Vol. 43 (2023)
No. 3

数 学 杂 志
J. of Math. (PRC)

SHARP WEIGHTED ESTIMATES FOR Q-VARIATIONS OF SINGULAR OPERATORS ON THE SPACES OF HOMOGENEOUS TYPE

GONG Chen-xi

(*School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China*)

Abstract: In this work we extend the theorems of the sharp A_p weights to the q-variation of average operators and Calderón–Zygmund operators on the spaces of homogeneous type. These results make use of the new sparse dominating techniques given by Lerner and Omisboand on Euclidean spaces [1], and Lorist [2] in the setting of homogeneous spaces. In particular, we establish the sparse pointwise estimates for the parabolic operators.At last, we also discuss some applications of our theorems.

Keywords: sharp weighted estimates; q-variational inequalities; Calderón–Zygmund operators; sparse operators

2010 MR Subject Classification: 42B20; 42B25

Document code: A **Article ID:** 0255-7797(2023)03-0213-16

1 Introduction

In the last ten years, the sharp dependence of A_p constant of weighted inequalities has been wildly studied in Harmonic analysis. The so-called A_2 conjecture asserted that the sharp dependence of the $L^2(w)$ norm of a Calderón–Zygmund operator T on the A_2 constant of the weight w was linear, that was

$$\|Tf\|_{L^p(w)} \leq C_p\,[w]_{A_2}\|f\|_{L^2(w)}. \tag{1.1}$$

Hytönen proved in full generality the A_2 theorem in [3]. A further improvement was obtained in [4] by Hytönen, Lacey and Perez. In [4], the authors also replaced a Calderón–Zygmund operator T by the q-variation operator V_qT in a smooth cut-off setting. Shortly after, Lerner gave a simpler proof [5] relying on the pointwise dyadic domination by using a formula of local mean oscillation decomposition. Since then, there are numerous research papers on A_2 theorem associated to different operators by using Lerner's local mean oscillation decomposition.

After that, Lacey [6] discovered a new approach via weak type endpoint estimates. And soon it was developed by Hytönen, Roncal, and Tapiola [7]. Currently, the method of sparse domination which is most widely used is the one given by Lerner in [8].

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* **Received date:** 2021-05-16 **Accepted date:** 2021-09-09

Biography: Gong Chenxi (1996–), female, postgraduate, major in harmonic analysis and its applications. E-mail: 672255791@qq.com.

Vert recently, Lerner and Ombrosi had a more refined argument for the sparse domination in [1]. And then Lorist [2] proved a general sparse domination theorem in the spaces of homogeneous type. Lorist also obtained a series of applications in harmonic analysis. Especially, the A_2-theorem for Calderón-Zygmund operators in a space of homogeneous type was proved.

In order to develop the theory of Calderón-Zygmund operators in a more general setting, Coifman and Weiss [9]originally introduced the spaces of homogeneous type in the 1970s. Let us first recall the following notion of spaces of homogeneous type from [9,10]. Suppose that $\mathcal{X}$ is a non-empty set equipped with a quasi-metric d with a constant $c_d \geq 1$, namely, for any $x,\ y,\ z \in \mathcal{X}$,

(i) $d(x,y) = d(y,x)$;

(ii) $d(x,y) = 0$ if and only if $x = y$;

(iii) and $d(x,y) \leq c_d\,(d(x,z) + d(z,y))$.

If a Borel measure μ on the quasi-metric space $(\mathcal{X}, d)$ satisfies the doubling condition: there exists a constant $c_\mu \geq 1$, such that $\mu(B(x,2r)) \leq C_\mu\,\mu(B(x,r))$ for any ball $B_r(x) = B(x,r) := \{y \in \mathcal{X} : d(x,y) < r\}$, $r > 0$, then the triple $(\mathcal{X}, d, \mu)$ is called a space of homogeneous type. To simplify, we just assume that any ball B is a Borel set and that $0 < \mu(B) < \infty$.

In what follows, we need the following Ahlfors D-regular condition on the measure μ, namely, there exists an integer $D \geq 1$, such that, for any $x \in \mathcal{X}$ and $r > 0$, we have

$$\mu(B(x,r)) \sim r^D.$$

Whenever we have the notion of balls, the Hardy-Littlewood maximal function Mf for a $f \in L^1_{loc}(\mathcal{X})$ can be defined by

$$Mf(x) = \sup_{r>0} A_r(f)(x) := \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|d\mu(y), \quad x \in \mathcal{X}.$$

Indeed, since the Vitali covering lemma still holds true on spaces of homogeneous type by the doubling condition, M is bounded of $L^p(\mathcal{X}, \mu)$ for $1 < p \leq \infty$ and weak type $(1,1)$. And here $\mathcal{A} = (A_t)_{t>0}$ is called the average operator (or differential operator).

The distance d on $\mathcal{X}$ satisfies the Hölder condition if

$$|\rho(x,z) - \rho(y,z)| \leq C \max(\rho(x,z), \rho(y,z))^{1-\eta} \rho(x,y)^\eta$$

for some $0 < \eta \leq 1$. For more notions, properties and examples on the spaces of homogeneous type, we refer the reader to [11]. Throughout this article, we always assume that the space of homogeneous type $(\mathcal{X}, d, \mu)$ satisfies $\mathrm{diam}(\mathcal{X}) = \infty$. It is known that $\mathrm{diam}(\mathcal{X}) = \infty$ implies that $\mu(\mathcal{X}) = \infty$ (see, for instance, [2,Lemma 8.1]).

In the standard case of $\mathbb{R}^n$, the cubes play an important role. Christ originally characterized the dyadic cubes on then spaces of homogeneous type ([13]), which will be presented in Section 2. In recent years it has been known that a lot of results of Calderón-Zygmund operators hold true in the setting of spaces of homogeneous type. Indeed the spaces of homogeneous type were much adapted to the theory of Calderón-Zygmund operators.

In the present article, we will focus on establishing the sparse domination and sharp weighted inequalities for variational operators associated to average operators and singular integrals of non-convolution type. Let us then describe what is the variation norm of a continuous function, which plays a central role throughout the paper. Given a family $(a_t)_{t>0}$ be a sequence of complex numbers. Define $\|a\|_{v_q} = \sup \left(\sum_{j=0}^{\infty} |a_{t_j} - a_{t_{j+1}}|^q \right)^{\frac{1}{q}}$, where the supremum runs over all increasing sequences $\{t_j\}$ of positive real numbers. There is another inhomogeneous definition of variation

$$\|a\|_{\dot{v}_q} = |a_0| + \sup \left(\sum_{j=0}^{\infty} |a_{t_j} - a_{t_{j+1}}|^q \right)^{\frac{1}{q}}.$$

Then we can call the first one a homogeneous variation. We then define the q-variation of the average operator $A = (A_t)_{t>0}$ by $V_q A(f)(x) = \|A(f)(x)\|_{v_q}$.

