Vol. 43 (2023)
No. 3

数 学 杂 志
J. of Math. (PRC)

CACCIOPPOLI INEQUALITY AND HIGHER INTEGRABILITY FOR DIFFERENTIAL FORMS WITH NON-STANDARD GROWTH CONDITIONS

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Abstract: In this paper, we mainly study the related inequalities for differential forms. By using the properties of A-harmonic equation, the weak inverse Holder inequality associated with the equation and the properties of a class of Young functions satisfying non-standard growth conditions, we obtain the Caccopoli inequality and its high-order integrability for a special differential form (i.e., the non-homogeneous A-harmonic tensor) under the action of this kind of Young functions. This conclusion extends the Caccopoli inequality for differential form from Lp space to Orlicz space composed of young functions of this kind, and verifies that the Caccopoli inequality can be used for quantitative estimation and qualitative analysis of differential forms.

Keywords: Caccioppoli inequality; higher integrability; differential forms; non-standard growth conditions; *A*-harmonic equation

2010 MR Subject Classification: 58A10; 42B37

Document code: A **Article ID: 0255-7797(2023)03-0189-13**

1 Introduction

As generalizations of functions, differential forms have been widely used in many fields including potential theory, partial differential equations(PDE), quasi-conformal mappings, and nonlinear analysis[1].In recent decades, the inequalities for differential forms equipped with the L^p-norm have been very well studied, while that with Orlicz norms have not been fully developed [2,3].

In 2004, Buckley and Koskela first introduced a kind of Young function $G(p,q,C)$, since then some mathematicians devoted themselves to study the inequalities with L^φ-norm for differential forms and operators, where $\varphi \in G(p,q,C)$[4−9]. In 2013, Lu and Bao redefined a Young function φ satisfying the non-standard growth conditions, write $\varphi \in NG(p,q)$ for short, which can be traced back to [10], and obtained the Poincaré inequalities and the sharp maximal inequalities for differential forms with L^φ-norm[11−12]. However, it still remains unanswered on how to distinguish these two kinds of Young functions.

It is well known that Caccioppoli inequality and Poincaré inequality are two kinds of important inequalities in differential forms, which can be used in PDE and potential theory[13−14]. In recent

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* **Received date:** 2022-10-12 **Accepted date:** 2023-03-06

Foundation item: Supported by Shanxi Provincial Education Department(21JK0671).

Biography: Dai Zhimin(1982–), male, born at Xiantao, Hubei, lecturer, major in inequalities for differential forms. E-mail: daizhimin@xatu.edu.cn years, some versions of Poincaré inequalities for differential forms with L^p and Orlicz norms have been established[1,5−9,15−17]. Although Caccioppoli inequalities for differential forms with L^p-norm have been well obtained, that with Orlicz norm still needs further studies[1,17−18]. In this paper, we devoted ourselves to developing Caccioppoli inequalities for differential forms satisfying the nonhomogeneous A-harmonic equation with the Young function lies in $NG(p,q)$ and proving the higher integrability of the exterior derivative of the nonhomogeneous A-harmonic tensors.

Now we introduce some notations and definitions. Let Θ be an open subset of $\mathbb{R}^n (n \geq 2)$ and O be a ball with the center at the origin in $\mathbb{R}^n$. Let ρO denote the ball with the same center as O and $diam(\rho O) = \rho diam(O) (\rho > 0)$. $|\Theta|$ is used to denote the Lebesgue measure of a set $\Theta \subset \mathbb{R}^n$. Let $\wedge^\ell = \wedge^\ell(\mathbb{R}^n), \ell = 0, 1, \ldots, n$, be the linear space of all ℓ-forms $u(x) = \sum_I u_I(x)\mathrm{d}x_I = \sum_I u_{i_1 i_2 \cdots i_\ell}(x)\mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \cdots \wedge \mathrm{d}x_{i_\ell}$ in $\mathbb{R}^n$, where $I = (i_1, i_2, \ldots, i_\ell), 1 \leq i_1 < i_2 < \cdots < i_\ell \leq n$, are the ordered ℓ-tuples and the wedge outer product satisfies the relationship that

$$\mathrm{d}x_i \wedge \mathrm{d}x_j = \begin{cases} -\mathrm{d}x_j \wedge \mathrm{d}x_i, & i \neq j; \\ 0, & i = j. \end{cases} \tag{1.1}$$

Moreover, if each of the coefficient $u_I(x)$ of $u(x)$ is differential on Θ, then we call $u(x)$ a differential ℓ-form on Θ and use $D^{'}(\Theta, \wedge^\ell)$ to denote the space of all differential ℓ-forms on Θ. $C^\infty(\Theta, \wedge^\ell)$ denotes the space of smooth ℓ-forms on Θ. The exterior derivative $d : D^{'}(\Theta, \wedge^\ell) \to D^{'}(\Theta, \wedge^{\ell+1})$, $\ell = 0, 1, \ldots, n-1$, is given by

$$\mathrm{d}u(x) = \sum_I \sum_{j=1}^{n} \frac{\partial u_{i_1 i_2 \cdots i_\ell}(x)}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_\ell} \tag{1.2}$$

