Vol. 43 (2023)
No. 3

数 学 杂 志
J. of Math. (PRC)

ENDPOINT WEAK-TYPE ESTIMATES OF BI-PARAMETER SINGULAR INTEGRAL OPERATORS ON MIXED LEBESGUE SPACES

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Abstract: In this paper, the boundedness of the bi-parameter singular integral operators on mixed Lebesgue spaces is studied. Using the boundedness of the bi-parameter singular integral operators on Lebesgue spaces and a vector-valued extension theory. We obtain the endpoint weak-type estimates and strong-type estimates for a bi-parameter singular integral operators on mixed Lebesgue spaces. An application to a non-convolution singular integral operators on product spaces is also given. These results extend the conclusion of [3] to the mixed norm case.

Keywords: bi-parameter singular integrals; mixed norms; endpoint weak-type estimates
2010 MR Subject Classification: 42B20
Document code: **A** **Article ID:** **0255-7797(2023)03-0202-11**

1 Introduction

A bi-parameter singular integral operator is an operator on product spaces $\mathbb{R}^n \times \mathbb{R}^m$. An example of a bi-parameter singular integral operator is defined by $T_n \otimes T_m$, where T_n and T_m are linear singular integral operators on $\mathbb{R}^n$ and $\mathbb{R}^m$. $T_n \otimes T_m(f \otimes g)(x) = T_n(f(x_1))T_m(g(x_2))$ where $x = (x_1 \in \mathbb{R}^n, x_2 \in \mathbb{R}^m)$. Then by Fubini theorem, the bi-parameter singular integral operator $T_n \otimes T_m$ is bounded on L^p. But for a non-tensor form bi-parameter singular integral operator, it is not an easy iteration argument. So many works have been done for non-tensor form bi-parameter singular integral operators. In 1982, Fefferman-stein [1] studied the convolution form singular integral on product spaces. Journé[2] dealt with the general kernels in 1985. Rencently, Pott-Villarroya[3] gave the T1 theorem of the bi-parameter singular integral operators. Later, the representation theorem and non-homogeneous T1 theorem were studied by Martikainen[4] and Hytönen-Martikainen[5]. Ou[6] proved the Tb theorem and Li-Martikainen-Vuorinen[7] studied the Bloom type inequality for bi-parameter singular integral.

The purpose of this paper is to study such operators on mixed Lebesgue spaces. The following is the definition of the mixed Lebesgue spaces.

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* **Received date:** 2022-03-25 **Accepted date:** 2022-10-10

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Definition 1.1 Let $\overrightarrow{p} := (p_1, ..., p_n) \in (0, \infty]^n$. The mixed Lebesgue space $L^{\overrightarrow{p}}(\mathbb{R}^n)$ is defined to be the set of all measure function f such that their quasi-norms

$$\|f\|_{L^{\overrightarrow{p}}(\mathbb{R}^n)} := \{\int_{\mathbb{R}} ...[\int_{\mathbb{R}} |f(x_1, ...x_n)|^{p_1} dx_1]^{\frac{p_2}{p_1}} ...dx_n\}^{\frac{1}{pn}} < \infty$$

with the usual modifications made when $p_i = \infty$ for some $i \in \{1, ..., n\}$.

Benedek and Panzone in[8] first studied the Lebesgue space with mixed norm and proved that such spaces have similar properties as ordinary Lebesgue spaces, Related works refer to [9–12].

In this paper, we consider bi-parameter singular integral operators on mixed Lebesgue spaces. We prove the endpoint weak-type estimates for the bi-parameter singular integral operators on mixed Lebesgue spaces and also obtain the boundedness of these operators on $L^p(L^q)$.

This paper is organized as follows. We introduce some basic definitions in sect.2. And we collect some preliminary results in sect.3. Then in Sect.4, we give the main theorem and proof. And in sect.5 we apply this result to a non-convolution operator.

2 Definitions

In this section, we will introduce the definition of the bi-parameter singular integral operators as stated in Martikainen[4].

Definition 2.1 Let V be a cube in $\mathbb{R}^n$ (or $\mathbb{R}^m$). We say that a function u_v is V-adapted with zero mean in $\mathbb{R}^n$ (or $\mathbb{R}^m$) if it satisfies that $spt(u_v) \subset V$, $|u_v| \le 1$ and $\int u_v = 0$.