The first variation inequality was proved by Lépingle [14] for martingales which improves the classical Doob maximal inequality. Simple and different proofs have been given by Pisier and Xu [15]. Thirteen years later, Bourgain [16] proved the variation inequality for the ergodic averages of a dynamic system. Bourgains work has inaugurated a new research direction in ergodic theory and harmonic analysis.

In particular, Campbell, Jones, Reinhold and Wierdl in [17,18] studied unweighted norm inequalities for the singular integral operators. In [19] and [20], the authors proved the weighted version of q-variation inequalites for Calderón-Zygmund operators. However, those results did not reflect the quantitative dependence of the $L^p(w)$ operator norm in terms of the relevant constant involving the weights.

To obtain the sparse dominations for variations of in a space of homogeneous type, we will rely on Lorist's theorems as follows in [2]. Then we first define the following notion of the sharp grand maximal trunction operator.

$$\mathcal{M}^{\#}_{T,\alpha} f(x) := \sup_{x\in B} \operatorname{ess} \sup_{x',x''\in B} \|T(f\mathbb{1}_{\mathcal{X}\setminus\alpha B})(x') - T(f\mathbb{1}_{\mathcal{X}\setminus\alpha B})(x'')\|$$

Theorem 1.1 Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type and let X and Y be Banach spaces. Take $p_1, p_2, r \in [1, \infty)$ and set $p_0 := \max p_1, p_2$. Take $\alpha \geq 3c_d^2/\delta$, where c_d is the quasi-metric constant and δ is a constant. Assume the following conditions:

1. T is a bounded linear operator from $L^{p_1}(\mathcal{X}; X)$ to $L^{p_1,\infty}(\mathcal{X}; Y)$.
2. $\mathcal{M}^{\#}_{T,Q}$is bounded from $L^{p_2}(\mathcal{X}; X)$ to $L^{p_1,\infty}(\mathcal{X}; Y)$.

3. For disjointly supported $f_1, \cdots, f_n \in L^{p_0}(\mathcal{X}; X)$ we have

$$\|T(\sum_{k=1}^{n} f_k)(x)\| \leq C_r(\sum_{k=1}^{n} \|Tf_k(x)\|_Y^r)^{1/r}, \qquad x \in \mathcal{X}.$$

Then there exists an $\eta \in (0,1)$ such that for any compactly supported $f_n \in L^{p_0}(\mathcal{X}; X)$there is an η-sparse collection of cubes $\mathcal{S}$ such that

$$\|Tf(x)\|_Y \lesssim_{\mathcal{X},\alpha} C_T C_r(\sum_{Q\in\mathcal{X}} \langle \|f\|_X \rangle_{p_0,\alpha Q}^r \mathbb{1}_Q(x))^{1/r}, \quad x \in \mathcal{X},$$

where $C_T = \|T\|_{L^{p_1} \to L^{p_1,\infty}} + \|\mathcal{M}_{T,\alpha}^{\#}\|_{L^{p_2} \to L^{p_2,\infty}}$

It is well known that by the sparse domination, one can deduce the sharp weighted inequalities based on the following theorem which first belongs to Lerner.

Theorem 1.2 Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type, let $\mathcal{S}$ be an η-sparse collection of cubes and take $0 < p_0, r < \infty$. For $p \in (p_0, \infty), w \in A_{p/p_0}, f \in L^p(\mathcal{X}, w)$, we have

$$\|(\sum_{Q\in\mathcal{S}} \langle f \rangle_{p_0,Q}^r \mathbb{1}_Q)^{1/r}\|_{L^p(\mathcal{X},w)} \lesssim [w]_{A_{p/p_0}}^{\max\{\frac{1}{p-p_0}, \frac{1}{r}\}} \|f\|_{L^p(\mathcal{X},w)},$$

and for $w \in A_1$ and $f \in L^{p_0}(\mathcal{X}, w)$

$$\|(\sum_{Q\in\mathcal{S}} \langle f \rangle_{p_0,Q}^r \mathbb{1}_Q)^{1/r}\|_{L^{p_0,\infty}(\mathcal{X},w)} \lesssim [w]_{A_1}^{1/p_0} \log(e + [w]_{A_\infty})^\beta \|f\|_{L^{p_0}(\mathcal{X},w)}.$$

Our first main result is the sparse domination of q-variation associated to average operators in a space of homogeneous type.

Theorem 1.3 Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type with the christ system.Let $A = (A_t)$ be the average operator. Then for any compactly supported $f \in L^1(\mathbb{R}^n)$, there exists a sparse family $\mathcal{S}$, for *a.e.* $x \in \mathbb{R}^n$,

$$|V_q Af(x)| \leq c_{\mathcal{X},q} \mathcal{A}_{\mathcal{S}} |f|(x). \tag{1.2}$$

Our second main result is the sparse domination of q-variation associated to the Calderón-Zygmund operators (see the definitions 3.1 and 3.2).

Theorem 1.4 Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type with the christ system.Let T be a Calderón–Zygmund operator defined in 3.2, and let $V_q T$ be weak type $(1,1)$, $q > 2$. Then for any compactly supported $f \in L^1(\mathbb{R}^n)$, there exists a spase family $\mathcal{S}$, for *a.e.* $x \in \mathbb{R}^n$,

$$|V_q Tf(x)| \leq c_{\mathcal{X},q} \mathcal{A}_{\mathcal{S}} |f|(x). \tag{1.3}$$

As a consequence of the previous results in Theorem 1.3 and Theorem 1.4, we get the following sharp weighted bound for the q-variation operator.

Theorem 1.5 Follow the conditions in Theorem 1.4 and let $w \in A_p$, $1 < p < \infty$. Then we have

$$||V_q T(f)||_{L^p(w)} \leq c_{\mathcal{X},q} [w]_{A_p}^{\max(1, \frac{1}{p-1})} ||f||_{L^p(w)}. \tag{1.4}$$

The outline of this paper will be as follows. In Section 2 we establish the notation and some assumption that we are going to study in this work. In Section 3, we give the definition of the Calderón-Zygmund operators. In Section 4 and 5, we present our proofs of the theorems 1.3 and 1.4 respectively. In Section 6, we outline some applications of our theorems.