for all $u \in D^{'}(\Theta, \wedge^\ell)$. The Hodge star operator $\star : \wedge^\ell \to \wedge^{n-\ell}$ is defined as follows. If $u = u_{i_1 i_2 \cdots i_\ell}(x_1, x_2, \ldots, x_n)\mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_\ell} = u_I \mathrm{d}x_I$, $i_1 < i_2 < \cdots < i_\ell$, is a differential ℓ-form, then $\star u = \star(u_{i_1 i_2 \cdots i_\ell}\mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_\ell}) = (-1)^{\Sigma(I)}\omega_I \mathrm{d}x_J$, where $I = (i_1, i_2, \cdots, i_\ell)$, $J = \{1, 2, \ldots, n\} - I$, and $\sum(I) = \frac{\ell(\ell+1)}{2} + \sum_{j=1}^{\ell} i_j$. The Hodge codifferential operator $d^* : D^{'}(\Theta, \wedge^{l+1}) \to D^{'}(\Theta, \wedge^\ell)$is defined by $d^* = (-1)^{n\ell+1} \star d \star$ on $D'(\Theta, \wedge^{\ell+1})$, $\ell = 0, 1, \ldots, n-1$. For all $u \in D^{'}(\Theta, \wedge^\ell)$, we have $d(\mathrm{d}u) = d^*(d^* u) = 0$. $L^p(\Theta, \wedge^\ell)(1 \leq p < \infty)$ is a Banach space with the norm $||u||_{p,\Theta} = (\int_\Theta |u(x)|^p \mathrm{d}x)^{1/p} = (\int_\Theta (\sum_I |u_I(x)|^2)^{p/2}\mathrm{d}x)^{1/p} < \infty$. Similarly, the notations $L^p_{loc}(\Theta, \wedge^\ell)$ and $W^{1,p}_{loc}(\Theta, \wedge^\ell)$ are self-explanatory[19−20].

From [19-20], u is a differential form in a bounded convex domain Θ, then there is a decomposition

$$u = \mathrm{d}(Tu) + T(\mathrm{d}u), \tag{1.3}$$

where T is called a homotopy operator, and exists

$$||Tu||_{p,O} \leq C|O|diam(O)||u||_{p,O} \tag{1.4}$$

for any differential form $u \in L^p_{loc}(\Theta, \bigwedge^\ell), \ell = 1, 2, \ldots, n, 1 < p < \infty$. Furthermore, we can define the ℓ-form $u_\Theta \in D^{'}(\Theta, \bigwedge^\ell)$ by

$$u_\Theta = \begin{cases} |\Theta|^{-1} \int_\Theta u(x)\mathrm{d}x, & \ell = 0 \\ \mathrm{d}T(u), & \ell = 1, 2, \ldots, n \end{cases} \tag{1.5}$$

for all $u \in L^p(\Theta, \bigwedge^\ell), 1 \le p < \infty$.

The nonhomogeneous A-harmonic equation for differential forms is defined as follows

$$d^\star A(x, du) = B(x, du), \tag{1.6}$$

where $A : \Theta \times \wedge^\ell(\mathbb{R}^n) \to \wedge^\ell(\mathbb{R}^n)$ and $B : \Theta \times \wedge^\ell(\mathbb{R}^n) \to \wedge^{\ell-1}(\mathbb{R}^n)$ satisfy the conditions:

$$|A(x,\xi)| \le a|\xi|^{p-1}, \quad A(x,\xi)\cdot\xi \ge |\xi|^p, \quad |B(x,\xi)| \le b|\xi|^{p-1} \tag{1.7}$$

for almost every $x \in \Theta$ and all $\xi \in \wedge^\ell(\mathbb{R}^n)$. Here $a, b > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with (1.6). The solutions of the nonhomogeneous A-harmonic equation for differential forms are called as the nonhomogeneous A-harmonic tensor, see [1] for more details about these kinds of equations.

2 Caccioppoli Inequality for Differential Forms with Non-Standard Growth Conditions

It is precisely because Caccioppoli inequality plays an important role in the theoretical research of PDE. The purpose of this section is to prove the following Caccioppoli estimates for the solutions to the nonhomogeneous A-harmonic equation with the non-standard growth conditions, which are more general than the well-known $p(x)-$ growth[21]. First, we introduce some existing definitions and lemmas.

A continuously increasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ is so-called as an Orlicz function. The Orlicz space $L^\varphi(\Theta)$ consists of all measurable functions u on Θ such that $\int_\Theta \varphi(\frac{|u|}{\chi})dx < \infty$ for some $\chi = \chi(u) > 0$. $L^\varphi(\Theta)$ is equipped with the nonlinear Luxemburg functional

$$\|u\|_{\varphi,\Theta} = \inf\left\{\chi > 0 : \fint_\Theta \varphi\left(\frac{|u|}{\chi}\right)dx \le 1\right\}, \tag{2.1}$$

where $\fint_\Theta$ denotes the integral mean over Θ, that is, $\fint_\Theta dx = \frac{1}{|\Theta|}\int_\Theta dx$. A convex Orlicz function φ is often called a Young function. If φ is a Young function, then $\|\cdot\|_{\varphi,\Theta}$ defines a norm in $L^\varphi(\Theta)$, which is called the Orlicz norm or Luxemburg norm, see [17].

The Young function $\varphi : [0,\infty) \to [0,\infty)$ belongs to $NG(p,q)$, if φ satisfies the following non-standard growth conditions[4,6]:

$$p\varphi(s) \le s\varphi'(s) \le q\varphi(s), \quad 1 < p \le q < \infty. \tag{2.2}$$

For $\varphi \in NG(p,q)$, we note that the first inequality in (2.2) is equivalent to that $\frac{\varphi(s)}{s^p}$ is increasing, and the second inequality in that is equivalent to Δ_2-condition, i.e., $\varphi(2s) \le C\varphi(s)$ for all $s > 0$, where $C > 1$, and $\frac{\varphi(s)}{s^q}$ is decreasing with s. As examples of the Young function of the class $NG(p,q)$, one can take $\varphi(s) = s^p, s^p\log(1+s)(q \ge p+1)$, see [10-12]for more details about this kind of Young funciton.

Lemma 2.1 (see [10]) Suppose φ is a continuous function in the class $NG(p,q)$ with $p > \frac{nq}{n+q} = q^*$, $1 < p \le q < \infty$. For any $s > 0$, let

$$E(s) = \int_0^s \left(\frac{\varphi(\tau^{\frac{1}{q}})}{\tau}\right)^{\frac{n+q}{q}} d\tau, \quad F(s) = \frac{\left(\varphi(s^{\frac{1}{q}})\right)^{\frac{n+q}{q}}}{s^{\frac{n}{q}}} \tag{2.3}$$

Then $E(s)$ is a concave function, and there exists a constant $C > 1$, such that

$$F(s) \leq E(s) \leq CF(s), \quad \forall s > 0. \tag{2.4}$$

The following inequalities (2.5) and (2.6) are called respectively Hölder inequality and Jensen inequality, which appeared in [22].

