), \tag{2.1}$$

where

$$\Lambda(\lambda) = -\frac{\delta(\lambda(b_1 - b_0) + b_0 - \varphi(\lambda))}{4}, \tag{2.2}$$

$$H(\lambda) = -\frac{\delta}{2}\log\left(\frac{1}{2}\big(1 + h(\lambda)\big)\right), \tag{2.3}$$

$$R_T(\lambda) = -\frac{\delta}{2}\log\left(1 + \frac{1-h(\lambda)}{1+h(\lambda)}e^{\varphi(\lambda)T}\right), \tag{2.4}$$

and

$$\mathcal{D}_\Lambda = \left\{\lambda \in \mathbb{R}\colon \lambda(b_1^2 - b_0^2) + b_0^2 > 0,\ \lambda(b_1 - b_0) + b_0 - \sqrt{\lambda(b_1^2 - b_0^2) + b_0^2} < 0\right\}.$$

(b) The remainder $R_T(\lambda)$ satisfies

$$R_T(\lambda) = \mathcal{O}\big(\exp(\varphi(\lambda)T)\big).$$

Proof By using Girsanov formula,

$$\begin{aligned}\Lambda_T(\lambda) &= \log E_{\delta,\varphi}\left[\exp\left(\lambda\log\frac{dP_{\delta,b_1}}{dP_{\delta,b_0}}\right)\frac{dP_{\delta,b_0}}{dP_{\delta,\varphi}}\right]\\ &= \log E_{\delta,\varphi}\left[\exp\left(\frac{\lambda(b_1-b_0)+b_0-\varphi}{4}(X_T-\delta T) - \frac{\lambda(b_1^2-b_0^2)+b_0^2-\varphi^2}{8}\int_0^T X_t dt\right)\right].\end{aligned}$$

If $\lambda(b_1^2 - b_0^2) + b_0^2 > 0$, we take $\varphi(\lambda) = -\sqrt{\lambda(b_1^2 - b_0^2) + b_0^2}$, then

$$\Lambda_T(\lambda) = -\frac{\delta T\left[\lambda(b_1-b_0)+b_0-\varphi(\lambda)\right]}{4} + \log E_{\delta,\varphi}\left[\exp\left(\frac{\lambda(b_1-b_0)+b_0-\varphi(\lambda)}{4}X_T\right)\right],$$

according to Pitman and Yor [16],

$$\log E_{\delta,\varphi}\left[\exp\left(\frac{\lambda(b_1-b_0)+b_0-\varphi(\lambda)}{4}X_T\right)\right] = -\frac{\delta}{2}\log\left(1 - \frac{\lambda(b_1-b_0)+b_0-\varphi(\lambda)}{2\varphi(\lambda)}\left(e^{\varphi(\lambda)T}-1\right)\right).$$

So, for any $\lambda \in \mathcal{D}_\Lambda$,

$$\Lambda_T(\lambda) = -\frac{\delta T\left[\lambda(b_1-b_0)+b_0-\varphi(\lambda)\right]}{4} - \frac{\delta}{2}\log\left(1 - \frac{\lambda(b_1-b_0)+b_0-\varphi(\lambda)}{2\varphi(\lambda)}\left(e^{\varphi(\lambda)T}-1\right)\right),$$

where

$$\mathcal{D}_\Lambda = \left\{\lambda \in \mathbb{R}\colon \lambda(b_1^2 - b_0^2) + b_0^2 > 0,\ \lambda(b_1 - b_0) + b_0 - \sqrt{\lambda(b_1^2 - b_0^2) + b_0^2} < 0\right\}.$$

Finally, set $h(\lambda) = \frac{\lambda(b_1-b_0)+b_0}{\varphi(\lambda)}$, we obtain that

$$\begin{aligned}\Lambda_T(\lambda) &= -\frac{\delta T\left[\lambda(b_1-b_0)+b_0-\varphi(\lambda)\right]}{4} - \frac{\delta}{2}\log\left(1 - \frac{1}{2}(h(\lambda)-1)(e^{\varphi(\lambda)T}-1)\right)\\ &= -\frac{\delta T\left[\lambda(b_1-b_0)+b_0-\varphi(\lambda)\right]}{4} - \frac{\delta}{2}\log\left(\frac{1}{2}\big(1+h(\lambda)\big)\right) - \frac{\delta}{2}\log\left(1 + \frac{1-h(\lambda)}{1+h(\lambda)}e^{\varphi(\lambda)T}\right)\\ &= T\Lambda(\lambda) + H(\lambda) + R_T(\lambda).\end{aligned}$$

And the remainder $R_T(\lambda)$ satisfies

$$R_T(\lambda) = \mathcal{O}(\exp(\varphi(\lambda)T)).$$

Let $\Delta_{\Lambda_T} = \{z \in \mathbb{C}, Re(z) \in \mathcal{D}_{\Lambda_T}\}$. Now, we prove the following lemma by a similar method as in Appendix D in Bercu, Coutin, and Savy [13].