Throughout this paper $A \lesssim B$ will denote $A \le CB$, where C will denote a positive constant independent of the weight constant which may change from one line to another.

2 Preliminaries

Recall that the standard dyadic system $\mathscr{D}$ on $\mathbb{R}^n$ consisting of the cubes

$$2^{-k}([0,1)^n + j), \quad k \in \mathbb{Z}, j \in \mathbb{Z}^n.$$

Then we can define the dyadic maximal operator $M_{\mathscr{D}}$,

$$M_{\mathscr{D}} f(x) = \sup_{Q \in \mathscr{D}} \langle |f| \rangle_Q \mathbb{1}_Q(x), \quad x \in \mathbb{R}^n$$

for any $f \in L^1_{loc}(\mathbb{R}^n)$. $M_{\mathscr{D}}$ enjoys some better properties than the Hardy-Littlewood maximal operator M itself. $M_{\mathscr{D}}$ actually is a kind of special martingale maximal operator, so that many boundedness properties of $M_{\mathscr{D}}$ could be obtained from the corresponding properties of the martingale maximal operators (see, for instance, [21], P.32]).

Motivated by the indispensable role of the system of the dyadic cubes on Euclidean spaces, Christ [13] constructed the analogous system on doubling metric spaces by an axiomatic approach, and also proved that every space of homogeneous type admits a system of dyadic cubes defined as follows.

2.1 System of Dyadic Cubes on Spaces of Homogeneous Type

In a space of homogeneous type $(\mathcal{X}, d, \mu)$, a family $\mathscr{D}$ of Borel sets Q_j^k, $k \in \mathbb{Z}$, $j \in J_k$, with parameters $0 < c_0 \le C_0 < \infty$ and $0 < \delta < 1$ is called a system of dyadic cubes if it satisfies the following properties:

1. for any $k \in \mathbb{Z}$, $\mathcal{X} = \bigcup_{j \in J_k} Q_j^k$;
2. if $l \le k$, either $Q_j^k \subset Q_j^l$ or $Q_j^k \cap Q_j^l = \emptyset$;
3. there is a collection $\{z_j^k\}_{j \in J_k}$ of points, such that $B(z_j^k, c_0\delta^k) \subset Q_j^k \subset B(z_j^k, C_0\delta^k)$ for any $k \in \mathbb{Z}$, $j \in J_k$.

Q_j^k is called a dyadic cube of generation k with center point z_j^k and side length δ^k. Let $\mathscr{D}_k$ denote the collection that consists of all dyadic cubes of generation k. For a cube $Q \in \mathscr{D}$, we then define the restricted dyadic system $\mathscr{D}(Q) = \{P \in \mathscr{D} : P \subset Q\}$. Moreover, for a cube $Q_j^k \in \mathscr{D}$, we define the dilations αQ_j^k for $\alpha \ge 1$ as the dilations of the ball that contains Q_j^k in the axiom (3) above, that is $\alpha Q_j^k := B(z_j^k, \alpha C_0 \delta^k)$. We say that a family $\mathscr{D}$

of subsets of $\mathcal{X}$ has the small boundary property if there exist $\eta > 0$ and $C_3 < \infty$ such that for every $Q \in \mathscr{D}$ and every $0 < \tau \leq 1$,

$$\mu(\partial_{\tau \operatorname{diam}(Q)} Q) \leq C_3 \tau^{\eta} \mu(Q),$$

where $\partial_\tau(Q) = \{x \in Q : \operatorname{dist}(x, X \setminus Q) \leq \tau\} \cup \{x \in X \setminus Q : \operatorname{dist}(x, Q) \leq \tau\}$. We call $\mathscr{D}$ a Christ system, if the dyadic system $\mathscr{D}$ has the small boundary property. In what follows, when we give a space of homogeneous type $(\mathcal{X}, d, \mu)$, it always admits a Christ system.

We introduce the notation of martingale associated with the system of dyadic cubes $\mathscr{D}$. The conditional expectation $\mathbb{E}_k$ associated with the σ-algebra generated by $\mathscr{D}_k$ is defined by

$$\mathbb{E}_k f = \sum_{Q \in \mathscr{D}_k} \langle f \rangle_Q \mathbb{1}_Q, \quad f \in L^1_{loc}(\mathcal{X}),$$

where $\langle f \rangle_Q := \frac{1}{\mu(Q)} \int_Q f(x) d\mu(x)$. Then the martingale generated by f is just the sequence $(\mathbb{E}_k f)_{k \in \mathbb{Z}}$ and for each $k \in \mathbb{Z}$, $\mathbb{D}_k = \mathbb{E}_k - \mathbb{E}_{k+1}$ denotes the martingale difference.

We also have a Calderón-Zygmund decomposition on spaces of homogeneous type (see [9] or [11]). It says the following. Let $f \in L^1(\mathcal{X})$, and $\lambda > 0$. Then f can be decomposed as $f = g + b$ $\|g\|_{L^2}^2 \leq C\lambda \|f\|_{L^1}$ and $b = \sum_j b_j$, where each b_j is supported on some cube $Q_j \in \mathscr{D}$ with $\int_{Q_j} b_j(y) d\mu(y) = 0$. Here all Q_j are disjoint cubes and $\sum_j \mu(Q_j) \leq \|f\|_{L^1}$.

In a space of homogeneous type , we can find a finite collection of dyadic systems, such that any ball is contained in a cube of comparable size from one of these dyadic systems as the following proposition (see [22,Theorem 4]).

2.2 Sparse Family

Let $\mathscr{D}$ be a system of dyadic cubes. A family $\mathcal{S} \subset \mathscr{D}$ is η-sparse if for every $Q \in \mathcal{S}$ there exists a measurable set $E_Q \subset Q$ such that $\mu(E_Q) \geq \eta \mu(Q)$ and such that the E_Q's are pairwise disjoint.

For a given sparse family $\mathcal{S}$, the so-call sparse operator is defined by

$$\mathcal{A}_{\mathcal{S}}(f)(x) = \sum_{Q \in \mathcal{S}} \langle f \rangle_Q \mathbb{1}_Q(x).$$

When $\mathcal{X} = \mathbb{R}^n$, these operators verify the following linear A_p theorem that was proved in [23] and that be very useful in the A_2 theorems.