Lemma 2.2 Let $0 < p, q < \infty$ and $1 = p^{-1} + q^{-1}$. If f and g are measurable functions on $\mathbb{R}^n$, then

$$\fint_\Theta |fg| \mathrm{d}x \leq \left(\fint_\Theta |f|^p \mathrm{d}x \right)^{\frac{1}{p}} \left(\fint_\Theta |g|^q \mathrm{d}x \right)^{\frac{1}{q}} \tag{2.5}$$

for any $\Theta \subset \mathbb{R}^n$.

Lemma 2.3 Let μ be positive measurable on a σ-algebra M in a set Ω, so that $\mu(\Omega) = 1$. If f is a real function in $L^1(\mu)$, $a < f(x) < b$ for all $x \in \Omega$ and ϕ is convex on (a, b), then

$$\phi\left(\int_\Omega |f| d\mu \right) \leq \int_\Omega \phi(|f|) d\mu. \tag{2.6}$$

Remark If the function ϕ is a concave function, the inequality is inverse.

The following inequalities (2.7) and (2.8) are called respectively the Sobolev-Poincaré inequality and the weak reverse Hölder inequality, which appeared in [20,23].

Lemma 2.4 Let $u \in D'(O, \wedge^\ell)$ be a differential form and $\mathrm{d}u \in L^p(O, \wedge^{\ell+1})$, then $u - u_O$ is in $L^{\frac{np}{(n-p)}}(O, \wedge^\ell)$ and

$$\left(\fint_O |u - u_O|^{\frac{np}{n-p}} \mathrm{d}x \right)^{\frac{(n-p)}{np}} \leq C_p(n) |O|^{\frac{1}{n}} \left(\fint_O |\mathrm{d}u|^p \mathrm{d}x \right)^{\frac{1}{p}} \tag{2.7}$$

for a cube or a ball O in $\mathbb{R}^n$, $\ell = 0, 1, \ldots, n-1$ and $1 < p < n$.

Lemma 2.5 Let u be a solution to the nonhomogeneous A-equation (1.6) in Θ, $\sigma > 1$ be some constant, and $0 < r, s < \infty$ be any constants. Then there exists a constant C, independent of u, such that

$$\left(\fint_O |u|^s \mathrm{d}x \right)^{\frac{1}{s}} \leq C \left(\fint_{\sigma O} |u|^r \mathrm{d}x \right)^{\frac{1}{r}} \tag{2.8}$$

for all cubes or balls O with $\sigma O \subset \Theta$.

Remark (i)when u is the solution of the A-harmonic equation (1.6), du is also the solution of Equation (1.6), then by (2.8), we have

$$||\mathrm{d}u||_{s,O} \leq C|O|^{\frac{r-s}{rs}} ||\mathrm{d}u||_{r,\sigma O}. \tag{2.9}$$

(ii)For u is the solution of Equation (1.6) and c is a closed form, i.e., $dc = 0$, $u - c$ is also the solution of Equation (1.6), similarly, we have

$$||u - c||_{s,O} \leq C|O|^{\frac{r-s}{rs}} ||u - c||_{r,\sigma O}. \tag{2.10}$$

Lemma 2.6 (see [18]) Let $u \in D'(\Theta, \wedge^\ell)$, $\ell = 0, 1, \ldots, n$, be a solution of the nonhomogeneous A-harmonic equation (1.6) in a bounded domain $\Theta \subset \mathbb{R}^n$, and assume that $1 < p < \infty$ is a fixed exponent associated with the equation (1.6) and $\sigma > 1$ is a constant. Then there exists a constant C, independent of u and du, such that

$$||du||_{p,O} \le C diam(O)^{-1} ||u - c||_{p,\sigma O} \tag{2.11}$$

for all balls or cubes O with $\sigma O \subset \Theta$ and any closed form c.

Remark From Lemma 2.6, we know that p is related with the condition of the nonhomogeneous A-harmonic equation (1.6). Combining (2.9)-(2.11), we can get the generalized version of Caccioppoli inequality (2.12), i.e., Theorem 2.1.

Theorem 2.1 Let $u \in D'(\Theta, \wedge^\ell), \ell = 0, 1, \ldots, n$, be a solution of the nonhomogeneous A-harmonic equation (1.6) in a bounded domain $\Theta \subset \mathbb{R}^n$, and assume that $0 < s < \infty$. Then there exists a constant C, independent of u and du, such that

$$||du||_{s,O} \le C diam(O)^{-1} ||u - c||_{s,\sigma O} \tag{2.12}$$

for all balls or cubes O with $\sigma O \subset \Theta (\sigma > 1)$ and any closed form c.

Proof For any $s > 0$, we have

$$\begin{aligned} ||du||_{s,O} &\le C_1 |O|^{\frac{p-s}{ps}} ||du||_{p,\sigma_1 O} \\ &\le C_2 |O|^{\frac{p-s}{ps}} diam(O)^{-1} ||u - c||_{p,\sigma_2 O} \\ &\le C_3 |O|^{\frac{p-s}{ps}} diam(O)^{-1} |O|^{\frac{s-p}{sp}} ||u - c||_{s,\sigma_3 O} \\ &= C_3 diam(O)^{-1} ||u - c||_{s,\sigma_3 O}, \end{aligned} \tag{2.13}$$

where $\sigma_3 > \sigma_2 > \sigma_1 > 1$. Write (2.13) as the form of integral mean, we get

$$\left(\fint_O |du|^s dx \right)^{\frac{1}{s}} \le C diam(O)^{-1} \left(\fint_{\sigma O} |u - c|^s dx \right)^{\frac{1}{s}}. \tag{2.14}$$

Theorem 2.2 Let φ be a Young function in the class $NG(p, q)$ with $p > \frac{nq}{n+q} = q^*$ and Θ be a bounded domain in $\mathbb{R}^n$. Assume that $\varphi(|u|) \in L^1_{loc}(\Theta)$ and u is a solution of the nonhomogeneous A-harmonic equation (1.6) in Θ, Then there exists a constant C, independent of u, such that

$$\int_O \varphi(|du|) dx \le C \int_{\sigma O} \varphi \left(diam(O)^{-1} |u - c| \right) dx \tag{2.15}$$

for all balls O with $\sigma O \subset \Theta$ and c is any closed form, where $\sigma > 1$ is a constant.