Definition 2.2 We say that the bi-parameter operator T has full kernel representation with kernel K if following holds. If $f = f_1 \otimes f_2$ and $g = g_1 \otimes g_2$ with $f_1, g_1 : \mathbb{R}^n \to \mathbb{C}$, $f_2, g_2 : \mathbb{R}^m \to \mathbb{C}$, $spt f_1 \cap spt g_1 = \emptyset$ and $spt f_2 \cap spt g_2 = \emptyset$, we have the kernel representation

$$\langle Tf, g\rangle = \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^{n+m}} K(x, y) f(y) g(x) dy dx,$$

where the kernel is a function

$$K : (\mathbb{R}^{n+m} \times \mathbb{R}^{n+m}) \setminus \{(x_1, x_2; y_1, y_2) \in \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} : x_1 = x_2 \quad or \quad y_1 = y_2\} \to \mathbb{C}.$$

Note that this implies full kernel representation for T^*, $\widetilde{T}$ and $\widetilde{T}^*$ where T^*, $\widetilde{T}$ and $\widetilde{T}^*$ are bi-parameter operators with kernel K^*, $\widetilde{K}$ and $\widetilde{K}^*$ respectively and

$$K^*(x, y) = K(y_1, y_2; x_1, x_2), \widetilde{K}(x, y) = K(y_1, x_2; x_1, y_2), \widetilde{K}^*(x, y) = K(x_1, y_2; y_1, x_2).$$

Definition 2.3 The kernel K satisfied the full standard estimates if the following holds. We have the size condition

$$|K(x, y)| \le C \frac{1}{|x_1 - y_1|^n} \frac{1}{|x_2 - y_2|^m} \tag{2.1}$$

the Hölder condition

$$|K(x,y)-K(x,(y_1,y_2'))-K(x,(y_1',y_2))+K(x,y')|\le C\frac{|y_1-y_1'|^\delta}{|x_1-y_1|^{n+\delta}}\frac{|y_2-y_2'|^\delta}{|x_2-y_2|^{m+\delta}} \tag{2.2}$$

whenever $2|y_1-y_1'|\le|x_1-y_1|$ and $2|y_2-y_2'|\le|x_2-y_2|$, and the mixed Hölder condition and size condition

$$|K(x,y)-K(x,(y_1,y_2'))|\le C\frac{|y_1-y_1'|^\delta}{|x_1-y_1|^{n+\delta}}\frac{1}{|x_2-y_2|^{m}} \tag{2.3}$$

whenever $2|y_1-y_1'|\le|x_1-y_1|$. The same condition are imposed on K^*, $\widetilde{K}$ and $\widetilde{K}^*$.

Definition 2.4 We say that the bi-parameter operator T has partial kernel representations if the following holds. If $f=f_1\otimes f_2$ and $g=g_1\otimes g_2$ with $sptf_1\cap sptg_1=\emptyset$, we have

$$\langle Tf,g\rangle=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}K^2_{f_2,g_2}(x_1,y_1)f_1(y_1)g_1(x_1)dy_1dx_1. \tag{2.4}$$

Here $K^2_{f_2,g_2}:(\mathbb{R}^n\times\mathbb{R}^n)\setminus\{(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n: x_1=x_2\}\to\mathbb{C}$. Moreover, we assume that $K^2_{f_2,g_2}$ satisfied the standard one-parameter kernel estimates and $C(f_2,g_2)$ is the minimal constant (depending on f_2 and g_2) with which those estimates are satisfied. This constant is assumed to satisfy

$$C(\chi_V,\chi_V)+C(\chi_V,g_V)+C(g_V,\chi_V)\le C|V| \tag{2.5}$$

whenever $V\subset\mathbb{R}^m$ is a cube and g_V is a V-adapted function with zero mean in $\mathbb{R}^m$.

We assume the analogous representation and properties with a kernel $K^1_{f_1,g_1}$ whenever $sptf_2\cap sptg_2=\emptyset$.

Definition 2.5 We say that a bi-parameter operator T is a bi-parameter SIO in the sense of Martikainen[4] if the following holds:

- The operator T has a full kernel representation with a kernel satisfying the full standard estimates.
- The operator T has a partial kernel representation.

3 Preliminaries

In this section, we collect some preliminary results which are used in the proof.

The boundedness of the non-convolution singular integral operators is well known. The proof of the proposition refers to [13].