Lemma 2.2 For T large enough and for any $(a, u) \in \mathbb{R}^2$ such that $a + iu \in \Delta_{\Lambda_T}$,

$$\begin{aligned} |\exp(\Lambda_T(a+iu) - \Lambda_T(a))|^2 \leq 4^\delta l^\delta(a)\left(1+\frac{u^2(b_1^2-b_0^2)^2}{\varphi^4(a)}\right)^{\frac{\delta}{4}} \\ \times \exp\left(\frac{\delta T u^2(b_1^2-b_0^2)^2}{32\varphi^3(a)}\left(1+\frac{u^2(b_1^2-b_0^2)^2}{\varphi^4(a)}\right)^{-\frac{3}{4}}\right), \end{aligned} \tag{2.5}$$

where $\varphi(a) = -\sqrt{a(b_1^2 - b_0^2) + b_0^2}$, $l(a) = \max\left(1, \frac{|\varphi(a)+b_0|}{|\varphi(a)|}\right) \max\left(1, \frac{|\varphi(a)+b_1|}{|\varphi(a)|}\right)$.

Proof Step 1: For all $a \in \mathcal{D}_\Lambda$, $u \in \mathbb{R}$, we deduce from (2.2) that

$$\Lambda(a+iu) - \Lambda(a) = -\frac{\delta}{4}\Big(iu(b_1 - b_0) - \varphi(a+iu) + \varphi(a)\Big),$$

which clearly implies that

$$\Big|\exp\Big(T\big(\Lambda(a+iu) - \Lambda(a)\big)\Big)\Big| \leq \exp\left(\frac{\delta T}{4}\Big(Re\big(\varphi(a+iu) - \varphi(a)\big)\Big)\right).$$

Since

$$Re\big(\varphi(a+iu) - \varphi(a)\big) \leq \frac{u^2(b_1^2-b_0^2)^2}{8\varphi^3(a)}\left(1+\frac{u^2(b_1^2-b_0^2)^2}{\varphi^4(a)}\right)^{-\frac{3}{4}},$$

we have

$$\Big|\exp\Big(T\big(\Lambda(a+iu) - \Lambda(a)\big)\Big)\Big|^2 \leq \exp\left(\frac{\delta T u^2(b_1^2-b_0^2)^2}{32\varphi^3(a)}\left(1+\frac{u^2(b_1^2-b_0^2)^2}{\varphi^4(a)}\right)^{-\frac{3}{4}}\right). \tag{2.6}$$

Step 2: For all $a \in \mathcal{D}_\Lambda$, $u \in \mathbb{R}$, we deduce from (2.3) that

$$\big|\exp\big(H(a+iu) - H(a)\big)\big|^2 = \left|\frac{1+h(a)}{1+h(a+iu)}\right|^\delta.$$

Since

$$1 + h(a) = \frac{(b_0 + \varphi(a))(b_1 + \varphi(a))}{(b_1 + b_0)\varphi(a)},$$

we have

$$\begin{aligned} \left|\frac{1+h(a)}{1+h(a+iu)}\right| &\leq \frac{|\varphi(a+iu)|}{|\varphi(a)|}\max\left(1, \frac{|\varphi(a)+b_0|}{|\varphi(a)|}\right)\max\left(1, \frac{|\varphi(a)+b_1|}{|\varphi(a)|}\right) \\ &\leq \left(1+\frac{u^2(b_1^2-b_0^2)^2}{\varphi^4(a)}\right)^{\frac{1}{4}}\max\left(1, \frac{|\varphi(a)+b_0|}{|\varphi(a)|}\right)\max\left(1, \frac{|\varphi(a)+b_1|}{|\varphi(a)|}\right), \end{aligned}$$

then

$$\big|\exp\big(H(a+iu) - H(a)\big)\big|^2 \leq l^\delta(a)\left(1+\frac{u^2(b_1^2-b_0^2)^2}{\varphi^4(a)}\right)^{\frac{\delta}{4}}, \tag{2.7}$$

where

$$l(a) = \max\left(1, \frac{|\varphi(a)+b_0|}{|\varphi(a)|}\right)\max\left(1, \frac{|\varphi(a)+b_1|}{|\varphi(a)|}\right).$$