Proof Applying Hölder inequality (2.5), we have

$$\fint_O \varphi(|du|) dx = \fint_O \frac{\varphi(|du|)}{|du|^{\frac{nq}{n+q}}} |du|^{\frac{nq}{n+q}} dx \le \left(\fint_O \frac{\varphi^{\frac{n+q}{q}}(|du|)}{|du|^n} dx \right)^{\frac{q}{n+q}} \left(\fint_O |du|^q dx \right)^{\frac{n}{n+q}}. \tag{2.16}$$

Using the Δ_2-condition of φ and the concavity of E, which appeared in Lemma 2.1, using(2.8)and(2.14), (2.16) becomes

$$\begin{aligned}\fint_O \varphi(|\mathrm{d}u|)\mathrm{d}x &\le \left(\fint_O F(|\mathrm{d}u|^q)\mathrm{d}x\right)^{\frac{q}{n+q}}\left(\fint_O |\mathrm{d}u|^q\mathrm{d}x\right)^{\frac{n}{n+q}} \le \left(\fint_O E(|\mathrm{d}u|^q)\mathrm{d}x\right)^{\frac{q}{n+q}}\left(\fint_O |\mathrm{d}u|^q\mathrm{d}x\right)^{\frac{n}{n+q}}\\ &\le E^{\frac{q}{n+q}}\left(\fint_O |\mathrm{d}u|^q\mathrm{d}x\right)\left(\fint_O |\mathrm{d}u|^q\mathrm{d}x\right)^{\frac{n}{n+q}} \le C_1F^{\frac{q}{n+q}}\left(\fint_O |\mathrm{d}u|^q\mathrm{d}x\right)\left(\fint_O |\mathrm{d}u|^q\mathrm{d}x\right)^{\frac{n}{n+q}}\\ &= C_1\frac{\varphi((\fint_O |\mathrm{d}u|^q\mathrm{d}x)^{\frac{1}{q}})}{(\fint_O |\mathrm{d}u|^q\mathrm{d}x)^{\frac{n}{n+q}}}\left(\fint_O |\mathrm{d}u|^q\mathrm{d}x\right)^{\frac{n}{n+q}} = C_1\varphi\left((\fint_O |\mathrm{d}u|^q\mathrm{d}x)^{\frac{1}{q}}\right)\\ &\le C_1\varphi\left(C_2(\fint_{\sigma_1 O} |\mathrm{d}u|^p\mathrm{d}x)^{\frac{1}{p}}\right) \le C_1\varphi\left(C_3 diam(O)^{-1}(\fint_{\sigma_2 O} |u-c|^p\mathrm{d}x)^{\frac{1}{p}}\right)\\ &\le C_4\varphi\left(diam(O)^{-1}(\fint_{\sigma_2 O} |u-c|^p\mathrm{d}x)^{\frac{1}{p}}\right),\end{aligned}\tag{2.17}$$

where $\sigma_2 > \sigma_1 > 1$. Let $\Phi(s) = \int_0^s \frac{\varphi(\tau)}{\tau}d\tau$ and that $\frac{\varphi(s)}{s^p}$, $\frac{\varphi(s)}{s^q}$ are increasing and decreasing respectively with s in $[0, +\infty)$, so

$$\Phi(s) = \int_0^s \frac{\varphi(\tau)}{\tau^p}\tau^{p-1}d\tau \le \frac{\varphi(s)}{s^p}\int_0^s \tau^{p-1}d\tau = \frac{\varphi(s)}{p}.\tag{2.18}$$

Similarly, we have $\Phi(s) \ge \frac{\varphi(s)}{q}$. Therefore we have

$$\frac{\varphi(s)}{q} \le \Phi(s) \le \frac{\varphi(s)}{p}.\tag{2.19}$$

Let $\Psi(s) = \Phi(s^{\frac{1}{p}}), \Psi'(s) = \frac{1}{p}\frac{\varphi(s^{\frac{1}{p}})}{s}$ is increasing, so Ψ is a convex function. For all $\vartheta \in L^1(\Theta)$, by Jensen inequality (2.6), we obtain

$$\Phi\left(\left(\fint_\Theta \vartheta\mathrm{d}x\right)^{\frac{1}{p}}\right) = \Psi\left(\fint_\Theta \vartheta\mathrm{d}x\right) \le \fint_\Theta \Psi(\vartheta)\mathrm{d}x = \fint_\Theta \Phi(\vartheta^{\frac{1}{p}})\mathrm{d}x.\tag{2.20}$$

Replacing ϑ with $(\mathrm{diam}(O)^{-1}|u-c|)^p$ in (2.20), we get

$$\Phi\left(\left(\fint_{\sigma_2 O}\mathrm{diam}(O)^{-p}|u-c|^p\mathrm{d}x\right)^{\frac{1}{p}}\right) \le \fint_{\sigma_2 O}\Phi\left(\mathrm{diam}(O)^{-1}|u-c|\right)\mathrm{d}x.\tag{2.21}$$

Combining (2.19) and (2.21), (2.17) becomes

$$\begin{aligned}\fint_O \varphi(|\mathrm{d}u|)\mathrm{d}x &\le C_5\Phi\left(\mathrm{diam}(O)^{-1}(\fint_{\sigma_2 O}|u-c|^p\mathrm{d}x)^{\frac{1}{p}}\right)\\ &\le C_5\fint_{\sigma_2 O}\Phi\left(\mathrm{diam}(O)^{-1}|u-c|\right)\mathrm{d}x\\ &\le C_6\fint_{\sigma_2 O}\varphi\left(\mathrm{diam}(O)^{-1}|u-c|\right)\mathrm{d}x,\end{aligned}\tag{2.22}$$

which gives

$$\int_O \varphi(|\mathrm{d}u|)\mathrm{d}x \le C_7\int_{\sigma O}\varphi\left(\mathrm{diam}(O)^{-1}|u-c|\right)\mathrm{d}x.\tag{2.23}$$

The proof of Theorem 2.2 has been completed.

