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# THE NEUMANN PROBLEM FOR A SPECIAL LAGRANGIAN TYPE EQUATION WITH SUPERCRITICAL PHASE

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**Abstract:** In this paper, we explore the Neumann problem for special lagrangian type equations with supercritical phase in $\mathbb{R}^n$. We show the global $C^2$ a priori estimates of the solution and establish the existence of classical solutions by the methods of continuity.

**Keywords:** Neumann problem; special Lagrangian type equation; supercritical phase

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## 1 Introduction and main results

We consider the Neumann problem of a special Lagrangian equation

$$\arctan\{\Delta u \mathrm{I}_n - D^2u\} = \Theta(x), \text{ in } \Omega \subset \mathbb{R}^n, \tag{1.1}$$

where

$$\arctan\{\Delta u \mathrm{I}_n - D^2u\} =: \arctan\eta_1 + \arctan\eta_2 + \cdots + \arctan\eta_n.$$

Denote $\eta := (\eta_1, \eta_2, \cdots, \eta_n)$ which are the eigenvalues of the matrix $\Delta u I_n - D^2u$ in [1] with

$$\eta_i = \sum_{k\neq i} \lambda_k, \ \forall i = 1, 2, \cdots, n,$$

where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ are the eigenvalues of the Hessian matrix $D^2u$. Here $\Theta(x)$ is usually studied under three different types of two boundary value conditions: the phase, the critical phase, supercritical phase. More precisely, $\Theta(x) \in (-\frac{n\pi}{2}, \frac{n\pi}{2})$, $\Theta = \frac{(n-2)\pi}{2}$, $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$. In this paper, we consider the special Lagrangian equation (1.1) with supercritical phase, that is the third type.

The first boundary value problem (Dirichlet problem) for elliptic partial differential equations has been intensively studied many years. For the Laplace equation, results can be found in Gilbarg-Trudinger [2]. The Dirichlet problem for Monge-Ampère equations

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**Biography:** Jia Haohao(1995–), female, born at Handan, Hebei, postgraduate, major in partial differential equations. E-mail: 2101400103@nbu.edu.cn was investigated in Caffarelli-Nirenberg-Spruck [3] and Krylov [4]. They showed the global regularity of solutions. Caffarelli-Nirenberg-Spruck [5] studied the existence of admissible solutions and the global regularity of $k$-Hessian equations. The Hessian quotient equations which have different structure conditions were studied in Trudinger [6]. To the best of my knowledge, the special Lagrangian equation

$$\sum_i \arctan \lambda_i = \Theta(x), \quad x \in \Omega \subset \mathbb{R}^n,$$

was introduced in Harvey-Lawson [7] firstly and $\Theta$ is a constant called the phase angle. In their study, the graph $x \mapsto (x, Du(x))$ defines a calibrated, minimal submanifold of $\mathbb{R}^{2n}$. Collins-Picard-Wu [8] considered the Dirichlet problem to Lagrangian phase operator in both the real and complex setting. They solved the concavity of Lagrangian phase operator, the essential condition, to obtain the existence theorem by using the classical methods. Recently, Zhu [1] established the global $C^2$ estimates and showed the existence theorem of the Dirichlet problem to (1.1).

For the Neumann and oblique derivative problem of elliptic equations, there are many research results. A priori estimates and the existence theorem of the Laplace equation can be found in [2]. And, we can see more results about the Neumann and the oblique derivative problems of linear and quasilinear elliptic equations in Lieberman [9]. The Neumann problem of Monge-Ampère equations was solved in Lions-Trudinger-Urbas [10]. Ma-Qiu [11] studied the Neumann problem of $k$-Hessian equations in uniformly convex domain. And, Chen-Zhang [12] solved the Neumann problem of Hessian quotient equations, the general forms of $k$-Hessian equations. For the special Lagrangian equation with supercritical phase in strictly convex domain, Chen-Ma-Wei established the global $C^2$ estimates and obtained the existence theorem by the method of continuity in [13] recently.

It is worth mentioning that the key to solving of the existence and uniqueness of classical solutions for elliptic partial differential equations is to establish the global a priori estimates and the method of continuity in above works. To our best knowledge, the existence theorem of the classical Neumann problem to (1.1) with supercritical phase has not been studied before. In this paper, we apply the method used in [9, 10] and show the existence theorem of the Neumann problem of special Lagrangian equation following the classical idea (see for example [14] or [15]). More precisely, we get our theorem.

**Theorem 1.1** Suppose $\Omega \subset \mathbb{R}^n$ is a $C^4$ strictly convex domain and $\nu$ is outer unit normal vector of $\partial\Omega$. Let $\varphi \in C^3(\partial\Omega)$ and $\Theta(x) \in C^2(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$. Then there exists a unique constant $\beta$ such that the Neumann problem of special Lagrangian equation

$$\begin{cases} \arctan\{\Delta u \mathrm{I}_n - D^2 u\} = \Theta(x), & \text{in} \quad \Omega, \\ u_\nu = \beta + \varphi(x), & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

has admissible solutions $u \in C^{3,\alpha}(\overline{\Omega})$, which are unique up to a constant.

**Remark 1** Because a solution plus any constant is still a solution for the classical Neumann problem of special Lagrangian equation (1.2), we can't get a uniform bound of the solutions to (1.2), and we can't apply the continuity method to get the existence of the solution. Thanks to the perturbation arguments in [10, 15], we consider the solution $u^\varepsilon$ of the equation

$$\begin{cases} \arctan\{\Delta u^\varepsilon \mathrm{I}_n - D^2 u^\varepsilon\} = \Theta(x), & \text{in} \quad \Omega \subset \mathbb{R}^n, \\ u^\varepsilon_\nu = -\varepsilon u^\varepsilon + \varphi(x), & \text{on} \quad \partial\Omega, \end{cases} \tag{1.3}$$

for any small $\varepsilon > 0$. We need to establish a priori estimate of $u^\varepsilon$ which is independent of $\varepsilon$, and the strict convexity of $\Omega$ plays an important role. By taking the limit on $\varepsilon$ and the perturbation argument, we can obtain the existence of a solution of (1.2).

## 2 Preliminaries

In this section, we show some properties of the special Lagrangian equation with supercritical phase.

