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A SEMILINEAR PARTIAL DIFFERENTIAL EQUATION INDUCED BY HERMITIAN YANG-MILLS METRICS

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Abstract: In this paper, we investigate the boundary value problem and the radial symmetry of the global solution of a semilinear partial differential equation induced by studying the limiting behaviour of Hermitian Yang-Mills metrics. By applying maximum principle and Leray-Schauder fixed point theorem, we obtain the radial symmetry of the C^2 global solution in $\mathbb{R}^2$ and the existence of $C^{2,\alpha}$ solution of the Dirichlet problem in any bounded domain.

Keywords: Hermitian Yang-Mills metric; C^k-estimates; boundary value problems

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1 Introduction

Let X be a Kähler manifold with a family of Kähler metrics ω_ε, and let V be a slope stable holomorphic vector bundle over X. According to the Donaldson-Uhlenbeck-Yau theorem [1], V admits unique fully irreducible Hermitian-Yang-Mills metrics H_ε associated to each ω_ε. Similar to study the limiting behaviour Ricci flat metrics, Professor Jixiang Fu [2] studied the limiting behaviour of Hermitian Yang-Mills metrics H_ε when ω_ε goes to a large Kähler metric limit. A critical step in [2] is to explicitly construct a family of Hermitian-Yang-Mills metrics by solving the following semilinear partial differential equation on unit ball $B_1(0)$ of $\mathbb{R}^2$

$$\begin{cases} \Delta u = \varepsilon^{-2}\left(e^u - \left(x^2+y^2\right)e^{-u}\right) & \text{in } B_1(0), \\ u = 0 & \text{on } \partial B_1(0). \end{cases} \tag{1.1}$$

Here ε is a constant and (x, y) is the coordinate of $\mathbb{R}^2$. By the symmetry of the domain $B_1(0)$ and using reference [3], Jixiang Fu proved the equation (1.1) has a unique radially symmetric solution. However, this method can not be applied to non-symmetric domain Ω in $\mathbb{R}^2$.

In this paper, we first study the following equation defined a bounded connected domain $\Omega \subset \mathbb{R}^2$ with Dirichlet boundary value

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$$\begin{cases} \Delta u = \varepsilon^{-2}\left(e^{u} - \left(x^{2}+y^{2}\right)e^{-u}\right) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

For existence and uniqueness of the solution of (1.2), we have the following theorem.

Theorem 1.1 If $\partial\Omega$ is $C^{2,\alpha}$ and $g \in C^{2,\alpha}(\partial\Omega)$, there is a unique solution $u \in C^{2,\alpha}(\Omega)$ to equation (1.2) . Especially if $\partial\Omega$ and g are smooth, the solution u is smooth.

This theorem will give a Hermitian Yang-Mills metrics on a certain Kähler manifold given by [2]. On the other hand, the equation (1.1) can be defined on whole space $\mathbb{R}^2$. It is natural to explore whether the global solution of (1.1) is radially symmetric. The symmetry of global solutions of some semilinear equations has been investigated in [4] and [3] under the assumption $u(x,y)$ decays to zero at a certain rate as $r^2 = x^2 + y^2 \to +\infty$. But they do not fit the equation(1.1) since one can see the global solution u is not bounded. Similar to [3, 4], by using moving plane method and maximum principle, we get the following theorem.

Theorem 1.2 For any given constant c, if the global C^2 solution u of

$$\Delta u = \varepsilon^{-2}\left(e^{u} - \left(x^{2}+y^{2}\right)e^{-u}\right) \qquad \text{in } \mathbb{R}^2 \tag{1.3}$$

satisfies

$$u(s) - u(t) \to 0 \quad \text{as} \quad |s|, |t| \to \infty \quad \text{and} \quad |s| - |t| = c,$$

then u is radially symmetric and $\frac{\partial u}{\partial r} \geqslant 0$. Here $s, t \in \mathbb{R}^2$.

One may observe $\frac{1}{2}\log(x^2+y^2)$ is a singular solution to the equation (1.3) and also satisfies $\log(|s|) - \log(|t|) \to 0$ as $|t| - |s| = c$ and $|s|, |t| \to \infty$, so the assumption of Theorem1.2 is natural and reasonable.

The next part of this paper will give the detailed proof of Theorem1.1 and Theorem1.2.

2 Existence of Solution of the Dirichlet Boundary Value Problem

In this section we will prove Theorem1.1. One can use Chapter 14 in [5] to show the existence of the equation 1.2 by using the variational method. Here we take Leray-Schauder existence theorem to prove it.

Let Ω be a $C^{2,\alpha}$ bounded domain in $\mathbb{R}^2$ and $g \in C^{2,\alpha}(\partial\Omega)$ with $\alpha \in (0,1)$. We first have a $C^0(\Omega)$ estimate.