Lemma 2.1 Suppose that $1 < p < \infty$ and $w \in A_p$. Then $\|\mathcal{A}_{\mathcal{D},\mathcal{S}}(f)\|_{L^p(w)} \lesssim [w]_{A_P}^{\max(1, \frac{1}{p-1})} \|f\|_{L^p(w)}$.

3 Calderón–Zygmund Operators on Homogeneous spaces

We follow the notion of Calderón–Zygmund operators from [9] (see also [11]).

Definition 3.1 A function $K \in L^1_{loc}(\{\mathcal{X} \times \mathcal{X}\} \backslash \{(x,x) : x \in \mathcal{X}\})$ is called a Calderón–Zygmund kernel if there exists a constant $C > 0$, such that

(i) for any x, $y \in \mathcal{X}$ with $x \neq y$,

$$|K(x,y)| \leq \frac{C}{V(x,y)}, \tag{3.1}$$

where $V(x,y) := \mu(B(x, d(x,y)))$.

(ii) there exists a constant $s \in (0,1]$, such that for any x, x', $y \in \mathcal{X}$,

$$|K(x,y) - K(x',y)| \leq C \left(\frac{d(x,x')}{d(x,y)}\right)^s \frac{1}{V(x,y)} \tag{3.2}$$

for $d(x,y) \geq 2d(x,x') > 0$;

$$|K(x,y) - K(x,y')| \leq C \left(\frac{d(y,y')}{d(x,y)}\right)^s \frac{1}{V(x,y)}, \tag{3.3}$$

for $d(x,y) \geq 2d(y,y') > 0$;

(iii) the correlation condition: for $0 < r < R < \infty$,

$$\int_{r<\rho(x,y)<R} K(x,y)dy = \int_{r<\rho(x,y)<R} K(x,y)dx = 0. \tag{3.4}$$

Definition 3.2 Let $1 < p_0 < \infty$. A linear operator $T : L^{p_0}(\mathcal{X}) \to L^{p_0}(\mathcal{X})$ is called a Calderón-Zygmund operator if

1. $T : L^{p_0}(\mathcal{X}) \to L^{p_0}(\mathcal{X})$ is bounded.
2. Let K be a Calderón–Zygmund kernel, such that $Tf(x) = \int_{\mathcal{X}} K(x,y)f(y)d\mu(y), \quad x \in \operatorname{supp} f$, for any compactly continuous function f on $\mathcal{X}$.

We recall the following sparse domination theorem for Calderön–Zygmund operators in a space of homogeneous type from [2,Theorem 6.1]. For the convenience to the reader, we write down the proof. And we mention that the Calderón–Zygmund operators in [2] were taken in a general setting called the Dini condition. However, it is not much different in the proof with our presentation.

Theorem 3.3 Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type, and let $1 < p_0 < \infty$. Suppose T is a bounded Calderón–Zygmund operator from $L^{p_0}(\mathcal{X})$ to $L^{p_0}(\mathcal{X})$ with kernel K. Then for every boundedly supported $f \in L^1(\mathcal{X})$ there exists an η-sparse collection of cubes $\mathcal{S}$ such that

$$|Tf(x)| \lesssim_{S,p_0} C_T \sum_{Q\in\mathcal{S}} \langle |f| \rangle_{1,Q} \mathbb{1}_Q(x), \quad x \in \mathcal{X}.$$

Moreover, for all $p \in (1,\infty)$ and $w \in A_p$, we have

$$\|T\|_{L^p(\mathcal{X},w)\to L^p(\mathcal{X},w} \lesssim_{\mathcal{X},p,p_0} C_T \, [w]_{A_p}^{\max\left\{\frac{1}{p-1},1\right\}}$$

with $C_T := \|T\|_{L^{p_0}(\mathcal{X}) \to L^{p_0,\infty}(\mathcal{X})}$.

Proof Based on Theorem 1.1 and 1.2, We are left to check the assumptions of the weak L^1-boundedness of T and $M^{\#}_{T,\alpha}$. In fact, the weak L^1-boundedness of T has been originally considered by Coifman and Weiss [9], and also by Stein [24,Chapter 1], even in non-homogeneous metric spaces by Nazarov, Treil and Volberg [25].

Now, it suffices to show that $M^{\#}_{T,\alpha}$ is weak type $(1,1)$. Fix $x \in \mathcal{X}$ and a ball $B = B(z,r)$ with $x \in B$. Let $\alpha = 5\,c_d^2/\delta$. Then for any boundedly supported $f \in L^1(\mathcal{X})$, we have

$$\left|T(\mathbb{1}_{\mathcal{X}\setminus\alpha B} f)(x') - T(\mathbb{1}_{\mathcal{X}\setminus\alpha B} f)(x'')\right| \le \int_{\mathcal{X}\setminus\alpha B} |K(x',y) - K(x'',y)|\,|f(y)|\,d\mu(t).$$

Note that $d(x',x'') \le 2c_d r$ and $d(y,x') \ge \alpha r/c_d - r = 5c_d r/\delta - r$. Then for $c_d \ge 1, 0 < \delta < 1$, we have $2d(x',x'') \le d(y,x')$. Therefore, using the property (3.2), we deduce

$$\begin{aligned}
&\|T(\mathbb{1}_{\mathcal{X}\setminus\alpha B} f)(x') - T(\mathbb{1}_{\mathcal{X}\setminus\alpha B} f)(x'')\| \\
&\qquad \le C \int_{d(x',y) > 2c_d r} \left(\frac{d(x',x'')}{d(x',y)}\right)^s \frac{1}{V(x',y)}\,|f(y)|\,d\mu(y) \\
&\qquad \le C \sum_{j=0}^{\infty} \int_{2c_d r \le d(x',y) < 2^{j+1} 2c_d r} \left(\frac{2c_d r}{2^j 2c_d r}\right)^s \frac{c_\mu}{\mu(B(x', 2^{j+1} 2c_d r))}\,|f(y)|\,d\mu(y) \\
&\qquad \lesssim \sum_{j=0}^{\infty} \left(\frac{1}{2^j}\right)^s \frac{c_\mu}{\mu(B(x', 2^{j+1} 2c_d r))} \int_{d(x',y) \le 2^{j+1} 2c_d r} |f(y)|\,d\mu(y) \\
&\qquad \le \sum_{j=0}^{\infty} \left(\frac{1}{2^j}\right)^s M(f)(x) \lesssim M(f)(x). \qquad (3.5)
\end{aligned}$$

Taking the supremum over all x', x'' in B which is an arbitrary ball containing x, we get $M^{\#}_{T,\alpha} \lesssim M(f)$ pointwise on $\mathcal{X}$. We then deduce directly the weak L^1-boundedness of the operator $M^{\#}_{T,\alpha}$ for the Hardy-Littlewood operator. The proof of 3.3 is complete.