**Property 2.1** Let $\Omega \subset \mathbb{R}^n$ be a domain and $\Theta(x) \in C^0(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$. Suppose $u \in C^2(\Omega)$ is a solution of special Lagrangian equation (1.1) and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ are the eigenvalues of the Hessian matrix $D^2u$ with

$$\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n, \tag{2.1}$$

then we have some properties:

$$\eta_1 \leq \cdots \leq \eta_n, \tag{2.2}$$

$$\eta_1 + \eta_2 + \cdots + \eta_n > 0, \tag{2.3}$$

$$|\eta_{n-1}| \leq \eta_n, \tag{2.4}$$

$$|\eta_1| \leq C_0, \tag{2.5}$$

where $C_0 = \max\{\tan\{\frac{(n-1)\pi}{2} - \min\limits_{\overline{\Omega}} \Theta\}, \tan\big(\frac{\max\limits_{\overline{\Omega}} \Theta}{n}\big)\}$.

The proofs are analogous to Property 2.1 and Lemma 2.1 in [1, 13, 16, 17] and are omitted. The following property is Property 2.2 in [1] and we give the proof here for convenience.

**Property 2.2** [1] Suppose $\Omega \subset \mathbb{R}^n$ is a domain and $\Theta(x) \in C^2(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$. Let $u \in C^4(\Omega)$ be a solution of (1.1). Then for any $\xi \in \mathbb{S}^{n-1}$, we have

$$\sum_{ij=1}^{n} F^{ij} u_{ij\xi\xi} \geq \Theta_{\xi\xi} - A{\Theta_\xi}^2, \quad \text{in } \Omega, \tag{2.6}$$

where $F^{ij} = \frac{\partial \arctan \eta}{\partial u_{ij}}$ and $A = \frac{2}{\tan\left(\min\limits_{\overline{\Omega}} \Theta - \frac{(n-2)\pi}{2}\right)}$.

**Proof** By rotating the coordinates, for any $x \in \Omega$, we assume $D^2u$ is diagonal with $\lambda_i = u_{ii} (i = 1, 2, \cdots, n)$. By calculation, we have

$$F^{ij} =: \frac{\partial \arctan \eta}{\partial u_{ij}} = \begin{cases} \sum\limits_{p\neq i} \frac{1}{1+\eta_p^2}, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

and

$$F^{ij,kl} =: \frac{\partial^2 \arctan \eta}{\partial u_{ij} \partial u_{kl}} = \begin{cases} \sum\limits_{p\neq i,k} \frac{-2\eta_p}{(1+\eta_p^2)^2}, & \text{if } i = j, k = l, \\ -\frac{\eta_i+\eta_j}{(1+\eta_i^2)(1+\eta_j^2)}, & \text{if } i = l, j = k, i \neq j, \\ 0, & \text{otherwise }. \end{cases}$$

From the equation (1.1), we have $\sum_{ij=1}^n F^{ij} u_{ij\xi} = \Theta_\xi$, and

$$\begin{aligned} \sum_{ij=1}^n F^{ij} u_{ij\xi\xi} =& \Theta_{\xi\xi} - \sum_{ijkl=1}^n F^{ij,kl} u_{ij\xi} u_{kl\xi} = \Theta_{\xi\xi} - \sum_{i,k=1}^n F^{ii,kk} u_{ii\xi} u_{kk\xi} - \sum_{i\neq j} F^{ij,ji} u_{ij\xi}^2 \\ \geq& \Theta_{\xi\xi} - \sum_{i,k=1}^n F^{ii,kk} u_{ii\xi} u_{kk\xi}. \end{aligned} \tag{2.7}$$

From the concavity lemma (Lemma 2.2 in [8]), we know

$$\begin{aligned} -\sum_{i=1}^n F^{ii,kk} u_{ii\xi} u_{kk\xi} =& -\frac{\partial^2 \arctan \eta}{\partial \eta_p \partial \eta_q} \frac{\partial \eta_p}{\partial \lambda_i} \frac{\partial \eta_q}{\partial \lambda_k} u_{ii\xi} u_{kk\xi} \\ \geq& -\frac{2}{\tan\big(\min\limits_{\overline{\Omega}} \Theta - \frac{(n-2)\pi}{2}\big)} \Big(\sum_{p=1}^n \frac{\partial \arctan \eta}{\partial \eta_p} \frac{\partial \eta_p}{\partial \lambda_i} u_{ii\xi}\Big)^2 \\ =& -\frac{2}{\tan\big(\min\limits_{\overline{\Omega}} \Theta - \frac{(n-2)\pi}{2}\big)} {\Theta_\xi}^2. \end{aligned} \tag{2.8}$$

The proof is completed.

# 3 A priori estimates

## 3.1 $C^0$ estimate

The $C^0$ estimate is easy to prove following the idea of Trudinger [18].

**Theorem 3.2** Let $\Omega \subset \mathbb{R}^n$ be a $C^1$ bounded domain and $\varphi \in C^0(\partial\Omega)$. Suppose $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is the solution of special Lagrangian equation (1.3) for any small $\varepsilon > 0$ and $\Theta(x) \in C^0(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$, then we have

$$\sup_{\overline{\Omega}} |\varepsilon u| \leq M_0, \tag{3.1}$$

where $M_0$ depends only on $n$, $diam(\Omega)$, $\max\limits_{\partial\Omega} |\varphi|$ and $\max\limits_{\overline{\Omega}} \Theta$.

**Proof** By(2.3), we have $\Delta u > 0$. So $u$ attains its maximum at some boundary point $x_0 \in \partial\Omega$. Then we have

$$0 \le u_\nu(x_0) = -\varepsilon u(x_0) + \varphi(x_0), \tag{3.2}$$

and

$$\varepsilon u \le \varepsilon u(x_0) \le \varphi(x_0) \le \max_{\partial\Omega} |\varphi|. \tag{3.3}$$

We can assume $0 \in \Omega$, and denote $B = \frac{1}{2(n-1)} \tan\big(\frac{\max_{\overline{\Omega}} \Theta}{n}\big) < +\infty$. Then we have

$$\arctan \eta(\lambda(D^2 u)) = \Theta \le \max_{\overline{\Omega}} \Theta = \arctan(D^2(B|x|^2)). \tag{3.4}$$

Using the comparison principle, we get $u - B|x|^2$ to attain its minimum at a boundary point $y_0 \in \partial\Omega$. Therefore,

$$\begin{aligned} 0 &\ge (u - B|x|^2)_\nu(y_0) = u_\nu(y_0) - 2By_0 \cdot \nu \\ &\ge -\varepsilon u(y_0) - \varphi(y_0) - 2B\mathrm{diam}(\Omega). \end{aligned} \tag{3.5}$$

Then, we have

$$\begin{aligned} \varepsilon u &\ge \varepsilon(u - B|x|^2) \\ &\ge \varepsilon(u(y_0) - B|x|^2) \\ &\ge -2B\mathrm{diam}(\Omega) - \max_{\partial\Omega} \varphi - B\mathrm{diam}(\Omega)^2. \end{aligned} \tag{3.6}$$

### 3.2 Global $C^1$ estimate

In this subsection, we prove the $C^1$ estimate of solutions for the special Lagrangian equation (1.3) with supercritical phase. We show the following theorem.