Lemma 2.1 Let Φ be the $C^{2,\alpha}$solution of Dirichlet boundary value problem

$$\begin{cases} \Delta \Phi = \varepsilon^{-2}(1 - \left(x^{2}+y^{2}\right)) & \text{in } \Omega, \\ \Phi = g & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Then a solution u to (1.2) satisfy

$$\sup_{\bar{\Omega}} |u| \leqslant \sup_{\bar{\Omega}} 2|\Phi|. \tag{2.2}$$

Proof The existence of Φ is from Green formula (one can see [6]). We consider $u-\Phi$ for $u>0$. We note that on $\mathcal{O}=\{x\in\Omega: u(x)>0\}$

$$\Delta(u-\Phi)=\varepsilon^{-2}(e^{u}-1+\left(x^{2}+y^{2}\right)\left(1-e^{-u}\right))>0.$$

By maximum principle, we have

$$\sup_{\mathcal{O}}(u-\Phi)=\sup_{\partial\mathcal{O}}(u-\Phi)\leqslant\sup_{\Omega}\{-\Phi,0\}\leqslant\sup_{\Omega}|\Phi|.$$

It follows

$$\sup_{\Omega}u\leqslant\sup_{\Omega}2|\Phi|. \tag{2.3}$$

Similarly, if $u<0$, $\Delta(u-\Phi)=\varepsilon^{-2}(e^{u}-1+(x^{2}+y^{2})(1-e^{-u}))<0$. Hence we obtain on $\mathcal{O}^{-}=\{x\in\Omega: u(x)<0\}$

$$\sup_{\mathcal{O}^{-}}(\Phi-u)=\sup_{\partial\mathcal{O}^{-}}(\Phi-u)\leqslant\sup_{\Omega}\{\Phi,0\}\leqslant\sup_{\Omega}|\Phi|$$

which implies

$$\sup_{\Omega}-u\leqslant\sup_{\Omega}2|\Phi|. \tag{2.4}$$

Therefore from (2.3) and (2.4) one can get the estimate (2.2).

Second, we give the gradient estimate of u.

Lemma 2.2 Suppose $u\in C^{2}(\Omega)$ satisfies the equation (1.2) in Ω, then there is positive constant C depending only on Ω and g such that

$$\sup_{\bar{\Omega}}|\nabla u|\leqslant C. \tag{2.5}$$

Proof From the equation (1.2) and by the standard regularity, one can see u is C^{4} since u and g are C^{2}. Then we have

$$\begin{aligned}\Delta|\nabla u|^{2}&=<\nabla\Delta u,\nabla u>+|\nabla^{2}u|^{2}\\&=\varepsilon^{-2}(e^{u}+r^{2}e^{-u})|\nabla u|^{2}-\varepsilon^{-2}e^{-u}<\nabla r^{2},\nabla u>+|\nabla^{2}u|^{2}.\end{aligned} \tag{2.6}$$

If $|\nabla u|^{2}$ attains its maximum on the boundary $\partial\Omega$, we have $\sup|\nabla u|=\sup|\nabla g|$ which leads to (2.5). Now we assume $|\nabla u|^{2}$ attains its maximum at $z_{0}\in\Omega$. Then from (2.6), at the point z_{0} we have

$$\varepsilon^{-2}(e^{u}+r^{2}e^{-u})|\nabla u|^{2}-\varepsilon^{-2}e^{-u}<\nabla r^{2},\nabla u>\leqslant 0$$

or

$$(e^{u}+r^{2}e^{-u})|\nabla u|^{2}\leqslant e^{-u}|\nabla r^{2}||\nabla u|.$$

Since $|u|$ is bounded from Lemma2.1, there is a constant C dependent on Ω and g such that $|\nabla u|(z_{0})\leqslant C$. Then we finish the proof.

Now we give the proof of Theorem1.1.

Proof Let $\sigma \in [0,1]$, we claim if $u_\sigma \in C^{2,\alpha}(\Omega)$ is the solution of boundary value problem

$$\begin{cases} \Delta u = \sigma\varepsilon^{-2}\left(e^u - \left(x^2+y^2\right)e^{-u}\right) & \text{in } \Omega, \\ u = \sigma g & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

then there is a constant M independent of u_σ and σ such that

$$||u_\sigma||_{C^{1,\alpha}(\bar{\Omega})} \leqslant M. \tag{2.8}$$

Then one can use the Leray-Schauder existence theorem (see Theorem 6.23 in [6]) to show the Dirichlet problem (1.2) is solvable in $C^{2,\alpha}(\bar{\Omega})$.