Since $\varphi \in NG(p,q)$ satisfies the Δ_2-condition, from the proof of Theorem 2.2 or directly from (2.23), we have

$$\fint_O \varphi(\frac{|\mathrm{d}u|}{\chi})\mathrm{d}x \le C_7 \fint_{\sigma O} \varphi\left(\mathrm{diam}(O)^{-1}\frac{|u-c|}{\chi}\right)\mathrm{d}x \tag{2.24}$$

for all balls O with $\sigma O \subset \Theta$ and any constant $\chi > 0$. From the definition of the Orlicz norm and (2.23), the following inequality with the Orlicz norm

$$||\mathrm{d}u||_{\varphi,O} \le C||\mathrm{diam}(O)^{-1}(u-c)||_{\varphi,\sigma O} \tag{2.25}$$

holds if the conditions described in Theorem 2.2 are satisfied.

Noticing that in Theorem 2.2, c is any closed form. Hence, we may choose $c = u_{\sigma O}$ in Theorem 2.2 and obtain the following version of L^{φ}-integral inequality which may be convenient to be used later.

Corollary 2.1 Let φ be a Young function in the class $NG(p,q)$ with $p > \frac{nq}{n+q} = q^*$ and Θ be a bounded domain in $\mathbb{R}^n$. Assume that $\varphi(|u|) \in L^1_{loc}(\Theta)$ and u is a solution of the nonhomogeneous A-harmonic equation (1.6) in Θ. Then, there exists a constant C, independent of u, such that

$$\int_O \varphi(|\mathrm{d}u|)\mathrm{d}x \le C\int_{\sigma O}\varphi\left(\mathrm{diam}(O)^{-1}|u-u_{\sigma O}|\right)\mathrm{d}x \tag{2.26}$$

for all balls O with $\sigma O \subset \Theta$, where $\sigma > 1$ is a constant.

3 Higher Integrability for Differential Forms with Non-Standard Growth Conditions

In [24], Bogelein, Duzaar, Korte and Scheve established that the gradient of weak solutions to porous medium-type systems admits the self-improving property of higher integrability. In recent years, the higher integrability of integrals with operators acting on differential forms is becoming a research hotspot [25–27]. Therefore for the self-improving property, we will prove the higher integrability of the external differential of the nonhomogeneous A-harmonic tensors u, which are also essentially differential forms. So the title of this section is named as above.

Now we first introduce the extension of Gehring lemma for functions φ be in the class $NG(p,q)$, then we will prove the higher integrability of the exterior differential of differential forms satisfying the nonhomogeneous A-harmonic equation under non-standard growth conditions with the Caccioppoli inequality.

Lemma 3.1 (see [10]) If φ be in the class $NG(p,q)$ and f is an $L^1_{loc}(\Theta)$ function, $f \ge 0$, such that, for any cube $Q \subset \Theta$ for which $2Q \subset\subset \Theta$,

$$\fint_Q \varphi(f)\mathrm{d}x \le b_1\varphi\left(\fint_{2Q} f\mathrm{d}x\right) + b_2, \tag{3.1}$$

then there exist c_l, $c_2 > 0$, $r > 1$, depending only on b_1, b_2, n, k, p such that, for any $2Q \subset\subset \Theta$,

$$\fint_Q \varphi^r(f)\mathrm{d}x \le c_1\varphi^r\left(\fint_{2Q} f\mathrm{d}x\right) + c_2, \tag{3.2}$$

Remark Noticing that in Lemma 3.1, cubes in inequalities (3.1) and (3.2) may easily be replaced by balls and $2Q$ can be replaced with σQ, $\sigma > 1$. Moreover, it is clear from the proof that if $b_2 = 0$, then also $c_2 = 0$.

Theorem 3.1 Let φ be a Young function in the class $NG(p, q)$ with $1 < \frac{nq}{n+q} = q^* < p$ and Θ be a bounded domain in $\mathbb{R}^n$. Assume that $\varphi(|u|) \in L^1_{loc}(\Theta)$ and u is a solution of the nonhomogeneous A-harmonic equation (1.6) in Θ, then there exist $r > 1, C > 0$ such that

$$\fint_O \varphi^r(|\mathrm{d}u|)\mathrm{d}x \le C\left(\fint_{\sigma O} \varphi(|\mathrm{d}u|)\mathrm{d}x\right)^r \tag{3.3}$$

for all balls O with $\sigma O \subset \Theta$, where $\sigma > 1$ is a constant.

Proof Let $t = \left|\frac{u-u_{\sigma O}}{\mathrm{diam}(O)}\right|$, from Theorem 2.2 or Corollary 2.1, and Hölder inequality (2.5), we deduce that

$$\fint_O \varphi(|\mathrm{d}u|)\mathrm{d}x \le C_1 \fint_{\sigma O} \frac{\varphi(t)}{t^{\frac{nq}{n+q}}} t^{\frac{nq}{n+q}} \mathrm{d}x \le C_1 \left(\fint_{\sigma O} \frac{\varphi^{\frac{n+q}{q}}(t)}{t^n}\mathrm{d}x\right)^{\frac{q}{n+q}} \cdot \left(\fint_{\sigma O} t^q \mathrm{d}x\right)^{\frac{n}{n+q}}. \tag{3.4}$$