Step 3: For all $(a,u)\in\mathbb{R}^2$ such that $a+iu\in\Delta_{\Lambda_T}$, we deduce from (2.4) that

$$\left|\exp\big(R_T(a+iu)-R_T(a)\big)\right|^2 = \left|\frac{1+r(a)\exp(\varphi(a)T)}{1+r(a+iu)\exp(\varphi(a+iu)T)}\right|^\delta,$$

where

$$r(a) = \frac{1-h(a)}{1+h(a)}.$$

Since

$$\left|\frac{1+r(a)\exp(\varphi(a)T)}{1+r(a+iu)\exp(\varphi(a+iu)T)}\right| \le 4,$$

we have

$$\left|\exp\big(R_T(a+iu)-R_T(a)\big)\right|^2 \le 4^\delta. \tag{2.8}$$

Finally, together with (2.1), (2.6), (2.7) and (2.8), we can complete the proof of Lemma 2.2.

3 Sharp Large Deviations for the Log-Likelihood Ratio

In this section, we mainly prove the sharp large deviations for the log-likelihood ratio. If $b_1 < b_0$, for $c < \frac{\delta(b_1-b_0)^2}{8b_0}$, let

$$a_c = \frac{\delta^2(b_1^2-b_0^2)}{4\left(4c+\delta(b_1-b_0)\right)^2} - \frac{b_0^2}{b_1^2-b_0^2},$$

consider the change of probability:

$$\frac{dQ_{\delta,T}}{dP_{\delta,b_0}} = \exp\left\{a_cV_T - \Lambda_T(a_c)\right\},$$

and denote by E_Q the expectation under $Q_{\delta,T}$. We obtain that

$$\begin{aligned} P_{\delta,b_0}(V_T\le cT) &= E_Q\left(\exp(\Lambda_T(a_c) - ca_cT + ca_cT - a_cV_T)I_{\{V_T\le cT\}}\right) \\ &= \exp(\Lambda_T(a_c)-ca_cT)E_Q\left(\exp(-a_c\sigma_cU_T\sqrt{T})I_{\{U_T\le 0\}}\right), \end{aligned}$$

where

$$U_T = \frac{V_T-cT}{\sigma_c\sqrt{T}}\ ,\ \sigma_c^2 = \frac{\left(4c+\delta(b_1-b_0)\right)^3}{2\delta^2(b_0^2-b_1^2)}.$$

Let

$$\begin{aligned} A_T &= \exp(\Lambda_T(a_c)-ca_cT), \\ B_T &= E_Q\left(\exp(-a_c\sigma_cU_T\sqrt{T})I_{\{U_T\le 0\}}\right), \end{aligned}$$

then we have

$$P_{\delta,b_0}(V_T\le cT) = A_TB_T. \tag{3.1}$$

Now we consider the asymptotic expansion of A_T and B_T.

3.1 Asymptotic Expansion of A_T

Lemma 3.1 For all $c < \frac{\delta(b_1-b_0)^2}{8b_0}$, T tends to infinity,

$$A_T = \exp\big(-I(c)T + H(a_c)\big)\big(1+\mathcal{O}\big(e^{\varphi(\lambda)T}\big)\big).$$

Proof It follows from Lemma 2.1 that

$$\begin{aligned} A_T &= \exp(\Lambda_T(a_c) - ca_cT) \\ &= \exp(T\Lambda(a_c) + H(a_c) + R_T(a_c) - ca_cT) \\ &= \exp\big(-I(c)T + H(a_c)\big)\big(1+\mathcal{O}\big(e^{\varphi(\lambda)T}\big)\big). \end{aligned}$$

Thus we can complete the proof of Lemma 3.1.