**Theorem 3.3** Suppose $\Omega \subset \mathbb{R}^n$ is a $C^3$ uniformly convex domain and $\varphi \in C^2(\partial\Omega)$. Let $\Theta(x) \in C^1(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$ and $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ be a solution of special Lagrangian equation (1.3) for any small $\varepsilon > 0$, then we have

$$\sup_{\overline{\Omega}} |Du| \le M_1, \tag{3.7}$$

where $M_1$ depends only on $n$, $\Omega$, $\max_{\overline{\Omega}} \Theta$, $\min_{\overline{\Omega}} \Theta$, $M_0$, $|\Theta|_{C^1}$ and $|\varphi|_{C^2}$.

**Proof** We just have to prove

$$D_\xi u(x) \le M_1, \ \forall (x, \xi) \in \bar{\Omega} \times \mathbb{S}^{n-1}, \tag{3.8}$$

where $\xi = (\xi_1, \cdots, \xi_n)$, $|\xi| = 1$. Choose

$$w(x, \xi) = D_\xi u(x) - \langle \nu, \xi \rangle (-\varepsilon u + \varphi) + \varepsilon^2 u^2 + K|x|^2, \tag{3.9}$$

where $\nu$ is a $C^2(\overline{\Omega})$ extension of the outer unit normal vector field on $\partial\Omega$, $\varepsilon$ is a small positive constant in (1.3) and $K$ is a large positive constant. Note that here $\varphi \in C^2(\overline{\Omega})$ is an extension with universal $C^2$ norm. Suppose $w(x,\xi)$ attains its maximum at $(x_0,\xi_0) \in \bar{\Omega} \times S^{n-1}$. In the following, we divide (3.8) into two steps.

1. We claim that $x_0 \in \partial\Omega$. Assume $x_0 \in \Omega$, and we will prove $F^{ii}\partial_{ii}w(x,\xi_0)|_{x=x_0} > 0$ to establish a contradiction. For $x_0 \in \Omega$, we can assume $D^2u(x_0)$ is diagonal with $\lambda_i = u_{ii}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ by rotating the coordinate $(e_1, \cdots, e_n)$, then $F^{ij}(x_0)$ is diagonal. Then we have

$$F^{ij} =: \frac{\partial \arctan \eta}{\partial u_{ij}} = \begin{cases} \sum\limits_{p\neq i} \frac{1}{1+\eta_p^2}, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{3.10}$$

Hence we can get from Property 2.1,

$$F^{11} \leq F^{22} \leq \cdots \leq F^{nn}; \tag{3.11}$$

$$F^{nn} = \sum_{p\neq n} \frac{1}{1+\eta_p^2} \geq c_0 > 0; \tag{3.12}$$

$$F^{ii}u_{ii} = \sum_{p=1}^{n} \frac{\eta_p}{1+\eta_p^2} \in (-\frac{1}{2}, \frac{n}{2}); \tag{3.13}$$

where $c_0 = \frac{1}{1+\ \max\{\tan\left(\frac{(n-1)\pi}{2} - \min\limits_{\overline{\Omega}} \Theta\right), \tan\left(\frac{\max\limits_{\overline{\Omega}} \Theta}{n}\right)\}^2}$.

We suppose the maximum of $w$ fixed in some direction $\xi_0$, all the calculations are at $x_0$ in the following. We get

$$\begin{aligned}
0 \geq& \sum_i F^{ii}\partial_{ii}w(x,\xi_0)|_{x=x_0} \\
=& \sum_i F^{ii}[u_{ii\xi_0} - \langle\nu,\xi_0\rangle_{ii}(-\varepsilon u + \varphi) - 2\langle\nu,\xi_0\rangle_i(-\varepsilon u_i + \varphi_i) - \langle\nu,\xi_0\rangle(-\varepsilon u_{ii} + \varphi_{ii}) + 2\varepsilon^2(u_i^2 + uu_{ii}) + 2K] \\
=& \sum_i F^{ii}u_{ii\xi_0} + \sum_i F^{ii}[\langle\nu,\xi_0\rangle\varepsilon u_{ii} + 2\varepsilon^2 uu_{ii}] + \sum_i F^{ii}[2\varepsilon^2 u_i^2 + 2\langle\nu,\xi_0\rangle_i \varepsilon u_i] \\
&+ \sum_i F^{ii}[2K - \langle\nu,\xi_0\rangle_{ii}(-\varepsilon u + \varphi) - 2\langle\nu,\xi_0\rangle_i\varphi_i - \langle\nu,\xi_0\rangle\varphi_{ii}] \\
\geq& \Theta_{\xi_0} + \sum_i F^{ii}[\langle\nu,\xi_0\rangle\varepsilon u_{ii} + 2\varepsilon^2 uu_{ii}] + \sum_i F^{ii}(\varepsilon u_i + \langle\nu,\xi_0\rangle_i)^2 \\
&+ \sum_i F^{ii}[2K - \langle\nu,\xi_0\rangle_{ii}(-\varepsilon u + \varphi) - 2\langle\nu,\xi_0\rangle_i\varphi_i - \langle\nu,\xi_0\rangle\varphi_{ii} - \langle\nu,\xi_0\rangle_i^2] \\
\geq& -|\nabla\Theta| - |\sum_p \frac{\eta_p}{1+\eta_p^2}| + \sum_i F^{ii}[2K - C_1 - |D\langle\nu,\xi_0\rangle|^2] \\
\geq& -|\nabla\Theta| - \frac{n}{2} + \frac{1}{1+C_0^2}(2K - C_2),
\end{aligned}$$

where $C_0$ is defined in (2.5), $C_1$ is a positive constant depending only on $n$, $M_0$, $|\nu|_{C^2}$, $|\varphi|_{C^2}$. $C_2$ is positive constant depending only on $C_1$ and $|\nu|_{C^1}$.

Choose $K > \frac{1+C_0^2}{2}(|\nabla\Theta| + \frac{n}{2}) + \frac{C_2}{2}$, we have $F^{ii}\partial_{ii}w(x,\xi_0)|_{x=x_0} > 0$. This is a contradiction. Thus $x_0 \in \partial\Omega$.