4 Proof of Theorem 1.3

In order to deduce the results in Theorem 1.3, we need the weak L^1-boundedness of $V_q A$ and $M^{\#}_{V_q A,\alpha}$. Set the variational intervals $I_j = (t_j, t_{j+1}]$, and put $A_{I_j} = A_{t_{j+1}} - A_{t_j}$ for any $j \ge 0$. Then $V_q A(f) = \sup_{t_j} (\sum_j |A_{I_j} f|^q)^{1/q}$.

Lemma 4.1 Let $2 < q < \infty$. Then for $1 < p < \infty$, $V_q A$ is bounded on $L^p(\mathcal{X})$ and weak type $(1,1)$. If $\mathcal{X} = \mathbb{R}^n$, variational estimates go back to [16] (see also [26]). Weighted variational estimates for averaging operators were firstly considered in [19,20]. Taking vector-valued cases, variational estimates for averaging operators were established in [27]. In fact, even in $\mathbb{R}^n$ it is long and technical to estimate variation norms for averaging operators. However, in a space of homogeneous type, the proof of variational estimates for averaging operators is similar to that in a Euclidean space. Let us sketch the proof of Lemma 4.1.

Let $\mathscr{D}$ be a system of dyadic cubes on $(\mathcal{X}, d, \mu)$, and $\mathscr{D}_k$ the system of the k-th generation. Then there is a σ-algebra generated by $\mathscr{D}_k$ on which we denote by $\mathbb{E}_k$ the conditional expectation. For an integrable function f on $(\mathcal{X},d,\mu)$, $f_k=\mathbb{E}_kf$, $k\in\mathbb{Z}$ forms a sequence of martingale, which means that $\mathbb{E}_k(f_k-f_{k+1})=0$ for every $k\in\mathbb{Z}$. Thus by the Lépingle martingale inequality, we have that V_q is bounded on $L^p(\mathcal{X})$ and weak type $(1,1)$ for the martingales.

Proof As for the variation of average operators $V_qA(f)$ on an integrable function f, we can compare it with the variation of the martingale generated by f. Then we divide the $V_qA(f)$ into three parts, the short variation, the long variation and the martingale variation. Noticing that by the structure of the system of dyadic cubes, we can deal with the short variation and long variation with analogous geometry methods. Generally, we first deduce the L^2 boundedness of $V_qA(f)$. Finally, the weak type $(1,1)$ bound follows by the Calderón-Zygmund decomposition.

Lemma 4.2 Let $2<q<\infty$, $\alpha\geq 5\,c_d^2/\delta$. Then for $1<p<\infty$, $M^{\#}_{V_qA,\alpha}$ is bounded on $L^p(\mathcal{X})$ and weak type $(1,1)$.

Proof Fix $x\in\mathcal{X}$ and a ball $B=B(z,r)$ with $x\in B$. Then by the triangle inequality, for any boundedly supported $f\in L^1(\mathcal{X})$, we have

$$\begin{aligned}&\big|V_qA(\mathbb{1}_{\mathcal{X}\backslash\alpha B}f)(x')-V_qA(\mathbb{1}_{\mathcal{X}\backslash\alpha B}f)(x'')\big|\\&\leq\sup_{t_j}\left(\sum_{j=0}^{\infty}\big|A_{I_j}(\mathbb{1}_{\mathcal{X}\backslash\alpha B}f)(x')-A_{I_j}(\mathbb{1}_{\mathcal{X}\backslash\alpha B}f)(x'')\big|^q\right)^{1/q}.\end{aligned}\tag{4.1}$$

Note that if $d(y,x')\leq 2r$, $y\in\alpha B$. Then $A_{t_j}(\mathbb{1}_{\mathcal{X}\backslash\alpha B}f)(x')=0$ for $t_j\leq 2r$. And it is also the fact for x''. So we can set $\{t_j\}_{j\geq0}$ be an increasing sequence with $t_0>2r$. Let us now go back to (4.1). We have

$$\begin{aligned}&A_{I_j}(\mathbb{1}_{\mathcal{X}\backslash\alpha B}f)(x')-A_{I_j}(\mathbb{1}_{\mathcal{X}\backslash\alpha B}f)(x'')\\=&\frac{1}{\mu(B_{t_{j+1}})}\int_{\mathcal{X}}(\mathbb{1}_{\mathcal{X}\backslash\alpha B}f)(\mathbb{1}_{B_{t_{j+1}}(x')}-\mathbb{1}_{B_{t_j}(x')})(y)dy\\&-\frac{1}{\mu(B_{t_{j+1}})}\int_{\mathcal{X}}(\mathbb{1}_{\mathcal{X}\backslash\alpha B}f)(\mathbb{1}_{B_{t_{j+1}}(x'')}-\mathbb{1}_{B_{t_{j+1}}(x'')})(y)dy\\&+\left(\frac{1}{\mu(B_{t_j})}-\frac{1}{\mu(B_{t_{j+1}})}\right)\int_{\mathcal{X}}(\mathbb{1}_{\mathcal{X}\backslash\alpha B}f)\left(\mathbb{1}_{B_{t_j}(x'')}-\mathbb{1}_{B_{t_j}(x')}\right)(y)dy\\=:&I_j+II_j+III_j.\end{aligned}\tag{4.2}$$

In fact, we have to show that for some $1<s\leq q$,

$$\big|V_qA(\mathbb{1}_{\mathcal{X}\backslash\alpha B}f)(x')-V_qA(\mathbb{1}_{\mathcal{X}\backslash\alpha B}f)(x'')\big|\leq M(|f|^s)^{1/s}.\tag{4.3}$$

To deal with the parts I_j and II_j, we introduce the following sets which divide all variational intervals into two parts,

$$J_1=\big\{j\ :\ t_{j+1}-t_j\leq d(x',x'')\big\}\quad\text{and}\quad J_2=\big\{j\ :\ t_{j+1}-t_j>d(x',x'')\big\}.$$