Noticing that $q = \frac{nq^*}{n-q^*}$ and using the Sobolev-Poincaré inequality (2.7), we have

$$\begin{aligned}\left(\fint_{\sigma O} t^q \mathrm{d}x\right)^{\frac{1}{q}} &\le \mathrm{diam}(O)^{-1} C_2 |\sigma O|^{\frac{1}{n}} \left(\fint_{\sigma O} |\mathrm{d}u|^{q^*}\mathrm{d}x\right)^{\frac{1}{q^*}} \\ &\le C_2 \mathrm{diam}(O)^{-1} \sigma C_3 \mathrm{diam}(O) \left(\fint_{\sigma O} |\mathrm{d}u|^{q^*}\mathrm{d}x\right)^{\frac{1}{q^*}} \\ &\le C_4 \left(\fint_{\sigma O} |\mathrm{d}u|^{q^*}\mathrm{d}x\right)^{\frac{1}{q^*}}.\end{aligned} \tag{3.5}$$

Therefore we obtain

$$\left(\fint_{\sigma O} t^q \mathrm{d}x\right)^{\frac{n}{n+q}} \le \left(C_4 \left(\fint_{\sigma O} |\mathrm{d}u|^{q^*}\mathrm{d}x\right)^{\frac{1}{q^*}}\right)^{\frac{nq}{n+q}} \le C_5 \fint_{\sigma O} |\mathrm{d}u|^{q^*}\mathrm{d}x. \tag{3.6}$$

Combining (3.4)and (3.6), using the concavity of E and the Δ_2-condition of φ, we get

$$\begin{aligned}\fint_O \varphi(|\mathrm{d}u|)\mathrm{d}x &\le C_6 \left(\fint_{\sigma O} F\left(t^q\right)\mathrm{d}x\right)^{\frac{q}{n+q}} \cdot \fint_{\sigma O} |\mathrm{d}u|^{q^*}\mathrm{d}x \le C_6 \left(\fint_{\sigma O} E\left(t^q\right)\mathrm{d}x\right)^{\frac{q}{n+q}} \cdot \fint_{\sigma O} |\mathrm{d}u|^{q^*}\mathrm{d}x \\ &\le C_6 E^{\frac{q}{n+q}} \left(\fint_{\sigma O} t^q \mathrm{d}x\right) \cdot \fint_{\sigma O} |\mathrm{d}u|^{q^*}\mathrm{d}x \le C_6 F^{\frac{q}{n+q}} \left(\fint_{\sigma O} t^q \mathrm{d}x\right) \cdot \fint_{\sigma O} |\mathrm{d}u|^{q^*}\mathrm{d}x \\ &\le C_7 F^{\frac{q}{n+q}} \left(\left[\fint_{\sigma O} |\mathrm{d}u|^{q^*}\mathrm{d}x\right]^{\frac{q}{q^*}}\right) \cdot \fint_{\sigma O} |\mathrm{d}u|^{q^*}\mathrm{d}x \\ &= C_8 \frac{\varphi\left(\left[\fint_{\sigma O} |\mathrm{d}u|^{q^*}\mathrm{d}x\right]^{\frac{1}{q^*}}\right)}{\fint_{\sigma O} |\mathrm{d}u|^{q^*}\mathrm{d}x} \cdot \fint_{\sigma O} |\mathrm{d}u|^{q^*}\mathrm{d}x \\ &= C_8 \varphi\left(\left[\fint_{\sigma O} |\mathrm{d}u|^{q^*}\mathrm{d}x\right]^{\frac{1}{q^*}}\right).\end{aligned} \tag{3.7}$$

If we set $\phi(s) = \varphi(s^{\frac{1}{q^*}})$, then $\phi'(s) = \frac{1}{q^*}s^{\frac{1}{q^*}-1}\varphi'(s^{\frac{1}{q^*}})$. Considering that $\varphi \in NG(p, q)$, we have

$$\frac{p}{q^*}\varphi(s^{\frac{1}{q^*}}) \le s\left(\frac{1}{q^*}s^{\frac{1}{q^*}-1}\varphi'(s^{\frac{1}{q^*}})\right) \le \frac{q}{q^*}\varphi(s^{\frac{1}{q^*}}). \tag{3.8}$$

That is

$$p'\phi(s) \le s\phi'(s) \le q'\phi(s). \tag{3.9}$$

Since $p > q^*$, we obtain that $\phi(s) \in NG(p', q')$, where $p' = \frac{p}{q^*}$, $q' = \frac{q}{q^*} > 1$. By (3.7) with $f = |\mathrm{d}u|^{q^*}$, we obtain that

$$\fint_O \phi(f)\mathrm{d}x \le C_8\phi\left(\fint_{\sigma O} f\mathrm{d}x\right), \tag{3.10}$$

from which it follows that ϕ and f satisfy the assumptions of Lemma 3.1. Thus there exists $r > 1$ such that

$$\fint_O \phi^r(f)\mathrm{d}x \le C_9\phi^r\left(\fint_{\sigma O} f\mathrm{d}x\right), \tag{3.11}$$

i.e., for any $\sigma O \subset \Theta$,

$$\fint_O \varphi^r(|\mathrm{d}u|)\mathrm{d}x \le C_9\varphi^r\left(\left(\fint_{\sigma O} |\mathrm{d}u|^{q^*}\mathrm{d}x\right)^{\frac{1}{q^*}}\right). \tag{3.12}$$

Replacing ϑ with $|\mathrm{d}u|^p$ in (2.20) and using (2.19), we have

$$\Phi\left(\left(\fint_{\sigma O} |\mathrm{d}u|^p\mathrm{d}x\right)^{\frac{1}{p}}\right) \le \fint_{\sigma O} \Phi(|\mathrm{d}u|)\mathrm{d}x \le \frac{1}{p}\fint_{\sigma O} \varphi(|\mathrm{d}u|)\mathrm{d}x. \tag{3.13}$$

Noticing that $p > q^*$, we obtain from (2.19) and (3.13) that

$$\frac{1}{q}\varphi\left(\left(\fint_{\sigma O} |\mathrm{d}u|^{q^*}\mathrm{d}x\right)^{\frac{1}{q^*}}\right) \le \frac{1}{q}\varphi\left(\left(\fint_{\sigma O} |\mathrm{d}u|^p\mathrm{d}x\right)^{\frac{1}{p}}\right) \le \Phi\left(\left(\fint_{\sigma O} |\mathrm{d}u|^p\mathrm{d}x\right)^{\frac{1}{p}}\right) \le \frac{1}{p}\fint_{\sigma O} \varphi(|\mathrm{d}u|)\mathrm{d}x. \tag{3.14}$$

From (3.12) and (3.14), we get

$$\fint_O \varphi^r(|\mathrm{d}u|)\mathrm{d}x \le C_{10}\left(\fint_{\sigma O} \varphi(|\mathrm{d}u|)\mathrm{d}x\right)^r. \tag{3.15}$$

The proof of Theorem 3.1 has been completed.