Definitions 3.1 We say that $K:\mathbb{R}^n\times\mathbb{R}^n\setminus\Delta\to\mathbb{C}$ is a standard kernel if there exists $\delta>0$ such that

$$|K(x,y)|\le\frac{A}{|x-y|^n}, \tag{3.1}$$

$$|K(x,y)-K(x,y')|\le A\frac{|y-y'|^\delta}{|x-y|^{n+\delta}}\quad if\quad |x-y|>2|y-y'|, \tag{3.2}$$

$$|K(x,y)-K(x',y)|\leq A\frac{|x-x'|^{\delta}}{|x-y|^{n+\delta}} \quad if \quad |x-y|>2|x-x'|. \tag{3.3}$$

The class of all standard kernels with constants δ, A is denoted by $SK(\delta, A)$.

Definitions 3.2 If T is associated with $K\in SK(\delta,A)$ and admits a bounded extension on $L^2(\mathbb{R}^n)$, that is, it satisfies

$$\|Tf\|_{L^2}\leq B\|f\|_{L^2},$$

$$Tf(x)=\int_{\mathbb{R}^n}K(x,y)f(y)dy, \quad x\notin spt(f)$$

for $f\in L^2$ with compact suppport, then T is called a Calderón-Zygmund operator associated with the standard kernel k. We denote by $CZO(\delta,A,B)$ the class of all Calderòn-Zygmund operator associated with the standard kernel in $SK(\delta,A)$.

Proposition 3.3 Assume that $K(x,y)$ is in $SK(\delta,A)$ and let T be an element of $CZO(\delta,A,B)$ assosiated with the kernel K. Then T has a bounded extension that maps $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ with norm

$$\|T\|_{L^1\to L^{1,\infty}}\leq C_n(A+B)$$

and also maps $L^p(\mathbb{R}^n)$ to itself for $1<p<\infty$ with norm

$$\|T\|_{L^p\to L^p}\leq C_n\max(p,(p-1)^{-1})(A+B)$$

where C_n is a dimensional constant.

Next we want to prove the boundedness of the bi-parameter singular integral operators under the weak boundedness condition and the cancellation conditions on L^p, $1<p<\infty$.

Proposition 3.4[14] If T is a bi-parameter SIO in the sense of Martikainen/Pott-Villarroya, then T is a bi-parameter SIO in the sense of Journé[6]. The converse statement is clear.

In 2011, Pott and Villarroya[3] proved the L^2 boundedness of the bi-parameter singular integral operators and extended to L^p spaces under the special cancellation hypotheses.

Proposition 3.5[3] (L^p boundedness) Let Λ be a bilinear Calderón-Zygmund form satisfying the mixed WB-CZ conditions. We also assume that Λ satisfies the weak boundedness condition, and the special cancellation conditions:

(a) $T(1), T^*(1), T_1(1), T_1^*\in BMO(\mathbb{R}^n\times\mathbb{R}^m)$;

(b) $\langle T(\phi_I\otimes 1),\varphi_I\otimes\cdot\rangle, \langle T(1\otimes\phi_I),\cdot\otimes\varphi_I\rangle, \langle T^*(\phi_I\otimes 1),\varphi_I\otimes\cdot\rangle, \langle T^*(1\otimes\phi_I),\cdot\otimes\varphi_I\rangle\in BMO$ on $\mathbb{R}^n$ or $\mathbb{R}^m$ for all ϕ_I, φ_I bump functions adapted to I with norms uniformly bounded in I.

Then Λ and Λ_i for $i=1,2$ are bounded bilinear forms on L^p.

Remark 3.6 See [3, Definition 2.1-2.8]. A bilinear Calderón-Zygmund form Λ is a bilinear form associated with a product Calderón-Zygmund kernel and has some integral representations, $\Lambda(f,g)=\langle T(f),g\rangle=\langle f,T^*(g)\rangle$, $\Lambda_1(f,g)=\Lambda(g_1\otimes f_2,f_1\otimes g_2)$, $\Lambda_2(f,g)=\Lambda(f_1\otimes g_2,g_1\otimes f_2)$ and $\Lambda_i(f,g)=\langle T_i(f),g\rangle=\langle f,T_i^*(g)\rangle$, $i=1,2$. We need to point out that T is a bi-parameter SIO, if a bilinear Calderón-Zygmund form Λ satisfies the mixed WB-CZ conditions.