Then we can deal with the sums in the parts I_j and II_j on J_1 and J_2. On J_1, we only need to consider I, since II_j can be done in the same way. Then by the Hölder inequality and the D-regular property of $\mathcal{X}$, we have

$$\begin{aligned}
\sum_{j\in J_1}|I_j|^q &\le \sum_{j\in J_1}\frac{1}{\mu(B_{t_{j+1}})^s}\Big|\int_{\mathcal{X}\setminus\alpha B} f(y)(\mathbb{1}_{B_{t_{j+1}}(x')}-\mathbb{1}_{B_{t_j}(x')})(y)dy\Big|^s\\
&\lesssim \sum_{j\in J_1}\frac{1}{t_{j+1}^{Ds}}(t_{j+1}^D-t_j^D)^{s-1}\int_{\mathcal{X}\setminus\alpha B}|f(y)|^s\mathbb{1}_{B_{t_{j+1}}(x')\setminus B_{t_j}}(y)dy\\
&\lesssim d(x',x'')^{s-1}\sum_{j\in J_1}\frac{1}{t_{j+1}^{D+s-1}}\int_{\mathcal{X}\setminus\alpha B}|f(y)|^s\mathbb{1}_{B_{t_{j+1}}(x')\setminus B_{t_j}}(y)dy\\
&\lesssim d(x',x'')^{s-1}\int_{\mathcal{X}\setminus\alpha B}|f(y)|^s\sum_{j\in J_1}\frac{1}{t_{j+1}^{D+s-1}}\mathbb{1}_{B_{t_{j+1}}(x')\setminus B_{t_j}}(y)dy\\
&\lesssim d(x',x'')^{s-1}\int_{\mathcal{X}\setminus\alpha B}|f(y)|^s\frac{1}{t_{j(y)+1}^{D+s-1}}dy\\
&\lesssim (2r)^{s-1}\int_{d(y,x')\ge 2r}|f(y)|^s\frac{1}{d(y,x')^{D+r-1}}dy, \qquad (4.4)
\end{aligned}$$

where $j(y)$ denotes the unique j such that $t_{j(y)}\le d(y,x')<t_{j(y)+1}$ for given $y\in\mathcal{X}\setminus\alpha B$. Therefore, in a similar standard method of integration as in (3.5), we get

$$\sum_{j\in J_1}|I_j|^q\lesssim(2r)^{s-1}\int_{d(y,x')>2r}|f(y)|^s\frac{1}{d(y,x')^{D+s-1}}dy\lesssim M(|f|^s)(x).$$

To deal with the parts I_j and II_j on J_2, we have to put I_j and II_j together. For an interval $I=(a,b]$ and $x\in\mathcal{X}$, let $R_I(x)=R_{(a,b]}(x)$ denote the annulus $\{y: a<d(y,x)\le b\}$. For every $j\in J_2$, we then have

$$I_j+II_j=\left(\frac{1}{\mu(B_{t_{j+1}})}\int_{\mathcal{X}\setminus\alpha B}f(y)(\mathbb{1}_{R_{(t_j,t_{j+1}]}(x')}-\mathbb{1}_{R_{(t_j,t_{j+1}]}(x'')})(y)dy\right). \qquad (4.5)$$

Observe that

$$\begin{aligned}
|(\mathbb{1}_{R_{(t_j,t_{j+1}]}(x')}-\mathbb{1}_{R_{(t_j,t_{j+1}]}(x'')})(y)| \le{}&\mathbb{1}_{R_{(t_j,\,t_j+d(x',x'')]}(x')}(y)+\mathbb{1}_{R_{(t_{j+1},\,t_{j+1}+d(x',x'')]}(x')}(y)\\
&+\mathbb{1}_{R_{(t_j,\,t_j+d(x',x'')]}(x'')}(y)+\mathbb{1}_{R_{(t_{j+1},\,t_{j+1}+d(x',x'')]}(x'')}(y).
\end{aligned} \qquad (4.6)$$

However, for the analogous structures, it suffices to deal with the part I_j+II_j on J_2 in only one of the four above cases. Indeed,

$$\begin{aligned}
&\sum_{j\in J_2}\left|\frac{1}{\mu(B_{t_{j+1}})}\int_{\mathcal{X}\setminus\alpha B} f(y)\mathbb{1}_{R_{(t_j,\,t_j+d(x',x'')]}(x')}(y)dy\right|^s\\
&\lesssim \sum_{j\in J_2}\frac{\left((t_j+d(x',x''))^D-t_j^D\right)^{s-1}}{t_{j+1}^{Ds}}\int_{\mathcal{X}\setminus\alpha B}|f(y)|^s\,\mathbb{1}_{R_{(t_j,\,t_j+d(x',x'')]}(x')}(y)dy\\
&\lesssim \sum_{j\in J_2}\frac{(t_j+d(x',x''))^{(D-1)(s-1)}d(x',x'')^{s-1}}{t_{j+1}^{Ds}}\int_{\mathcal{X}\setminus\alpha B}|f(y)|^s\,\mathbb{1}_{R_{(t_j,\,t_j+d(x',x'')]}(x')}(y)dy\\
&\lesssim \sum_{j\in J_2}\frac{r^{s-1}}{t_{j+1}^{D+s-1}}\int_{\mathcal{X}\setminus\alpha B}|f(y)|^s\,\mathbb{1}_{R_{(t_j,\,t_j+d(x',x'')]}(x')}(y)dy\\
&\lesssim r^{s-1}\int_{d(y,x')\geq 2r}|f(y)|^s\sum_{j\in J_2}\frac{1}{t_{j(y)+1}^{D+s-1}}\mathbb{1}_{R_{(t_j,\,t_j+d(x',x'')]}(x')}(y)dy\\
&\lesssim r^{s-1}\int_{d(y,x')\geq 2r}|f(y)|^s\,\frac{1}{d(y,x')^{D+s-1}}dy,
\end{aligned}$$

where $j(y)$ still denotes the unique j such that $t_{j(y)}\leq d(y,x')<t_{j(y)+1}$ for given $y\in\mathcal{X}\setminus\alpha B$. And the last formula in (4.4) is the same as the one in (4.4). Thus, we also have

$$\left(\sum_{j\in J_2}|I_j+II_j|^q\right)^{1/q}\lesssim M(|f|^s)^{1/s}(x). \tag{4.7}$$

Now we are left to deal with III_j. Observing that, for $d(y,x'')\leq d(y,x')+d(x',x'')$, we have

$$\mathbb{1}_{B_{t_j}(x'')}(y)-\mathbb{1}_{B_{t_j}(x')}(y)=\mathbb{1}_{R_{(t_j,\,t_j+d(x',x'')]}(x')}(y)-\mathbb{1}_{R_{(t_j,\,t_j+d(x',x'')]}(x'')}(y).$$

Then using the argument similar to that of the preceding steps, we get

$$\begin{aligned}
\sum_j|III_j|^s&\lesssim\sum_{j\in J_2}\frac{d(x',x'')^{s-1}}{t_{j+1}^{D+s-1}}\int_{\mathcal{X}\setminus\alpha B}|f(y)|^s\left(\mathbb{1}_{R_{(t_j,\,t_j+d(x',x'')]}(x')}+\mathbb{1}_{R_{(t_j,\,t_j+d(x',x'')]}(x'')}\right)(y)dy\\
&\lesssim r^{s-1}\int_{d(y,x')\geq 2r}|f(y)|^s\,\frac{1}{d(y,x')^{D+s-1}}dy\\
&\lesssim M(|f|^s)(x).
\end{aligned}\tag{4.8}$$

Finally, combining this inequality and (4.1), (4.2) and (4.7), we obtain L^p boundedness for $p>1$.