2. We now consider the direction $\xi_0$ with the following three cases.

Case a: $\xi_0$ is normal to $\partial\Omega$ at $x_0$, then we have

$$w(x_0,\xi_0)=\varepsilon^2u^2(x_0)+K|x_0|^2\le C_3,\text{ or}$$
$$w(x_0,\xi_0)=2(-\varepsilon u+\varphi)+\varepsilon^2u^2(x_0)+K|x_0|^2\le C_4.$$

And,

$$\begin{aligned}D_\xi u(x)&=w(x,\xi)+\langle\nu,\xi\rangle(-\varepsilon u+\varphi)-\varepsilon^2u^2-K|x_0|^2\\&\le w(x_0,\xi_0)+C_5\\&\le C_6.\end{aligned}\tag{3.14}$$

Case b: $\xi_0$ is non-tangential but not normal to $\partial\Omega$ at $x_0$. We can find a tangential vector $\tau\in\mathbb{S}^{n-1}$ such that $\xi_0=\alpha\tau+\beta\nu$, with $\alpha=\xi_0\cdot\tau\ne0$, $\beta=\xi_0\cdot\nu\ne0$, $\alpha^2+\beta^2=1$ and $\tau\cdot\nu=0$. Without loss of generality, we take $0<\alpha<1$. Then we have

$$\begin{aligned}w(x_0,\xi_0)&=D_{\xi_0}u(x_0)-\langle\nu,\xi_0\rangle(-\varepsilon u+\varphi)+\varepsilon^2u^2(x_0)+K|x_0|^2\\&=\alpha D_\tau u(x_0)+\beta D_\nu u(x_0)-\beta(-\varepsilon u+\varphi)+\varepsilon^2u^2(x_0)+K|x_0|^2\\&=\alpha D_\tau u(x_0)+\varepsilon^2u^2(x_0)+K|x_0|^2\\&=\alpha w(x_0,\tau)+\alpha\langle\nu,\tau\rangle(-\varepsilon u+\varphi)+(1-\alpha)(\varepsilon^2u^2(x_0)+K|x_0|^2)\\&\le\alpha w(x_0,\xi_0)+(1-\alpha)(\varepsilon^2u^2(x_0)+K|x_0|^2).\end{aligned}$$

Then we get

$$w(x_0,\xi_0)\le\varepsilon^2u^2(x_0)+K|x_0|^2\le C_7.$$

Hence,

$$\begin{aligned}D_\xi u&=w(x,\xi)+\langle\nu,\xi\rangle(-\varepsilon u+\varphi)-\varepsilon^2u^2-K|x|^2\\&\le w(x_0,\xi_0)+C_8\\&\le C_9.\end{aligned}\tag{3.15}$$

Case c: $\xi_0$ is tangential to $\partial\Omega$ at $x_0$, then we have $\xi_0\cdot\nu=0$. Without loss of generality, we may assume $\xi_0=e_1$. Then,

$$\begin{aligned}0&\le D_\nu w(x_0,\xi_0)\\&=D_\nu D_1u(x_0)+2\varepsilon^2u(x_0)D_\nu u(x_0)+KD_\nu|x_0|^2\\&\le D_\nu D_1u(x_0)+C_{10}\\&=D_1(D_\nu u)-DuD_1\nu+C_{10}.\end{aligned}$$

On the one hand, $D_1(D_\nu u)=D_1(-\varepsilon u+\varphi)\le D_1\varphi\le|D\varphi|$. For $DuD_1\nu$, since $\Omega$ is a $C^2$ convex domain, we have distance function $d(x)=\operatorname{dist}(x,\partial\Omega)\in C^2$ such that

$$\begin{aligned}&d=0\text{ on }\partial\Omega,\quad d>0\text{ in }\Omega;\\&|\nabla d|=1\text{ on }\partial\Omega.\end{aligned}$$

And, the matrix $\{d_{ij}\}_{1\leq i,j\leq n-1} \sim -\kappa_{min}$, where $\kappa_{min}$ is the smallest principal curvature of the boundary. Because $w(x_0,\xi)$ attains its maximum at direction $\xi=\xi_0$, it is easy to get

$$u_1(x_0)\geq 0,\ u_2(x_0)=\cdots=u_{n-1}(x_0)=0,\ u_n(x_0)=-u_\nu(x_0)=\varepsilon u(x_0)-\varphi(x_0).$$

On the other hand, from $\nu_k=-d_k$,

$$\begin{aligned} DuD_1\nu &= D_1(\nu_k)D_ku \\ &= -\sum_{k=1}^{n}\frac{\partial^2 d}{\partial x_k\partial x_1}D_ku \\ &= -\sum_{k=1}^{n} d_{1k}u_k = -\sum_{k=1}^{n-1} d_{1k}u_k - d_{1n}u_n \\ &\geq -\sum_{k=1}^{n-1} d_{1k}u_k - C\left|D_\nu u(x_0)\right| \\ &\geq -d_{11}u_1 - C_{11} \end{aligned}$$

Hence, $-DuD_1\nu\leq d_{11}u_1+C_{11}\leq -\kappa_{\min}w(x_0,\xi_0)+C_{11}$. Then, we have

$$w(x_0,\xi_0)\leq \tfrac{|D\varphi|+C_{10}+C_{11}}{\kappa_{\min}}.$$

Similar to the (3.14) and (3.15), we have

$$D_\xi u(x)=w(x,\xi)-\varepsilon^2u^2-K|x|^2\leq w(x_0,\xi_0)+C_{12}\leq C_{13}.$$

The proof is complete.

### 3.3 Global second derivatives estimate

We consider now to the global second derivatives and we can get the following theorem

**Theorem 3.4** Suppose $\Omega\subset\mathbb{R}^n$ is a $C^4$ strictly convex domain and $\varphi\in C^3(\partial\Omega)$. Let $\Theta(x)\in C^2(\overline{\Omega})$ with $\frac{(n-2)\pi}{2}<\Theta(x)<\frac{n\pi}{2}$ in $\overline{\Omega}$ and $u\in C^4(\Omega)\cap C^3(\overline{\Omega})$ be a solution of special Lagrangian equation (1.3) for any small $\varepsilon>0$, then we have

$$\sup_{\overline{\Omega}}|D^2u|\leq M_2, \tag{3.16}$$

where $M_2$ depends only on $n$, $\Omega$, $\max\limits_{\overline{\Omega}}\Theta$, $\min\limits_{\overline{\Omega}}\Theta$, $M_0$, $M_1$, $|\Theta|_{C^2}$ and $|\varphi|_{C^3}$.