Theorem 3.7 A bi-parameter singular integral operator T in the sense of Martikainen satisfies weak boundedness condition and the cancellation conditions. Then T is L^p-bounded.

Proof From Proposition 3.5, the bi-parameter singular integral operator T in the sense of Pott-Villarroya is L^p bounded. By Proposition 3.4, the bi-parameter singular integral operators in the sense of Pott-Villarroya and Martikainen are equal. So the bi-parameter singular integral operator T in the sense of Martikainen under the weak boundedness condition and the cancellation conditions is L^p bounded.

In order to complete the proof of the main theorem, we need some Lemmas.

Lemma 3.8 [14] Let T be a bi-parameter operator that has partial kernel representation with the kernel $K^i_{f_i,g_i}$ for $i=1,2$ as defined in (2.4) and V a cube in $\mathbb{R}^m$(resp. in $\mathbb{R}^n$)

$$C(\chi_V, g_V) + C(g_V, \chi_V) \leq C\|g_V\|_\infty |V| \tag{3.4}$$

whenever $g_V \in L_\infty(V)$.

Definition 3.9 Let $\Delta := \{(x,x) : x \in \mathbb{R}^n\}$ denotes the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$ and $T : \mathcal{C}_0^\infty(\mathbb{R}^n) \to [\mathcal{C}_0^\infty(\mathbb{R}^n)]'$ be a continuous linear mapping . For each $\delta \in (0,1)$, the operator T is called a *δ-SIO* if there exists a kernel K on $(\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta$ and a positive constant C such that, for all $x, y, z \in \mathbb{R}^n$,

$$|K(x,y)| \leq \frac{C}{|x-y|^n} \quad if x \neq y,$$

$$|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \leq \frac{|y-z|^\delta}{|x-y|^{n+\delta}} \quad if |x-y| > 2|y-z|,$$

for any $f, g \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ having disjoints supports,

$$\langle g, Tf \rangle = \int\int g(x)K(x,y)f(y)dy.$$

Lemma 3.10 [14] Let T be a δ-SIO on $\mathbb{R}^n$ and K its kernel. If there exists a constant $A > 0$ such that for every cube $V \subset \mathbb{R}^n$

$$\|T\chi_V\|_{L^1(V)} \leq A|V| \tag{3.5}$$

and

$$\|T^*\chi_V\|_{L^1(V)} \leq A|V|. \tag{3.6}$$

Then T is a bounded operator on L^2 such that $\|T\|_{2\to 2} \leq C_{\delta,n}(A + |K|_\delta)$.

4 Main Theorem

A vector-valued extension of the theory is well known, we will need the following version of the original result of Benedek, Calderón, and Panzone[9]. The proof of this theory refers to [15].

Let A and B be two Banach spaces and let $\mathcal{L}(A,B)$ be the space of bounded linear operators from A to B. Suppose that K is a function defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ which takes values in $\mathcal{L}(A,B)$ and T is an operator which has K as its associated kernel: if $f \in L^\infty(A)$ and has compact support, then

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \quad x \notin spt(f).$$

We have the following result,

Proposition 4.1 [16] Let T be a bounded operator from $L^r(A)$ to $L^r(B)$ for some r, $1 < r < \infty$, with assosiated kernel K. If K satisfies

$$\int_{|x-y|>2|y-z|} \|K(x,y) - K(x,z)\|_{\mathcal{L}(A,B)}dx \le C, \tag{4.1}$$

$$\int_{|x-y|>2|x-w|} \|K(x,y) - K(w,y)\|_{\mathcal{L}(A,B)}dx \le C, \tag{4.2}$$

then T is bounded from $L^p(A)$ to $L^p(B)$, $1 < p < \infty$, and is weak(1,1), that is

$$|\{x \in \mathbb{R}^n : \|Tf(x)\|_B > \lambda\}| \le \frac{C}{\lambda}\|f\|_A.$$

Next we give the main theorem.