4 Applications

In [28], Skrzypczak derived Hardy inequalities in weighted Sobolev spaces via anticoercive partial differential inequalities of elliptic type involving φ-Laplacian $-\Delta_\varphi u = -div\varphi(\nabla u) \ge \Phi$, where Φ is a given locally integrable function and u is defined on an open subset $\Theta \subset \mathbb{R}^n$. By knowing solutions, he derived Caccioppoli inequalities for u. As a consequence, he obtained Hardy inequalities for compactly supported Lipschitz functions involving certain measures, having the form

$$\int_\Theta F_{\overline{\varphi}}(|\zeta|)\mu_1(\mathrm{d}x) \le \int_\Theta \overline{\varphi}(|\nabla\zeta|)\mu_2(\mathrm{d}x), \tag{4.1}$$

where $\overline{\varphi}(t) = t^p \ln^{\alpha}(2+t)(p > 1, \alpha \geq 0)$ is a Young function related to φ and $F_{\overline{\varphi}}(t) = \frac{1}{\overline{\varphi}(\frac{1}{t})}$. If let $\alpha = q - p > 0$, we can get the following theorem.

Theorem 4.1 The Young function $\varphi : [0, \infty) \to [0, \infty)$ belongs to $NG(p, q)$, if $\varphi(t) = t^p \ln^{\alpha}(2+t), \alpha = q - p > 0, 1 < p < q < \infty$.

Proof According to the equivalent definition of $NG(p, q)$, we only need to prove the following facts:

(1) $\frac{\varphi(t)}{t^p}$ is increasing:Let $f(t) = \frac{\varphi(t)}{t^p} = \ln^{q-p}(2+t)$, then $f'(t) = (q-p)\ln^{q-p-1}(2+t)/(2+t) > 0$. So the result is clear;

(2) $\varphi(t)$ satisfies the Δ_2-condition:

$$\varphi(2t) = (2t)^p \ln^{q-p}(2+2t) = 2^p t^p (\ln 2 + \ln(1+t))^{q-p} \leq 2^q t^p \ln^{q-p}(2+t) = 2^q \varphi(t), \tag{4.2}$$

Thus the conclusion is obvious;

(3) $\frac{\varphi(t)}{t^q}$ is decreasing: Let $g(t) = \frac{\varphi(t)}{t^q} = (\frac{\ln(2+t)}{t})^{q-p}$, and $g(t)$ can be seen as a a compound function by the increasing function $f(u) = u^{q-p}$ and the decreasing function $u = \frac{\ln(2+t)}{t}$, thus the assertion is established.

In the nonhomogeneous A-harmonic equation (1.6) , if we take A, B to be different operators, we will obtain different examples of A-harmonic equations. For example, assume $B = 0$,then the nonhomogeneous A-harmonic equation changes to the following homogeneous A-harmonic equation

$$\mathrm{d}^{\star} A(x, \mathrm{d}u) = 0. \tag{4.3}$$

Moreover, if we take $A(x, \zeta) = \zeta|\zeta|^{p-2}$ in formula (4.3), then the homogeneous A-harmonic equation becomes the following p-harmonic equation

$$\mathrm{d}^{\star}(\mathrm{d}u|\mathrm{d}u|^{p-2}) = 0. \tag{4.4}$$

Particulary, let $p = 2$ in formula (4.4), then it reduces to

$$\mathrm{d}^{\star}(\mathrm{d}u) = 0. \tag{4.5}$$

In addition, if u is a function (0-form), then (4.5) is equivalent to the classic Laplace equation $\Delta u = 0$. The function u satisfying Laplace equation is called the harmonic function. Obviously, the related conclusions in Sections 2 and 3 still hold for differential forms satisfying Equation (4.5). It is easy to see that u is a trivial solution of (1.6) if $\mathrm{d}u = 0$. But the expression of $\mathrm{d}u \neq 0$ sometimes may be quite complicated, and it would be very hard to evaluate the norm of $\mathrm{d}u$ directly. In this case, we may consider to use the Caccioppli inequality to obtain the upper bound for the $\int_O \varphi(|\mathrm{d}u|)\mathrm{d}x$ instead of calculating the integral directly . Let us see the following simple example in $\mathbb{R}^2$.

Example 4.1 Let $u(x, y)$ be a function defined in $\mathbb{R}^2$ by

$$u(x, y) = x^3 - 6x^2y - 3xy^2 + 2y^3. \tag{4.6}$$

It is easy to check that $u(x, y)$ is a harmonic function in the upper half plane. Let $r > 0$ be a constant, and $O = (x, y) : x^2 + y^2 \leq r^2$. To obtain the upper bound for the $\int_O \varphi(|\mathrm{d}u|)\mathrm{d}x$ with $\varphi(t) = t^p \ln^{q-p}(2+t)$, we can use Caccioppoli inequality (2.15) with $c = 0$, and $n = 2$ as follows. First, we know that $\mathrm{diam}(O) = 2r, |\sigma O| = \pi\sigma^2 r^2$, and

$$|u(x, y)| \leq |x|^3 + 6|x|^2|y| + 3|x||y|^2 + 2|y|^3 \leq 12r^3. \tag{4.7}$$

Applying (2.15), we have

$$\begin{aligned}\int_O |\mathrm{d}u|^p \ln^{q-p}(2 + |\mathrm{d}u|)\mathrm{d}x &\leq C \int_{\sigma O} (\mathrm{diam}(O)^{-1}|u|)^p \ln^{q-p}(2 + \mathrm{diam}(O)^{-1}|u|)\mathrm{d}x \\ &\leq C \int_{\sigma O} \left(\frac{12r^3}{2r}\right)^p \ln^{q-p}\left(2 + \frac{12r^3}{2r}\right)\mathrm{d}x \\ &= C6^p r^{2p} \ln^{q-p}(2 + 6r^2) \int_{\sigma O} \mathrm{d}x \\ &= C6^p \pi\sigma^2 r^{2p+2}(1 + 6r^2)^{q-p}.\end{aligned} \tag{4.8}$$