We firstly reduce global second derivatives to double normal second derivatives on boundary and then we show the estimate of double normal second derivatives on boundary following the standard method in Lions-Trudinger-Urbas [10] and Ma-Qiu [11].

**Lemma 3.5** Suppose $\Omega\subset\mathbb{R}^n$ is a $C^4$ convex domain and $\varphi\in C^3(\partial\Omega)$. Let $\Theta(x)\in C^2(\overline{\Omega})$ with $\frac{(n-2)\pi}{2}<\Theta(x)<\frac{n\pi}{2}$ in $\overline{\Omega}$ and $u\in C^4(\Omega)\cap C^3(\overline{\Omega})$ be a solution of special Lagrangian equation (1.3) for any small $\varepsilon>0$, then we have

$$\sup_{\overline{\Omega}}|D^2u|\leq C_{14}(1+\sup_{\partial\Omega}|u_{\nu\nu}|), \tag{3.17}$$

where $C_{14}$ depends only on $n$, $\Omega$, $\max\limits_{\overline{\Omega}}\Theta$, $\min\limits_{\overline{\Omega}}\Theta$, $M_0$, $M_1$, $|\Theta|_{C^2}$ and $|\varphi|_{C^3}$.

**Proof** There exists a small constant $\mu > 0$ such that $d(x) \in C^4(\overline{\Omega_\mu})$ and $\nu = -Dd$ on $\partial\Omega$ because $\Omega$ is a $C^4$ domain. Define $\widetilde{d} \in C^4(\overline{\Omega})$ such that $\widetilde{d} = d$ in $\overline{\Omega_\mu}$ and denote

$$\nu = -D\widetilde{d}, \quad \text{in } \Omega.$$

Note that $\nu$ is a $C^3(\overline{\Omega})$ extension of the outer unit normal vector field on $\partial\Omega$.

We assume $0 \in \Omega$ and consider the auxiliary function

$$v(x,\xi) = u_{\xi\xi} - v'(x,\xi) + K_1|x|^2, \tag{3.18}$$

where $v'(x,\xi) = 2(\xi\cdot\nu)\xi'\cdot(D\varphi - \varepsilon Du - u_l D\nu^l) = a^l u_l + b$, $\xi' = \xi - (\xi\cdot\nu)\nu$, $a^l = -2(\xi\cdot\nu)(\xi'\cdot D\nu^l) - 2(\xi\cdot\nu)\varepsilon(\xi')^l$, $b = 2(\xi\cdot\nu)(\xi'\cdot D\varphi)$, and $K_1 > 0$ is to be determined later. And we know that here $\varphi \in C^3(\overline{\Omega})$ is an extension with universal $C^3$ norm. Recall

$$F^{ij} = \frac{\partial \arctan \eta}{\partial u_{ij}}.$$

By rotating the coordinates, for any $x \in \Omega$, we assume $D^2u$ is diagonal with $\lambda_i = u_{ii} (i = 1, 2, \cdots, n)$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We know that (3.10)-(3.13) in the proof of Theorem 3.3 still hold. From Property 2.2, for any fixed $\xi \in \mathbb{S}^{n-1}$, we have

$$\begin{aligned}\sum_{ij=1}^n F^{ij}v_{ij} =& \sum_{ij=1}^n F^{ij}u_{ij\xi\xi} - \sum_{ij=1}^n F^{ij}[a^l u_{ijl} + 2D_i a^l u_{jl} + D_{ij}a^l u_l + b_{ij}] + 2K_1\sum_{i=1}^n F^{ii}\\ \geq& \Theta_{\xi\xi} - A{\Theta_\xi}^2 - a^l\Theta_l - \sum_{i=1}^n |D_i a^i| - \sum_{i=1}^n F^{ii}[D_{ii}a^l u_l + b_{ii}] + K_1\sum_{i=1}^n F^{ii}\\ >& 0\end{aligned}$$

provided by

$$K_1 = \frac{1}{c_0}\left[|D^2\Theta| + A|D\Theta|^2 + |a^l||D\Theta| + |Da^l|\right] + |D^2a^l||Du| + |D^2b| + 1.$$

So $\max\limits_{\overline{\Omega}} v(x,\xi)$ attains its maximum on $\partial\Omega$. Therefore, we can assume the $\max\limits_{\Omega\times\mathbb{S}^{n-1}} v(x,\xi)$ attains its maximum at some point $x_0 \in \partial\Omega$ and some direction $\xi_0 \in \mathbb{S}^{n-1}$. We consider two cases in the following and all the calculations are at the point $x_0$ and $\xi = \xi_0$.

Case a: $\xi_0$ is tangential to $\partial\Omega$ at $x_0$.

Naturally, $\xi_0\cdot\nu = 0$, $v'(x_0,\xi_0) = 0$, and $u_{\xi_0\xi_0}(x_0) > 0$. Recall the following formulas in the book [9],

$$|D\nu| \leq C, \quad \text{in } \overline{\Omega_\mu}, \tag{3.19}$$

$$\sum_{i=1}^n \nu^i D_j\nu^i = 0, \quad \sum_{i=1}^n \nu^i D_i\nu^j = 0, \quad |\nu| = 1, \quad \text{in } \overline{\Omega_\mu}, \tag{3.20}$$

where $C$ is a constant depending only on $n$ and $\Omega$. As in [9], we still define

$$c^{ij} = \delta_{ij} - \nu^i\nu^j, \quad \text{in } \overline{\Omega_\mu}, \tag{3.21}$$

and for a vector $\zeta \in \mathbb{R}^n$, we denote $\zeta'$ for the vector with $i$-th component $\sum_{j=1}^n c^{ij}\zeta^j$. Then we have

$$|(Du)'|^2 = \sum_{i,j=1}^n c^{ij}u_iu_j \quad \text{and} \quad |Du|^2 = |(Du)'|^2 + u_\nu^2. \tag{3.22}$$

And,

$$\begin{aligned} u_{li}\nu^l =&[c^{ij} + \nu^i\nu^j]\nu^l u_{lj} \\ =&c^{ij}[D_j(\nu^l u_l) - D_j\nu^l u_l] + \nu^i\nu^j\nu^l u_{lj} \\ =&-c^{ij}\varepsilon u_j + c^{ij}D_j\varphi - c^{ij}u_lD_j\nu^l + \nu^i\nu^j\nu^l u_{lj}. \end{aligned} \tag{3.23}$$