Theorem 4.2 A bi-parameter singular integral operator T satisfies the weak bounded conditions and cancellation conditions, then T extends as a bounded operator on $L^p(L^q)(\mathbb{R}^n \times \mathbb{R}^m)$ for all $1 < p, q < \infty$ and also from $L^1(L^q)(\mathbb{R}^n \times \mathbb{R}^m)$ into $L^{1,\infty}(L^q)(\mathbb{R}^n \times \mathbb{R}^m)$ for all $1 < q < \infty$, in the sense that

$$|\{x \in \mathbb{R}^n : \|Tf\|_{L^q(\mathbb{R}^m)} > \lambda\}| \le \frac{A}{\lambda}\|f\|_{L^1(L^q)(\mathbb{R}^n\times\mathbb{R}^m)} \quad \text{for all } \lambda > 0.$$

Proof Let

$$\begin{aligned}\langle Tf, g\rangle &= \int_{\mathbb{R}^{n+m}}\int_{\mathbb{R}^{n+m}} K(x,y)f(y)g(x)dydx\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K^2_{f_2,g_2}(x_1,y_1)f_1(y_1)g_1(x_1)dy_1dx_1\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \langle K_1(x_1,y_1)f_2, g_2\rangle f_1(y_1)g_1(x_1)dy_1dx_1\\ &= \langle \widehat{T}f_1, g_1\rangle,\end{aligned}$$

that is $\widehat{T}$ is a vector-valued operator with kernel K_1.

By the definition of mixed norm, we have $L^p(\mathbb{R}^n, L^p(\mathbb{R}^m)) = L^p(\mathbb{R}^n \times \mathbb{R}^m)$. And by Theorem 3.7, T is a bounded operator on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$, thus $\widehat{T}$ is a vector-valued operator on $L^p(\mathbb{R}^n, B)$ with $B = L^p(\mathbb{R}^m)$. From proposition 4.1, it is enough to prove

$$\int_{|x_1-y_1|>2|y_1-y_1'|} \|K_1(x_1,y_1) - K_1(x_1,y_1')\|_{L^q(\mathbb{R}^m)}dx \le C, \tag{4.3}$$

and

$$\int_{|x_1-y_1|>2|x_1-x_1'|}\|K_1(x_1,y_1)-K_1(x_1',y_1)\|_{L^q(\mathbb{R}^m)}dx \leq C. \tag{4.4}$$

For $|x_1-y_1|>2|y_1-y_1'|$ and $|x_1-y_1|>2|x_1-x_1'|$, we consider the operators

$$\begin{aligned}L_1(x_1,y_1)f_2 &= (K_1(x_1,y_1)-K_1(x_1,y_1'))f_2\\ &= \int_{\mathbb{R}^m}(K(x_1,x_2;y_1,y_2)-K(x_1,x_2;y_1',y_2))f_2(y_2)dy_2\end{aligned}$$

$$\begin{aligned}L_2(x_1,y_1)f_2 &= ((K_1(x_1,y_1)-K_1(x_1',y_1))f_2\\ &= \int_{\mathbb{R}^m}(K(x_1,x_2;y_1,y_2)-K(x_1',x_2;y_1,y_2))f_2(y_2)dy_2,\end{aligned}$$

and let

$$l_1(x_1,y_1)(x_2,y_2)=K(x_1,x_2;y_1,y_2)-K(x_1,x_2;y_1',y_2)$$

$$l_2(x_1,y_1)(x_2,y_2)=K(x_1,x_2;y_1,y_2)-K(x_1',x_2;y_1,y_2).$$

Next we only need to prove the boundedness of the operators L_1, L_2 with the kernels l_1, l_2 respectively on $L^q(\mathbb{R}^m)$.

By proposition 3.3, it is enough to prove that l_1, l_2 satisfy the standard estimates and L_1, L_2 are L^2-bounded. By mixed Hölder and size condition (2.3), we have

$$\begin{aligned}l_1(x_1,y_1)(x_2,y_2) &= K(x_1,x_2;y_1,y_2)-K(x_1,x_2;y_1',y_2)\\ &\leq \frac{C|y_1-y_1'|^\delta}{|x_1-y_1|^{n+\delta}}\frac{1}{|x_2-y_2|^m}\end{aligned}$$

for $|x_1-y_1|>2|y_1-y_1'|$, and

$$l_2(x_1,y_1)(x_2,y_2)\leq \frac{C|x_1-x_1'|^\delta}{|x_1-y_1|^{n+\delta}}\frac{1}{|x_2-y_2|^m}$$

for $|x_1-y_1|>2|x_1-x_1'|$.