When $p=1$, the inequlity of the weak type $(1,1)$ will be deduced by Calderón–Zygmund decomposition $f=g+b$, $f\in L^1(\mathcal{X})$. For the part g, the weak type $(1,1)$ inequality is from the L^2 boundedness of $M^{\#}_{V_qA,\alpha}$. As for the part $b=\sum_j b_j$, let $\tilde{Q}=\cup_j\alpha Q_j$. It suffices to show

$$\mu(\{x\in\mathcal{X}\setminus\tilde{Q}:M^{\#}_{V_qA,\alpha}(b)>\lambda\})\lesssim\sum_j\mu(Q_j).$$

In fact, the small boundary property of the dyadic system and the properties of b_j imply that we can derive the above inequality by comparing the argument used in the case of $\mathbb{R}^D$. We need to first establish a reverse Hölder inequality (see. [28,Lemma 3.5], the corresponding Euclidean case in [26,Lemma 4.2]); the details are omitted. This finishes the proof of Lemma 4.2.

5 Proof of Theorem 1.4

To prove Theorem 1.4, it suffices to show the results in the following lemma.And by the Theorem 1.2, we also deduce the sharp weighted theorem 1.5.

Lemma 5.1 Let $2 < q < \infty$, $\alpha \geq 5\,c_d^2/\delta$ and $x \in \mathcal{X}$. Let T is a Calderón-Zygmund operator. Then for $1 < p < \infty$, $M^{\#}_{V_qT,\alpha}$ is bounded on $L^p(\mathcal{X})$ and weak type $(1,1)$.

Proof For any interval $I = (s,\, t] \subset (0, \infty)$ and any boundedly supported $f \in L^1(\mathcal{X})$, denote

$$\mathcal{K}_I(f)(x) := \int_{\mathcal{X}} K(x,y)\mathbb{1}_{R_I(x)}(y)f(y)dy, \quad x \in \mathcal{X}.$$

Fix a ball $B = B(z,r)$ with $x \in B$, and let $\alpha = 5\,c_d^2/\delta$. Then by the Hölder inequality of V_q, we have

$$\begin{aligned}
&\left|V_qT(f\mathbb{1}_{\mathcal{X}\setminus\alpha B})(x') - V_qT(f\mathbb{1}_{\mathcal{X}\setminus\alpha B})(x'')\right| \\
&= \sup_{t_j} \left\|\mathcal{K}_{(t_j,t_{j+1}]}(f\mathbb{1}_{\mathcal{X}\setminus\alpha B})(x') - \mathcal{K}_{(t_j,t_{j+1}]}(f\mathbb{1}_{\mathcal{X}\setminus\alpha B})(x'')\right\|_{\ell^q} \\
&= \sup_{t_j} \left\|\mathcal{K}_{I_j}(f\mathbb{1}_{\mathcal{X}\setminus\alpha B})(x') - \mathcal{K}_{I_j}(f\mathbb{1}_{\mathcal{X}\setminus\alpha B})(x'')\right\|_{\ell^q},
\end{aligned}$$

where the supremum runs over all sequences $\{t_j\}$ with $0 < t_0 \leq t_1 \leq t_2 \ldots < \infty$, and $I_j = (t_j,\, t_{j+1}]$. We then have

$$\begin{aligned}
&\mathcal{K}_{I_j}(f\mathbb{1}_{\mathcal{X}\setminus\alpha B})(x') - \mathcal{K}_{I_j}(f\mathbb{1}_{\mathcal{X}\setminus\alpha B})(x'') \\
&= \int_{\mathcal{X}\setminus\alpha B} \left[K(x',y)f(y)\mathbb{1}_{R_{I_j}(x')}(y) - K(x'',y)f(y)\mathbb{1}_{R_{I_j}(x'')}(y)\right]f(y)d\mu(y) \\
&= \int_{\mathcal{X}\setminus\alpha B} \left[(K(x',y) - K(x'',y)))\mathbb{1}_{R_{I_j}(x')}(y) - K(x'',y)(\mathbb{1}_{R_{I_j}(x'')} - \mathbb{1}_{R_{I_j}(x')})(y)\right]f(y)d\mu(y) \\
&= \int_{\mathcal{X}\setminus\alpha B} \left[(K(x',y) - K(x'',y))f(y)\mathbb{1}_{R_{I_j}(x')}(y)d\mu(y)\right. \\
&\quad - \int_{\mathcal{X}\setminus\alpha B} K(x'',y)f(y)(\mathbb{1}_{R_{I_j}(x'')} - \mathbb{1}_{R_{I_j}(x')})(y)]d\mu(y) =: \Gamma_j + \Delta_j.
\end{aligned}$$

Since $q \geq 1$ and noticing $d(y, x') \geq 2d(x', x'')$, we have

$$\begin{aligned}
\Big(\sum_{j=0}^{\infty} |\Gamma_j|^q\Big)^{1/q} &\leq \sum_{j=0}^{\infty} |\Gamma_j| \\
&\leq \sum_{j=0}^{\infty} c \int_{\mathcal{X}\setminus\alpha B} |(K(x', y) - K(x'', y)| \, |f(y)| \mathbb{1}_{R_{I_j}(x')}(y) d\mu(y) \\
&\leq \int_{d(x',y)>2c_d r} \left(\frac{d(x', x'')}{d(x', y)}\right)^s \frac{1}{V(x', y)} |f(y)| d\mu(y),
\end{aligned}$$