The last equation used boundary conditions. Then,

$$\begin{aligned} u_{lip}\nu^l =&[c^{pq} + \nu^p\nu^q]u_{liq}\nu^l \\ =&c^{pq}[D_q(u_{li}\nu^l) - u_{li}D_q\nu^l] + \nu^p\nu^q u_{liq}\nu^l \\ =&c^{pq}D_q(-c^{ij}\varepsilon u_j + c^{ij}D_j\varphi - c^{ij}u_lD_j\nu^l + \nu^i\nu^j\nu^l u_{lj}) \\ &- c^{pq}u_{li}D_q\nu^l + \nu^p\nu^q\nu^l u_{liq}, \end{aligned} \tag{3.24}$$

hence we obtain

$$\begin{aligned} u_{\xi_0\xi_0\nu} =& \sum_{ipl=1}^n \xi_0^i\xi_0^p u_{lip}\nu^l \\ =& \sum_{ip=1}^n \xi_0^i\xi_0^p[c^{pq}D_q(-c^{ij}\varepsilon u_j + c^{ij}D_j\varphi - c^{ij}u_lD_j\nu^l + \nu^i\nu^j\nu^l u_{lj}) \\ & \qquad - c^{pq}u_{li}D_q\nu^l + \nu^p\nu^q\nu^l u_{liq}] \\ =& \sum_{i=1}^n \xi_0^i\xi_0^q[D_q(-c^{ij}\varepsilon u_j + c^{ij}D_j\varphi - c^{ij}u_lD_j\nu^l + \nu^i\nu^j\nu^l u_{lj}) - u_{li}D_q\nu^l] \\ =& -\xi_0^i\xi_0^q\varepsilon[c^{ij}u_{jq} - D_qc^{ij}u_j] + \xi_0^i\xi_0^qD_q(c^{ij}D_j\varphi) \\ & - \xi_0^i\xi_0^qD_q(c^{ij}D_j\nu^l)u_l - \xi_0^j\xi_0^qu_{lq}D_j\nu^l + \xi_0^i\xi_0^qD_q\nu^iu_{\nu\nu} - \xi_0^i\xi_0^qu_{lq}D_i\nu^l \\ \le& -u_{\xi_0\xi_0} - 2\xi_0^iu_{l\xi_0}D_i\nu^l + C_{15} + C_{15}|Du| + C_{15}|u_{\nu\nu}|. \end{aligned} \tag{3.25}$$

Without loss of generality, we assume $\xi_0 = e_1$, then we can get the bound for $u_{1i}(x_0)$ for $i > 1$ due to the maximum of $v$ at the $\xi_0$ direction. Next, we can assume $\xi(t) = \frac{(1,t,0,\cdots,0)}{\sqrt{1+t^2}}$. By calculating,

$$\begin{aligned} 0 =&\frac{dv(x_0,\xi(t))}{dt}|_{t=0} \\ =&2u_{ij}(x_0)\frac{d\xi^i(t)}{dt}|_{t=0}\xi^j(0) - \frac{dv'(x_0,\xi(t))}{dt}|_{t=0} \\ =&2u_{12}(x_0) - 2\nu^2(D_1\varphi - \varepsilon u_1 - u_lD_1\nu^l), \end{aligned} \tag{3.26}$$

then

$$|u_{12}(x_0)| = |\nu^2(D_1\varphi - \varepsilon u_1 - u_l D_1\nu^l)| \le C_{16} + C_{16}|Du|. \tag{3.27}$$

Analogously, for all $i > 1$, we have

$$|u_{1i}(x_0)| \le C_{16} + C_{16}|Du|. \tag{3.28}$$

From $\{D_i\nu^l\} \ge 0$, we have

$$\begin{aligned} u_{\xi_0\xi_0\nu} \le & - u_{\xi_0\xi_0} - D_1\nu^1 u_{\xi_0\xi_0} + C_{17}(1 + |u_{\nu\nu}|) \\ \le & - u_{\xi_0\xi_0} + C_{17}(1 + |u_{\nu\nu}|). \end{aligned} \tag{3.29}$$

On the other hand, from the Hopf lemma and (3.23), we have

$$\begin{aligned} 0 \le & v_\nu(x_0, \xi_0) \\ = & u_{\xi_0\xi_0\nu} - a^l u_{l\nu} - D_\nu a^l u_l - b_\nu + 2K_1(x \cdot \nu) \\ \le & - u_{\xi_0\xi_0} + C_{17}(1 + |u_{\nu\nu}|) + C_{18}. \end{aligned} \tag{3.30}$$

Hence,

$$u_{\xi_0\xi_0}(x_0) \le (C_{17} + C_{18})(1 + |u_{\nu\nu}|). \tag{3.31}$$

Since $u$ is the subharmonic function and (3.31), we obtain

$$\begin{aligned} \max_{\Omega\times\mathbb{S}^{n-1}} |u_{\xi\xi}(x)| \le & (n-1) \max_{\Omega\times\mathbb{S}^{n-1}} u_{\xi\xi}(x) \\ \le & (n-1)[\max_{\Omega\times\mathbb{S}^{n-1}} v(x, \xi) + C_{19}] \\ = & (n-1)[v(x_0, \xi_0) + C_{19}] \\ \le & (n-1)[u_{\xi_0\xi_0}(x_0) + 2C_{19}] \\ \le & C_{20}(1 + |u_{\nu\nu}|). \end{aligned} \tag{3.32}$$

Case b: $\xi_0$ is non-tangential.

In this case, we have $\xi_0 \cdot \nu \ne 0$. In fact, we can find a tangential vector $\tau \in \mathbb{S}^{n-1}$ such that $\xi_0 = \alpha\tau + \beta\nu$, with $\alpha = \xi_0 \cdot \tau \ge 0$, $\beta = \xi_0 \cdot \nu \ne 0$ and $\alpha^2 + \beta^2 = 1$. Naturally, $\tau \cdot \nu = 0$. So,

$$\begin{aligned} u_{\xi_0\xi_0}(x_0) = & \alpha^2 u_{\tau\tau}(x_0) + \beta^2 u_{\nu\nu}(x_0) + 2\alpha\beta u_{\tau\nu}(x_0) \\ = & \alpha^2 u_{\tau\tau}(x_0) + \beta^2 u_{\nu\nu}(x_0) + 2(\xi_0 \cdot \nu)[\xi_0 - (\xi_0 \cdot \nu)\nu][D\varphi - \varepsilon Du - u_l D\nu^l], \end{aligned} \tag{3.33}$$

then,

$$v(x_0, \xi_0) = \alpha^2 v(x_0, \tau) + \beta^2 v(x_0, \nu) \le \alpha^2 v(x_0, \xi_0) + \beta^2 v(x_0, \nu). \tag{3.34}$$

By the definition of $v(x_0,\xi_0)$, we know

$$v(x_0,\xi_0)\le v(x_0,\nu), \tag{3.35}$$

and

$$u_{\xi_0\xi_0}(x_0)\le v(x_0,\xi_0)+C_{21}\le v(x_0,\nu)+C_{21}\le |u_{\nu\nu}|+2C_{21}. \tag{3.36}$$

Similarly to (3.32), we complete the proof of the lemma.