By Hölder condition (2.2),

$$\begin{aligned}l_1(x_1,y_1)(x_2,y_2)-l_1(x_1,y_1)(x_2,y_2') &= K(x_1,x_2;y_1,y_2)-K(x_1,x_2;y_1',y_2)\\ &\quad -K(x_1,x_2;y_1,y_2')+K(x_1,x_2;y_1',y_2')\\ &\leq C\frac{|y_1-y_1'|^\delta}{|x_1-y_1|^{n+\delta}}\frac{|y_2-y_2'|^\delta}{|x_2-y_2|^{m+\delta}}\end{aligned}$$

for $|x_2-y_2|>2|y_2-y_2'|$ and $|x_1-y_1|>2|y_1-y_1'|$.

$$\begin{aligned}l_1(x_1,y_1)(x_2,y_2)-l_1(x_1,y_1)(x_2',y_2) &= K(x_1,x_2;y_1,y_2)-K(x_1,x_2;y_1',y_2)\\ &\quad -K(x_1,x_2';y_1,y_2)+K(x_1,x_2';y_1,y_2')\\ &\leq C\frac{|y_1-y_1'|^\delta}{|x_1-y_1|^{n+\delta}}\frac{|x_2-x_2'|^\delta}{|x_2-y_2|^{m+\delta}}\end{aligned}$$

for $|x_2-y_2|>2|x_2-x_2'|$ and $|x_1-y_1|>2|y_1-y_1'|$. And

$$l_2(x_1,y_1)(x_2,y_2)-l_2(x_1,y_1)(x_2,y_2')\le C\frac{|x_1-x_1'|^\delta}{|x_1-y_1|^{n+\delta}}\frac{|y_2-y_2'|^\delta}{|x_2-y_2|^{m+\delta}}$$

for $|x_2-y_2|>2|y_2-y_2'|$ and $|x_1-y_1|>2|x_1-x_1'|$.

$$l_2(x_1,y_1)(x_2,y_2)-l_2(x_1,y_1)(x_2',y_2)\le C\frac{|x_1-x_1'|^\delta}{|x_1-y_1|^{n+\delta}}\frac{|x_2-x_2'|^\delta}{|x_2-y_2|^{m+\delta}}$$

for $|x_2-y_2|>2|x_2-x_2'|$ and $|x_1-y_1|>2|x_1-x_1'|$. So l_1 and l_2 satisfy the standard estimates on $\mathbb{R}^m$.

In order to obtain the L^2-bounded, it is enough to verify the condition of the Lemma 3.10. Let $V\subset\mathbb{R}^m$ be a cube and $g:\mathbb{R}^m\to\mathbb{C}$ be such that $spt(g)\subset V$ and $|g|\le 1$.

By Lemma 3.8, it holds that

$$\begin{aligned}|\langle L_1(x_1,y_1)\chi_V,g_V\rangle|+|\langle g_V,L_1(x_1,y_1)\chi_V\rangle| &\le |K^2_{\chi_V,g_V}(x_1,y_1)-K^2_{\chi_V,g_V}(x_1,y_1')|\\ &\quad+|K^2_{g_V,\chi_V}(x_1,y_1)-K^2_{g_V,\chi_V}(x_1,y_1')|\\ &\le C\frac{|y_1-y_1'|^\delta}{|x_1-y_1|^{n+\delta}}|V|.\end{aligned}$$

Then $L_1(x_1,y_1)$ is L^2-bounded. Similarly, $L_2(x_1,y_1)$ is also L^2-bounded. Therefore, the operators $L_1(x_1,y_1)$ and $L_2(x_1,y_1)$ with operator norms are bounded by$AC\frac{|y_1-y_1'|^\delta}{|x_1-y_1|^{n+\delta}}$ and $AC\frac{|x_1-x_1'|^\delta}{|x_1-y_1|^{n+\delta}}$. The operator norms of $L_1(x_1,y_1)$ and $L_2(x_1,y_1)$ from $L^q(\mathbb{R}^m)$ to $L^q(\mathbb{R}^m)$ are

$$\begin{aligned}\|L_1(x_1,y_1)\|_{L^q\to L^q} &= \sup_{\|g(y)\|_{L^q(\mathbb{R}^m)}}\frac{\|(K_1(x_1,y_1)-K_1(x_1,y_1'))g(y)\|_{L^q(\mathbb{R}^m)}}{\|g(y)\|_{L^q(\mathbb{R}^m)}}\\ &\le AC\frac{|y_1-y_1'|^\delta}{|x_1-y_1|^{n+\delta}},\end{aligned}$$

and

$$\|L_2(x_1,y_1)\|_{L^q\to L^q}\le AC\frac{|x_1-x_1'|^\delta}{|x_1-y_1|^{n+\delta}}.$$

The estimates (4.3) and (4.4) then hold.