3.2 Asymptotic Expansion of B_T

Let $\Phi_T(\cdot)$ be the characteristic function of U_T under $Q_{\delta,T}$. For all $u \in \mathbb{R}$, we have

$$\begin{aligned} \Phi_T(u) &= E_Q\big(\exp(iuU_T)\big) \\ &= E_{\delta,b_0}\left(\exp\left(iu\frac{V_T - cT}{\sigma_c\sqrt{T}}\right)\exp\big(a_cV_T - \Lambda_T(a_c)\big)\right) \\ &= \exp\left(-\frac{iuc\sqrt{T}}{\sigma_c}\right)E_{\delta,b_0}\left(\exp\left(\left(\frac{iu}{\sigma_c\sqrt{T}} + a_c\right)V_T - \Lambda_T(a_c)\right)\right) \\ &= \exp\left(-\frac{iuc\sqrt{T}}{\sigma_c}\right)\exp\left(\Lambda_T\Big(\frac{iu}{\sigma_c\sqrt{T}} + a_c\Big) - \Lambda_T(a_c)\right). \end{aligned} \tag{3.2}$$

Lemma 3.2 For all $c < \frac{\delta(b_1-b_0)^2}{8b_0}$,

$$B_T = C_T + D_T,$$

where

$$C_T = -\frac{1}{2\pi a_c\sigma_c\sqrt{T}}\int_{|u|\le 2T^{\frac{1}{6}}}\left(1+\frac{iu}{a_c\sigma_c\sqrt{T}}\right)^{-1}\Phi_T(u)du,$$

$$D_T = -\frac{1}{2\pi a_c\sigma_c\sqrt{T}}\int_{|u|> 2T^{\frac{1}{6}}}\left(1+\frac{iu}{a_c\sigma_c\sqrt{T}}\right)^{-1}\Phi_T(u)du.$$

And for T large enough, there exist two positive constants d and D such that

$$|D_T| \le d\exp\left\{-DT^{\frac{1}{3}}\right\}.$$

Proof Applying Parseval formula, we obtain

$$\begin{aligned} B_T &= E_Q\left(\exp(-a_c\sigma_c\sqrt{T}U_T)I_{\{U_T\le 0\}}\right) \\ &= -\frac{1}{2\pi a_c\sigma_c\sqrt{T}}\int_{\mathbb{R}}\left(1+\frac{iu}{a_c\sigma_c\sqrt{T}}\right)^{-1}\Phi_T(u)du, \end{aligned}$$

let

$$C_T = -\frac{1}{2\pi a_c \sigma_c \sqrt{T}} \int_{|u| \le 2T^{\frac{1}{6}}} \left(1 + \frac{iu}{a_c \sigma_c \sqrt{T}}\right)^{-1} \Phi_T(u) du,$$

$$D_T = -\frac{1}{2\pi a_c \sigma_c \sqrt{T}} \int_{|u| > 2T^{\frac{1}{6}}} \left(1 + \frac{iu}{a_c \sigma_c \sqrt{T}}\right)^{-1} \Phi_T(u) du,$$

then $B_T = C_T + D_T$. Next we prove that D_T goes exponentially fast to zero.

We deduce from the Cauchy-Schwarz inequality that

$$|D_T|^2 = \frac{1}{4\pi^2 a_c^2 \sigma_c^2 T} \int_{|u| > 2T^{\frac{1}{6}}} \left(1 + \frac{u^2}{a_c^2 \sigma_c^2 T}\right)^{-1} du \int_{|u| > 2T^{\frac{1}{6}}} |\Phi_T(u)|^2 du. \tag{3.3}$$

First of all, by integration by substitution,

$$\int_{|u| > 2T^{\frac{1}{6}}} \left(1 + \frac{u^2}{a_c^2 \sigma_c^2 T}\right)^{-1} du \le \left|a_c \sigma_c \sqrt{T}\right| \int_{\mathbb{R}} \frac{1}{1+v^2} dv \le \left|a_c \sigma_c \sqrt{T}\right| \pi. \tag{3.4}$$

Secondly, let $\gamma_T = \frac{b_1^2 - b_0^2}{\left|\sigma_c \sqrt{T}\right| \varphi^2(a_c)}$, we deduce from Lemma 2.2 together with (3.2) that for T large enough,

$$|\Phi_T(u)|^2 \le 4^\delta l^\delta(a_c) \left(1 + u^2 \gamma_T^2\right)^{\frac{\delta}{4}} \exp\left(\frac{\delta T \varphi(a_c)}{32} u^2 \gamma_T^2 \left(1 + u^2 \gamma_T^2\right)^{-\frac{3}{4}}\right).$$