**Lemma 3.6** Suppose $\Omega\subset\mathbb{R}^n$ is a $C^3$ strictly convex domain and $\varphi\in C^3(\partial\Omega)$. Let $\Theta(x)\in C^2(\overline{\Omega})$ with $\frac{(n-2)\pi}{2}<\Theta(x)<\frac{n\pi}{2}$ in $\overline{\Omega}$ and $u\in C^3(\Omega)\cap C^2(\overline{\Omega})$ be a solution of special Lagrangian equation (1.3) for any small $\varepsilon>0$, then we have

$$\min_{\partial\Omega}u_{\nu\nu}\ge -C_{22}, \tag{3.37}$$

where $C_{22}$ depends on $n$, $\Omega$, $\max\limits_{\overline{\Omega}}\Theta$, $\min\limits_{\overline{\Omega}}\Theta$, $M_0$, $M_1$, $|\Theta|_{C^2}$ and $|\varphi|_{C^3}$.

**Proof** Since $\Omega$ is a $C^3$ strictly convex domain, we can find the defining function $\rho\in C^3(\overline{\Omega})$ for it such that

$$\begin{aligned}&\rho=0 \text{ on } \partial\Omega,\quad \rho<0 \text{ in } \Omega;\quad |D\rho|=1 \text{ on } \partial\Omega;\\ &D^2\rho\ge a_0 I_n,\end{aligned}$$

where $a_0$ is a positive constant depending only on $\Omega$, and $I_n$ is the $n\times n$ identity matric. And, $\nu=(\nu^1,\nu^2,\cdots,\nu^n)$ is a $C^2(\overline{\Omega})$ extension of the outer unit normal vector field on $\partial\Omega$ as in Lemma 3.5.

By the classical barrier technique in [11], we consider the function

$$P(x)=u_\nu+\varepsilon u(x)-\varphi(x)+K_2\rho, \tag{3.38}$$

where $K_2=\max\{\frac{2(1+{C_0}^2)}{a_0}(|D\Theta||\nu|+|D\nu|+\frac{1}{2}),\frac{2}{a_0}(|Du||D^2\nu|+|D^2\varphi|)\}$ and $C_0$ is the constant in (2.5). Also note that here $\varphi\in C^2(\overline{\Omega})$ is an extension with universal $C^2$ norm. Recall

$$F^{ij}=\frac{\partial\arctan\eta}{\partial u_{ij}}.$$

For any $x\in\Omega$, we can assume $D^2u$ is diagonal with $\lambda_i=u_{ii}$ and $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n$. We know that (3.10)-(3.13) in the proof of Theorem 3.3 still hold. Hence we can get

$$\begin{aligned}\sum_{ij=1}^n F^{ij}P_{ij}=&\sum_{i=1}^n F^{ii}\{\sum_l[u_{lii}\nu^l+2u_{li}(\nu^l)_i+u_l(\nu^l)_{ii}]+\varepsilon u_{ii}-\varphi_{ii}\}+K_2\sum_{i=1}^n F^{ii}\rho_{ii}\\ =&\sum_l\Theta_l\nu^l+2\sum_{i=1}^n F^{ii}u_{ii}(\nu^i)_i+\sum_{i=1}^n F^{ii}u_l(\nu^l)_{ii}+\varepsilon\sum_{i=1}^n F^{ii}u_{ii}-\sum_{i=1}^n F^{ii}\varphi_{ii}+K_2\sum_{i=1}^n F^{ii}\rho_{ii}\\ \ge&-|D\Theta||\nu|-|D\nu|-|Du||D^2\nu|\sum_{i=1}^n F^{ii}-\frac{\varepsilon}{2}-|D^2\varphi|\sum_{i=1}^n F^{ii}+K_2a_0\sum_{i=1}^n F^{ii}\\ \ge&-|D\Theta||\nu|-|D\nu|-\frac{1}{2}+\frac{K_2a_0}{2}\frac{1}{1+{C_0}^2}\\ &-|Du||D^2\nu|\sum_{i=1}^n F^{ii}-|D^2\varphi|\sum_{i=1}^n F^{ii}+\frac{K_2a_0}{2}\sum_{i=1}^n F^{ii}\\ \ge&0.\end{aligned} \tag{3.39}$$

Also, it is easy to know $P = 0$ on $\partial\Omega$. Hence $P$ attains its maximum on any boundary point. Then we can get for any $x \in \partial\Omega$,

$$\begin{aligned}0 \le P_\nu(x) =&[u_{\nu\nu} - \sum_j u_j d_{j\nu} + \varepsilon u_\nu - \varphi_\nu] + K_2\rho_\nu \\ \le& u_{\nu\nu} + |Du||D^2 d| + |Du| + |D\varphi| + K_2,\end{aligned} \tag{3.40}$$

hence (3.37) holds.

In the following, we establish the upper estimate of double normal second derivatives on boundary.

**Lemma 3.7** Suppose $\Omega \subset \mathbb{R}^n$ is a $C^3$ strictly convex domain and $\varphi \in C^3(\partial\Omega)$. Let $\Theta(x) \in C^2(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$ and $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ be a solution of special Lagrangian equation (1.3) for any small $\varepsilon > 0$, then we have

$$\max_{\partial\Omega} u_{\nu\nu} \le C_{23}, \tag{3.41}$$

where $C_{23}$ depends on $n$, $\Omega$, $\max\limits_{\overline{\Omega}} \Theta$, $\min\limits_{\overline{\Omega}} \Theta$, $M_0$, $M_1$, $|\Theta|_{C^2}$ and $|\varphi|_{C^3}$.