Then by proposition 4.1 the operator $\widehat{T}$ is a bounded operator on $L^p(\mathbb{R}^n,L^q(\mathbb{R}^m))$, and from $L^1(\mathbb{R}^n,L^q(\mathbb{R}^m))$ to $L^{1,\infty}(\mathbb{R}^n,L^q(\mathbb{R}^m))$, T is a bounded operator on $L^p(L^q)(\mathbb{R}^n\times\mathbb{R}^m)$, and from $L^1(L^q)(\mathbb{R}^n\times\mathbb{R}^m)$ to $L^{1,\infty}(L^q)(\mathbb{R}^n\times\mathbb{R}^m)$.

5 Application

In 1982, Fefferman[1] gave a singular integral on product space which kernel satisfied "cancellation" and "size" properties. Such kernels can be extended to the case of non-convolution kernels.

Definition 5.1 Let $k((x,y),(s,t))$ is integrable on $\mathbb{R}^n\times\mathbb{R}^m$, $(x,y),(s,t)\in\mathbb{R}^n\times\mathbb{R}^m$.

1. the kernel condition:

(a) $|k((x,y),(s,t))| \le A|x-s|^{-n}|y-t|^{-m}$.

(b) $|k((x',y),(s',t)) - k((x,y),(s,t))| \le A(|x-x'|+|s-s'|)^{\eta_1}|x-s|^{-n-\eta_1}|y-t|^{-m}$ whenever $|x-s| \ge 2(|x-x'|+|s-s'|)$.

(c) $|k((x,y'),(s,t')) - k((x,y),(s,t))| \le A(|y-y'|+|t-t'|)^{\eta_2}|x-s|^{-n}|y-t|^{-m-\eta_1}$ whenever $|y-t| \ge 2(|y-y'|+|t-t'|)$.

(d) $|k((x',y'),(s',t')) - k((x',y),(s',t)) - k((x,y'),(s,t')) + k((x,y),(s,t))| \le A(|x-x'|+|s-s'|)^{\eta_1}|x-s|^{-n-\eta_1}(|y-y'|+|t-t'|)^{\eta_2}|y-t|^{-m-\eta_1}$ whenever $|x-s| \ge 2(|x-x'|+|s-s'|)$ and $|y-t| \ge 2(|y-y'|+|t-t'|)$.

2. the cancellation condition:$|\int\int_{\alpha_1<|x-s|<\alpha_2,\beta_1<|y-t|<\beta_2} k((x,y),(s,t))dxdy| \le A$.

3. the mixed kernel-cancellation condition:

(a) if $k_1(x,s) = \int_{\beta_1<|y-t|<\beta_2} k((x,y),(s,t))dy$ then

  (i) $|k_1(x,s)| \le A|x-s|^{-n}$;

  (ii) $|k_1((x',s') - k_1((x,s)| \le A(|x-x'|+|s-s'|)^{\eta_1}|x-s|^{-n-\eta_1}$ whenever $|x-s| \ge 2(|x-x'|+|s-s'|)$.

(b) similar condition for $k_2(y,t) = \int_{\alpha_1<|x-s|<\alpha_2} k((x,y),(s,t))dx$.

Proposition 5.2 [1] Suppose k is integrable on $\mathbb{R}^n \times \mathbb{R}^m$ and satisfies all the conditions of Definition 5.1.Then

$$\|f * k\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \le A_p\|f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}$$

where A_p depending only on A and p.

Theorem 5.3 Let T be an operator associated with k satisfying all the conditions of Definition 5.1. Then T extends as bounded operators on $L^p(L^q)(\mathbb{R}^n\times\mathbb{R}^m)$ for all $1 < p, q < \infty$ and also from $L^1(L^q)(\mathbb{R}^n \times \mathbb{R}^m)$ into $L^{1,\infty}(L^q)(\mathbb{R}^n \times \mathbb{R}^m)$ for all $1 < q < \infty$, in the sense that

$$|\{x \in \mathbb{R}^n : \|Tf\|_{L^q(\mathbb{R}^m)} > \lambda\}| \le \frac{A}{\lambda}\|f\|_{L^1(L^q)(\mathbb{R}^n\times\mathbb{R}^m)} \quad \text{for all } \lambda > 0.$$

Proof By Theorem 4.2, we only need to prove the kernel K is a bi-parameter SIO, and the kernel satisfies the boundedness and cancellation assumption.