In fact, one can see $\sigma\Phi$ solves

$$\begin{cases} \Delta\Phi = \sigma\varepsilon^{-2}(1-(x^2+y^2)) & \text{in } \Omega, \\ \Phi = \sigma g & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

Then from Lemma2.1 , we have

$$||u_\sigma||_{C^0(\bar{\Omega})} \leqslant \sup_\Omega 2|\sigma\Phi| \leqslant \sup_\Omega 2|\Phi|. \tag{2.10}$$

Therefore, from (2.7), there is a constant C independent on σ and u_σ, such that $|\Delta u| \leqslant C$. This means $|\nabla^2 u|$ is also bounded. From Lemma2.2 and using interpolation inequality in Hölder space, there is a constant M independent on u and σ such that (2.8) is satisfied.

In the end, by standard bootstrap argument of the regularity we have u is smooth if Ω and g are smooth. For the uniqueness, one can use the method in Lemma 2.1 to prove that if there are two solutions u_1 and u_2, then $u_1 = u_2$.

3 Radial Symmetry of the Global C^2 Solution of the Equation in $\mathbb{R}^2$

In this section we will prove Theorem1.2. In [3] and [4], the radially symmetry of the C^2 positive solutions of the following second order elliptic equation is studied$\Delta u + f(u) = 0$ in $\mathbb{R}^n$ under the assumption on f and u. For example, they assumed $u(x) \to 0$ as $x \to \infty$. Obviously, our equation (1.2) is different from this type since $e^u - r^2e^{-u}$ has the term r^2. Also we cannot assume $|u| \to 0$ as $r \to +\infty$. In fact, it will lead $\Delta u \to -\infty$ and then u is unbounded. It contradicts the hypothesis. In this paper, we assume for any finite constant c,

$$u(s) - u(t) \to 0 \quad \text{and} \quad |s| - |t| = c, \qquad \text{as} \quad |s|, |t| \to \infty, \tag{3.1}$$

where $s, t \in \mathbb{R}^2$.

Proof of Theorem 1.2 Since the partial differential equation (1.3) is rotationally symmetric, we only have to prove the symmetry about a line across origin. Here we choose the line y axis. Define $\Sigma(\lambda) = \{(x,y) \in \mathbb{R}^2 \mid x < \lambda\}$ and let $v = u(2\lambda - x, y)$, $x^\lambda = 2\lambda - x$.

In $\Sigma(\lambda)$ we define $w(x,\lambda) = v - u$. When $\lambda = 0$ and $(x,y) \in \Sigma(\lambda)$, we have $x + x^\lambda = 0$ and $x < x^\lambda$. Then $x^2 = (x^\lambda)^2$ and

$$\Delta v - \varepsilon^{-2}\left(e^v - \left((x^\lambda)^2 + y^2\right)e^{-v}\right) = \Delta v - \varepsilon^{-2}\left(e^v - \left(x^2+y^2\right)e^{-v}\right) = 0. \tag{3.2}$$

By the mean value theorem, we have $\Delta w+\bar{c}w=0$ where $\bar{c}=-\int_0^1\varepsilon^{-2}(e^{u+tw}+r^2e^{-u-wt})dt<0$. Then from the assumption (3.1) and $w(0,0)=0$ on the y axis, we have by maximum principle and minimum principle $w=0$ in $\Sigma(0)$. That's to say the global solution of (1.3) in $\mathbb{R}^2$ is symmetric about y axis.

In the end, assuming $\lambda>0$ and $x\in\Sigma(\lambda)$, then we have $x+x^\lambda>0$ and $x<x^\lambda$. It follows $x^2<(x^\lambda)^2$ which implies

$$\Delta v-\varepsilon^{-2}\left(e^v-\left(x^2+y^2\right)e^{-v}\right)<0. \tag{3.3}$$

Then by the mean value theorem, in $\Sigma(\lambda)$, $\Delta w+\bar{c}w<0$ with $\bar{c}<0$. Using the infinite boundary condition (3.1) and $w(\lambda,\lambda)=0$, we have by maximum principle, in $\Sigma(\lambda)$, $w\geqslant 0$.

Then if $x>0$ and let $x_\lambda\to x$, we have $\frac{\partial u}{\partial x}\geqslant 0$. Since u is radially symmetric and from

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}=\frac{\partial u}{\partial r}\frac{x}{r},$$

it follows $\frac{\partial u}{\partial r}\geqslant 0$ and we finish the proof.

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一类由Hermitian Yang-Mills 度量导出的半线性偏微分方程

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摘要: 本本文研究了由一类Hermitian Yang-Mills度量的极限行为所导出的半线性方程的边值问题与全局解的径向对称性质. 使用极大值原理与Leray-Schauder 不动点定理, 我们得到了这个方程在$\mathbb{R}^2$平面中全局C^2解的径向对称性与这个方程的Dirichlet问题在任意有界区域内$C^{2,\alpha}$解的存在性.

关键词: Hermitian Yang-Mills 度量; C^k-估计; 边值问题

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