**Proof** Similar to the proof of Lemma 3.6, we now consider the test function

$$\tilde{P}(x) = u_\nu + \varepsilon u(x) - \varphi(x) - \tilde{K}\rho, \tag{3.42}$$

where $\tilde{K} = \max\{\frac{2(1+{C_0}^2)}{a_0}(|D\Theta||\nu| + |D\nu| + \frac{n}{2}), \frac{2}{a_0}(|Du||D^2\nu| + |D^2\varphi|)\}$ and $C_0$ is the constant in (2.5). And, here $\varphi \in C^2(\overline{\Omega})$ is an extension with universal $C^2$ norm. Denote

$$F^{ij} = \frac{\partial \arctan\eta}{\partial u_{ij}}.$$

For any $x \in \Omega$, we can assume $D^2u$ is diagonal with $\lambda_i = u_{ii}$ and $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. We know that (3.10)-(3.13) in the proof of Theorem 3.3 still hold. Hence we can get

$$\begin{aligned}\sum_{ij=1}^n F^{ij}\tilde{P}_{ij} =& \sum_{i=1}^n F^{ii}\Big\{\sum_l [u_{lii}\nu^l + 2u_{li}(\nu^l)_i + u_l(\nu^l)_{ii}] + \varepsilon u_{ii} - \varphi_{ii}\Big\} - \tilde{K}\sum_{i=1}^n F^{ii}\rho_{ii} \\ =& \sum_l \Theta_l\nu^l + 2\sum_{i=1}^n F^{ii}u_{ii}(\nu^i)_i + \sum_{i=1}^n F^{ii}u_l(\nu^l)_{ii} + \varepsilon\sum_{i=1}^n F^{ii}u_{ii} - \sum_{i=1}^n F^{ii}\varphi_{ii} - \tilde{K}\sum_{i=1}^n F^{ii}\rho_{ii} \\ \le& |D\Theta||\nu| + |D\nu| + |Du||D^2\nu|\sum_{i=1}^n F^{ii} + \frac{\varepsilon n}{2} + |D^2\varphi|\sum_{i=1}^n F^{ii} - \tilde{K}a_0\sum_{i=1}^n F^{ii} \\ \le& |D\Theta||\nu| + |D\nu| + \frac{n}{2} - \frac{\tilde{K}a_0}{2}\frac{1}{1+{C_0}^2} \\ &+ |Du||D^2\nu|\sum_{i=1}^n F^{ii} + |D^2\varphi|\sum_{i=1}^n F^{ii} - \frac{\tilde{K}a_0}{2}\sum_{i=1}^n F^{ii} \\ \le& 0.\end{aligned} \tag{3.43}$$

Similar to (3.40), for any $x \in \partial\Omega$, we have

$$\begin{aligned} 0 \geq \tilde{P}_\nu(x) =& [u_{\nu\nu} - \sum_j u_j d_{j\nu} + \varepsilon u_\nu - \varphi_\nu] - \tilde{K}\rho_\nu \\ \geq& u_{\nu\nu} - |Du||D^2 d| - |Du| - |D\varphi| - \tilde{K}, \end{aligned} \tag{3.44}$$

hence (3.41) holds.

## 4 Existence of the boundary problems

In this section we complete the proofs of the Theorem 1.1.

### 4.1 Existence of solutions for the problem (1.3)

In Section 3, we have established the a priori estimate for the Neumann problem of special Lagrangian equation (1.3). We know that the special Lagrangian equation (1.3) is uniformly elliptic in $\overline{\Omega}$ by the global $C^2$ priori estimate above. $-e^{-A\arctan\eta}$ is concave with respect to $\eta$ thanks to the concavity lemma (Lemma 2.2 in [8]), where $A$ is the constant in (2.6). Following the discussions in [19], we can get the global Hölder estimate,

$$|u|_{C^{2,\alpha}(\overline{\Omega})} \leq C, \tag{4.1}$$

where $C$ and $\alpha$ depend on $n$, $\Omega$, $|\Theta|_{C^0}$, $|\Theta|_{C^2}$ and $|\varphi|_{C^3}$. Using the above estimate and differentiating the equation (1.3), one also obtains $C^{3,\alpha}(\overline{\Omega})$ estimates and applies the classical Schauder theory for linear uniformly elliptic equations.

Applying the method of continuity (see [9]), the existence of the classical solution holds. Using the standard regularity theory of uniformly elliptic partial differential equations, we can obtain the higher regularity.

### 4.2 Proof of Theorem 1.1

By a similar proof of existence of solutions for the problem (1.3), we know there exists a unique solution $u^\varepsilon \in C^{3,\alpha}(\overline{\Omega})$ to (1.3) for any small $\varepsilon > 0$. Let $v^\varepsilon = u^\varepsilon - \frac{1}{|\Omega|}\int_\Omega u^\varepsilon$, and it is easy to know $v^\varepsilon$ satisfies

$$\begin{cases} \arctan\{\Delta v^\varepsilon \mathrm{I}_n - D^2 v^\varepsilon\} = \Theta(x), \quad x \in \Omega, \\ (v^\varepsilon)_\nu = -\varepsilon v^\varepsilon - \frac{1}{|\Omega|}\int_\Omega \varepsilon u^\varepsilon + \varphi(x), \quad x \in \partial\Omega. \end{cases} \tag{4.2}$$

By the gradient estimate (3.7), we know $\varepsilon \sup\limits_{\overline{\Omega}} |Du^\varepsilon| \to 0$. Naturally, there is a constant $\beta$ and a function $v \in C^2(\overline{\Omega})$, such that $-\varepsilon u^\varepsilon \to \beta$, $-\varepsilon v^\varepsilon \to 0$, $-\frac{1}{|\Omega|}\int_\Omega \varepsilon u^\varepsilon \to \beta$ and $v^\varepsilon \to v$ uniformly in $C^2(\overline{\Omega})$ as $\varepsilon \to 0$. It is easy to verify that $v$ is a solution of

$$\begin{cases} \arctan\{\Delta v \mathrm{I}_n - D^2 v\} = \Theta(x), \quad x \in \Omega, \\ v_\nu = \beta + \varphi(x), \quad x \in \partial\Omega. \end{cases} \tag{4.3}$$

If there exists another function $v_1 \in C^2(\overline{\Omega})$ and another constant $\beta_1$ satisfying

$$\begin{cases} \arctan\{\Delta v_1 \mathrm{I}_n - D^2 v_1\} = \Theta(x), & x \in \Omega, \\ (v_1)_\nu = \beta_1 + \varphi(x), & x \in \partial\Omega, \end{cases} \tag{4.4}$$

and we can know $\beta = \beta_1$ and $v - v_1$ is the constant by applying the maximum principle and Hopf Lemma. We obtain the higher regularity by using the standard regularity theory.

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# 具有超临界相位的特殊拉格朗日型方程的Neumann问题

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**摘要:** 本文研究了$\mathbb{R}^n$空间中具有超临界相位的特殊拉格朗日型方程的Neumann的问题. 得到解的全局$C^2$估计并通过连续性方法建立了古典解的存在性定理.

**关键词:** Neumann问题; 特殊拉格朗日型方程; 超临界相位

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