Obviously, T has a full kernel representation and a partial kernel representation , K satisfies the full standard estimates.

By Definition 5.1 (1)-(a) and (3)-(a)-(i), let $c(V)$ be a center of V, we have

$$\begin{aligned}
|k^2_{\chi_V,\chi_V}| &= \Big|\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} k((x,y),(s,t))\chi_V(y)\chi_V(t)dydt\Big| \\
&= \int_{|y-c(V)|<\frac{|V|}{2}}\int_{|t-c(V)|<\frac{|V|}{2}} k((x,y),(s,t))dydt \\
&= \int_{|y|<\frac{|V|}{2}}\int_{|t|<\frac{|V|}{2}} k((x,y+c(V)),(s,t+c(V)))dydt \\
&= \int_{|y|<\frac{|V|}{2}}\int_{|y-(y-t)|<\frac{|V|}{2}} k((x,y+c(V)),(s,t+c(V)))dydt \\
&= \int_{|y|<\frac{|V|}{2}}\int_{||y|-\frac{|V|}{2}|<|y-t|<|y|+\frac{|V|}{2}} k((x,y+c(V)),(s,t+c(V)))dydt \\
&\le A|V|\frac{1}{|x-s|^n}.
\end{aligned}$$

And

$$|k^2_{\chi_V,\chi_V}(x',s')-k^2_{\chi_V,\chi_V}(x,s)|\le A|V|(|x-x'|+|s-s'|)^{\eta_1}|x-s|^{-n-\eta_1},$$

for $|x-s|\ge 2(|x-x'|+|s-s'|)$. Similarly

$$|k^2_{\chi_V,u_V}|\le A|V|\frac{1}{|x-s|^n},$$

$$|k^2_{\chi_V,u_V}(x',s')-k^2_{\chi_V,u_V}(x,s)|\le A|V|(|x-x'|+|s-s'|)^{\eta_1}|x-s|^{-n-\eta_1}$$

for $|x-s|\ge 2(|x-x'|+|s-s'|)$. And

$$|k^2_{u_V,\chi_V}|\le A|V|\frac{1}{|x-s|^n},$$

$$|k^2_{u_V,\chi_V}(x',s')-k^2_{u_V,\chi_V}(x,s)|\le A|V|(|x-x'|+|s-s'|)^{\eta_1}|x-s|^{-n-\eta_1}$$

for $|x-s|\ge 2(|x-x'|+|s-s'|)$. So $k^2_{f_2,g_2}$ satisfied the standard one-parameter kernel estimates and $C(f_2,g_2)$ is the minimal constant (depending on f_2 and g_2) for which those estimates are satisfied. This constant satisfies

$$C(\chi_V,\chi_V)+C(\chi_V,g_V)+C(g_V,\chi_V)\le C|V| \tag{5.1}$$

whenever $V\subset\mathbb{R}^m$ is a cube and g_V is a V-adapted function with zero mean in $\mathbb{R}^m$. $k^1_{f_1,g_1}$ has analogous properties by a similar argument.

By Definition 5.1 (2), $T1, T^*1, T_1(1)$ and $T_1^*(1)$ belong to the product BMO on $\mathbb{R}^n\times\mathbb{R}^m$, and T satisfies the weak boundedness property.

By Definition 5.1 (3)-(a)-(ii) and (3)-(b) and a similar argument as before we have the diagonal BMO condition.

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混合勒贝格空间上双参数奇异积分算子的端点弱估计

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摘要: 本文研究了混合勒贝格空间上双参数奇异积分算子的有界性. 利用双参数奇异积分算子在勒贝格空间的有界性和一个向量值延拓理论. 获得了双参数奇异积分算子在混合勒贝格空间上的端点弱估计和强型估计. 并给出了乘积空间上非卷积型奇异积分算子的一个应用. 这些结果将文献[3]中的结论推广到混合范数情形.

关键词: 双参数奇异积分算子; 混合范数; 端点弱估计

MR(2010)**主题分类号:** 42B20 **中图分类号:** O174.2