Then

$$\begin{aligned}
\int_{|u| > 2T^{\frac{1}{6}}} |\Phi_T(u)|^2 du &\le 2 \times 4^\delta l^\delta(a_c) \int_{2T^{\frac{1}{6}}}^{+\infty} \left(1 + u^2 \gamma_T^2\right)^{\frac{\delta}{4}} \exp\left(\frac{\delta T \varphi(a_c)}{32} u^2 \gamma_T^2 \left(1 + u^2 \gamma_T^2\right)^{-\frac{3}{4}}\right) du \\
&\le \frac{2^{2\delta+1} l^\delta(a_c)}{\gamma_T} \int_{2T^{\frac{1}{6}} \gamma_T}^{+\infty} \left(1 + v^2\right)^{\frac{\delta}{4}} \exp\left(\frac{\delta T \varphi(a_c)}{32} v^2 \left(1 + v^2\right)^{-\frac{3}{4}}\right) dv,
\end{aligned}$$

if $\zeta_T = 2T^{\frac{1}{6}} \gamma_T$, we have

$$\begin{aligned}
\int_{|u| > 2T^{\frac{1}{6}}} |\Phi_T(u)|^2 du \le \frac{2^{2\delta+1} l^\delta(a_c)}{\gamma_T} &\exp\left(\frac{\delta T \varphi(a_c)}{64} \frac{\zeta_T^2}{(1 + \zeta_T^2)^{\frac{3}{4}}}\right) \\
&\times \int_{\zeta_T}^{+\infty} 2^{\frac{\delta}{4}} \max(1, v^{\frac{\delta}{2}}) \exp\left(\frac{\delta T \varphi(a_c)}{64} \frac{\zeta_T^{\frac{3}{2}}}{(1 + \zeta_T^2)^{\frac{3}{4}}} \sqrt{v}\right) dv.
\end{aligned}$$

On the one hand,

$$\begin{aligned}
\exp\left(\frac{\delta T \varphi(a_c)}{64} \frac{\zeta_T^2}{(1 + \zeta_T^2)^{\frac{3}{4}}}\right) &\le \exp\left(\frac{\delta T \varphi(a_c)}{108} \frac{\zeta_T^2}{(1 + \zeta_T^2)^{\frac{3}{4}}}\right) \\
&\le \exp\left(\frac{\delta T \varphi(a_c)}{108} \times \frac{4T^{\frac{1}{3}} (b_1^2 - b_0^2)^2}{T \sigma_c^2 \varphi^4(a_c)}\right) \\
&\le C_1 e^{-C_2 T^{\frac{1}{3}}},
\end{aligned}$$

where C_1 and C_2 are positive constants.
On the other hand, let

$$e_T = \frac{\delta T\varphi(a_c)}{64}\frac{\zeta_T^{\frac{3}{2}}}{(1+\zeta_T^2)^{\frac{3}{4}}},$$

we obtain that e_T goes to $-\infty$ as T tends to infinity, which implies that for T large enough, $e_T - 1 < 0$. Then, for T large enough,

$$\int_{\zeta_T}^{+\infty} \max(1, v^{\frac{\delta}{2}})\exp(e_T\sqrt{v})dv \le \int_{\zeta_T}^{+\infty} \exp\left((e_T-1)\sqrt{v}\right)dv \le \frac{2}{(1-e_T)^2},$$

which tends to zero.
Thus, we obtain that

$$\int_{|u|>2T^{\frac{1}{6}}} |\Phi_T(u)|^2\, du \le \frac{2^{2\delta+1}l^\delta(a_c)C_1}{\gamma_T} e^{-C_2T^{\frac{1}{3}}}. \tag{3.5}$$

Finally, we deduce from (3.3), (3.4) and (3.5) that there exist two positive constants d and D such that

$$|D_T| \le d\exp\left\{-DT^{\frac{1}{3}}\right\}.$$

Now we prove the Taylor expansion of $\Phi_T(\cdot)$. First of all, for any $k\in\mathbb{N}$, $R_T^{(k)}(a_c) = \mathcal{O}\left(T^k\exp\left\{-\left|\frac{\delta(b_1^2-b_0^2)}{2(4c+\delta(b_1-b_0))}\right|T\right\}\right)$. Then, we obtain from (2.1) that

$$\Lambda_T^{(k)}(a_c) = T\Lambda_k + H_k + \mathcal{O}\left(T^k\exp\left\{-\left|\frac{\delta(b_1^2-b_0^2)}{2(4c+\delta(b_1-b_0))}\right|T\right\}\right), \tag{3.6}$$

where $\Lambda_k = \Lambda^{(k)}(a_c)$, $H_k = H^{(k)}(a_c)$.