Compared with the situation in (3.5), the following estimate is immediately obtained

$$\Big(\sum_{j=0}^{\infty} |\Gamma_j|^q\Big)^{1/q} \lesssim M(f)(x). \tag{5.1}$$

Now we turn to the part Δ_j. Then we still consider the partition J_1 and J_2 which have been previously introduced in Lemma 4.2. For every $j \in J_1$, we have

$$\Big|\mathbb{1}_{R_{I_j}(x'')} - \mathbb{1}_{R_{I_j}(x')}\Big| \leq \mathbb{1}_{R_{I_j}(x'')} + \mathbb{1}_{R_{I_j}(x')}.$$

Thus using the size condition of K and the D-regular property of $\mathcal{X}$, we get

$$\begin{aligned}
&\left|\int_{\mathcal{X}\setminus\alpha B} K(x'', y) f(y) (\mathbb{1}_{R_{I_j}(x'')} - \mathbb{1}_{R_{I_j}(x')})(y)] d\mu(y)\right| \\
&\lesssim \int_{\mathcal{X}\setminus\alpha B} \frac{1}{d(y, x'')^D} |f(y)| (\mathbb{1}_{R_{I_j}(x'')}(y)| + |\mathbb{1}_{R_{I_j}(x')}(y)) d\mu(y) \\
&\lesssim \frac{1}{t_{j+1}^D} \int_{\mathcal{X}\setminus\alpha B} |f(y)| \mathbb{1}_{R_{I_j}(x'')}(y) d\mu(y) + \frac{1}{t_{j+1}^D} \int_{\mathcal{X}\setminus\alpha B} |f(y)| \mathbb{1}_{R_{I_j}(x')}(y) d\mu(y).
\end{aligned} \tag{5.2}$$

We only need to estimate the first integral on the right of the last inequality. Thus

$$\begin{aligned}
&\frac{1}{t_{j+1}^D} \int_{\mathcal{X}\setminus\alpha B} |f(y)| \mathbb{1}_{R_{I_j}(x'')}(y) d\mu(y) \\
&= \frac{1}{t_{j+1}^D} \int_{\mathcal{X}\setminus\alpha B} |f(y)| (\mathbb{1}_{B_{t_{j+1}}(x'')} - \mathbb{1}_{B_{t_j}(x'')})(y) d\mu(y) \\
&\lesssim \left(\frac{1}{t_{j+1}^D} \int_{\mathcal{X}\setminus\alpha B} |f(y)| \mathbb{1}_{B_{t_{j+1}}(x'')}(y) d\mu(y) - \frac{1}{t_j^D} \int_{\mathcal{X}\setminus\alpha B} |f(y)| \mathbb{1}_{B_{t_j}(x'')}(y) d\mu(y)\right) \\
&\quad + \left(\frac{1}{t_j^D} - \frac{1}{t_{j+1}^D}\right) \int_{\mathcal{X}\setminus\alpha B} |f(y)| \mathbb{1}_{B_{t_j}(x'')}(y) d\mu(y).
\end{aligned} \tag{5.3}$$

For $j \in J_2$, using some arguments similar to (4.6), we eventually obtain that

$$\begin{aligned}(\sum_j |\Delta_j|^q)^{1/q} &\lesssim \Big(\sum_j \Big|\frac{1}{t_{j+1}^D}\int_{\mathcal{X}\setminus\alpha B}|f(y)|\mathbb{1}_{B_{t_{j+1}}(x'')}(y)d\mu(y) \\ &\quad -\frac{1}{t_j^D}\int_{\mathcal{X}\setminus\alpha B}|f(y)|\mathbb{1}_{B_{t_j}(x'')}(y)d\mu(y)\Big|^q\Big)^{1/q} \\ &\quad +\Big(\sum_j \big(\frac{1}{t_j^D}-\frac{1}{t_{j+1}^D}\big)^q\Big|\int_{\mathcal{X}\setminus\alpha B}|f(y)|\mathbb{1}_{B_{t_j}(x'')}(y)d\mu(y)\Big|^q\Big)^{1/q} \\ &\lesssim V_qA(f)(x)+M(f)(x), \end{aligned} \tag{5.4}$$

which implies the desired conclusion of Lemma 5.1 by the boundedness of V_qA and M. Hence we complete the proof.

6 Applications

In this section, we discuss some applications of sharp weighted estimates on the spaces of homogeneous type in the study of harmonic analysis. Lorist in [2] has pointed out that many operators are proven to be vector-valued Calderón–Zygmund operators on concrete spaces of homogeneous type. Sharp weighted bounds are then often concluded using [29,Theorem III.1.3] or [30]. With Theorem 3.3, these kinds of results can be extended to the sharp A_p-characteristic.

Stinga and Torrea [31] first used the method of semigroups associated to the parabolic operator $L = \partial_t - \Delta$ in order to develop the regularity theory for solutions of space-time nonlocal equations. They developed the vector-valued Calderón-Zygmund theorem related to the fractional Poisson semigroups associated with this parabolic operator on the spaces $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ with the parabolic distance given by

$$d((x,t),(y,s)) = |x-y| + |t-s|^{1/2}, \qquad \text{for } (x,t),(y,s) \in \mathbb{R}^n \times \mathbb{R},$$

where the $|\cdot|$ denotes the Euclidean distance. Then $\mathbb{R}^{n+1}$ with the topology generated by the distance d and the compatible Borel measure forms a space of homogeneous type. Then the sharp A_p weighted bounds can be used to study the regularity theory for this parabolic operator.

Theorem 1.4 and 1.5 show that we can deduce the sparse domination and sharp weighted bounds of variations of Calderón–Zygmund operators whence we have the weak type $(1,1)$ bounds for these Calderón–Zygmund operators. In fact, the bounded have been extensively studied in harmonic analysis (cf. e.g. [32,18,33,7,19,20,34]). Then by the boundedness of variations of the related Calderón–Zygmund operators presented in these papers also imply the sparse domination of variations of these Calderón–Zygmund operators by Theorem 1.4 and 1.5.

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奇异算子Q-变差在齐型空间上的最优加权估计

龚晨茜

(武汉大学数学与统计学院, 湖北 武汉 430072)

摘 要: 我们将齐型空间上的A_p理论的最优权有界性推广到了平均算子和Calderón–Zygmund算子的q变差. 这些结果利用了Lorist和Omisboand 在齐型空间上给出的新的稀疏控制技术[1]以及[2]. 最后我们还讨论了这些理论的应用.

关 键 词: 最优权估计; q变差不等式; Calderón–Zygmund 算子; 稀疏算子; 齐型空间

MR(2010)**主 题 分 类 号:** 42B20; 42B25 **中 图 分 类 号:** O174.2