Lemma 3.3 For any $p>0$, and for any $c < \frac{\delta(b_1-b_0)^2}{8b_0}$, there exist integers $q(p)$ and a polynomial sequence (η_k) independent of p, such that, for T large enough,

$$\Phi_T(u) = e^{-\frac{u^2}{2}}\left(1+\sum_{k=1}^{q(p)}\frac{\eta_k(u)}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{\max(1,|u|^{6(p+1)})}{T^{p+1}}\right)\right), \tag{3.7}$$

where the remainder $\mathcal{O}$ is uniform as soon as $|u|\le 2T^{\frac{1}{6}}$. Moreover, the η_k are polynomials in odd powers of u for k odd and in even powers of u for k even. For example,

$$\eta_1(u) = -\frac{iu^3\Lambda_3}{6\sigma_c^3} + \frac{iuH_1}{\sigma_c},$$

$$\eta_2(u) = -\frac{u^2H_1^2}{2\sigma_c^2} - \frac{u^2H_2}{2\sigma_c^2} + \frac{u^4\Lambda_4}{24\sigma_c^4} + \frac{u^4\Lambda_3H_1}{6\sigma_c^4} - \frac{u^6\Lambda_3^2}{72\sigma_c^6}.$$

Proof We deduce from (3.2) and (3.6) that there exists $\xi\in\mathbb{R}$ such that, for any $p>0$,

$$\log\Phi_T(u) = -\frac{iuc\sqrt{T}}{\sigma_c} + \sum_{k=1}^{[2p+3]}\left(\frac{iu}{\sqrt{T}\sigma_c}\right)^k\left(\frac{T\Lambda_k}{k!}+\frac{H_k}{k!}\right) + \mathcal{O}\left(\frac{\max(1,|u|^{2p+4})}{T^{p+1}}\right).$$

One can observe that $\Lambda^{(1)}(a_c) = c$ and $\Lambda^{(2)}(a_c) = \sigma_c^2$, thus,

$$\log \Phi_T(u) = -\frac{u^2}{2} + T \sum_{k=3}^{[2p+3]} \left(\frac{iu}{\sqrt{T}\sigma_c}\right)^k \frac{\Lambda_k}{k!} + \sum_{k=1}^{[2p+1]} \left(\frac{iu}{\sqrt{T}\sigma_c}\right)^k \frac{H_k}{k!} + \mathcal{O}\left(\frac{\max(1, |u|^{2p+4})}{T^{p+1}}\right). \tag{3.8}$$

Finally, we obtain (3.7) by taking the exponential of both sides of (3.8), remarking that in the range $|u| \le 2T^{\frac{1}{6}}$ and for any $k \ge 3$, the quantity $\frac{Tu^k}{(\sqrt{T})^k}$ remains bounded in (3.7).

From Lemmas 3.2 and 3.3 together with standard calculus on the $N(0,1)$ distribution, we obtain the asymptotic expansion of B_T.

Lemma 3.4 For all $c < \frac{\delta(b_1-b_0)^2}{8b_0}$, there exists a sequence (ψ_k) such that, for any $p > 0$ and T large enough,

$$B_T = -\frac{1}{a_c\sigma_c\sqrt{2\pi T}}\left(1 + \sum_{k=1}^{p} \frac{\psi_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right).$$

Proof of Theorem 1.1 and 1.2 We complete the proof of Theorem 1.1 by Lemmas 3.1 and 3.4 together with (3.1). The proof of Theorem 1.2 is similar to Theorem 1.1.

4 Conclusion

The Cox-Ingersoll-Ross process is widely used to model the evolution of short-term interest rates in mathematical finance, which has many appealing advantages. In the stationary case, in testing the Cox-Ingersoll-Ross model, we obtain the expansion formula of the probability of the first kind and the second kind. The limiting distribution for the log-likelihood ratio in the non-stationary case is different from that in the stationary case, and we will investigate the sharp large deviations for the log-likelihood ratio of the Cox-Ingersoll-Ross process in the non-stationary case.

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Cox-Ingersoll-Ross 过程对数似然比的精细大偏差

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摘 要: 本文研究了平稳状态下 Cox-Ingersoll-Ross 过程在原假设和备择假设下对数似然比的精细大偏差问题. 利用测度变换和特征函数技巧, 本文得到了对数似然比的完全展开式.

关键词: Cox-Ingersoll-Ross 过程; 对数似然比; 精细大偏差

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