As to the solution of Equation (4.5), until now we can not get the whole solution space, but we can give a kind of solutions with some character in $\mathbb{R}^3$. A special kind of solutions of 1-form with 3-independent variables satisfying Equation (4.5) is constructed by simple calculation. Since $d^{\star}d = ((-1)^{3*1+1} \star d\star)d = (\star d)^2$, let $u = a\mathrm{d}x_1 + b\mathrm{d}x_2 + c\mathrm{d}x_3$, where a, b, c are functions with second-order continuous partial derivatives about three independent variables x_1, x_2, x_3.

$$\mathrm{d}u = (b_{x_1} - a_{x_2})\mathrm{d}x_1 \wedge \mathrm{d}x_2 + (c_{x_1} - a_{x_3})\mathrm{d}x_1 \wedge \mathrm{d}x_3 + (c_{x_2} - b_{x_3})\mathrm{d}x_2 \wedge \mathrm{d}x_3; \tag{4.9}$$

$$\star\mathrm{d}u = (b_{x_1} - a_{x_2})\mathrm{d}x_3 + (a_{x_3} - c_{x_1})\mathrm{d}x_2 + (c_{x_2} - b_{x_3})\mathrm{d}x_1; \tag{4.10}$$

$$\begin{aligned}\mathrm{d} \star \mathrm{d}u = (a_{x_1x_3} + b_{x_2x_3} - c_{x_1x_1} - c_{x_2x_2})\mathrm{d}x_1 \wedge \mathrm{d}x_2 + (b_{x_1x_1} + b_{x_3x_3} - a_{x_1x_2} - c_{x_2x_3})\mathrm{d}x_1 \wedge \mathrm{d}x_3 \\ + (b_{x_1x_2} + c_{x_1x_3} - a_{x_2x_2} - a_{x_3x_3})\mathrm{d}x_2 \wedge \mathrm{d}x_3;\end{aligned} \tag{4.11}$$

$$\begin{aligned}\star\mathrm{d} \star \mathrm{d}u = (a_{x_1x_3} + b_{x_2x_3} - c_{x_1x_1} - c_{x_2x_2})\mathrm{d}x_3 + (a_{x_1x_2} + c_{x_2x_3} - b_{x_1x_1} - b_{x_3x_3})\mathrm{d}x_2 \\ + (b_{x_1x_2} + c_{x_1x_3} - a_{x_2x_2} - a_{x_3x_3})\mathrm{d}x_1;\end{aligned} \tag{4.12}$$

Example 4.2 Let

$$\begin{aligned}a_{x_1x_3} + b_{x_2x_3} - c_{x_1x_1} - c_{x_2x_2} = 0, \\ a_{x_1x_2} + c_{x_2x_3} - b_{x_1x_1} - b_{x_3x_3} = 0, \\ b_{x_1x_2} + c_{x_1x_3} - a_{x_2x_2} - a_{x_3x_3} = 0,\end{aligned} \tag{4.13}$$

then the u with coefficients a, b, c satisfying Equation (4.13) is the solution of Equation (4.5). Let $a = x_2^4 + x_3^4 + 4(x_2 + x_3)x_1^3 + (x_2^3 + 6x_1x_2x_3 + x_3^3)$,$b = x_1^4 + x_3^4 + 4(x_1 + x_3)x_2^3 + (x_1^3 + 6x_1x_2x_3 + x_3^3)$, $c = x_1^4 + x_2^4 + 4(x_1 + x_2)x_3^3 + (x_1^3 + 6x_1x_2x_3 + x_2^3)$, and $u = a\mathrm{d}x_1 + b\mathrm{d}x_2 + c\mathrm{d}x_3$. Let $r > 0$ be a constant,and $\{x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 \leq r^2\}$. It is not hard to check that $\mathrm{d}u \neq 0$ and u satisfies Equation (4.13). To obtain the upper bound for the $\int_O \varphi(|\mathrm{d}u|)\mathrm{d}x$, we calculate with Caccioppoli inequality (2.15) with $c = 0$ as follows. First, we know that $\mathrm{diam}(O) = 2r$, $|\sigma O| = 4\pi\sigma^3 r^3/3$, and

$$\begin{aligned}|u(x_1, x_2, x_3)| = (|a|^2 + |b|^2 + |c|^2)^{\frac{1}{2}} &\leq \sqrt{3}(2r^4 + 4 * (2r)r^3 + (r^3 + 6r^3 + r^3)) \\ &= \sqrt{3}r^3(10r + 8).\end{aligned} \tag{4.14}$$

By (2.15), it follows that

$$\begin{aligned}\int_O |\mathrm{d}u|^p \ln^{q-p}(2+|\mathrm{d}u|)\mathrm{d}x &\le C\int_{\sigma O}\left(\mathrm{diam}(O)^{-1}|u|\right)^p \ln^{q-p}\left(2+\mathrm{diam}(O)^{-1}|u|\right)\mathrm{d}x\\ &\le C(\sqrt{3}r^2(10r+8)/2)^p \ln^{q-p}(2+\sqrt{3}r^2(10r+8)/2)\int_{\sigma O}\mathrm{d}x\\ &\le 4C\pi\sigma^3(\sqrt{3}/2)^p r^{2p+3}(10r+8)^p(1+\sqrt{3}r^2(10r+8)/2)^{q-p}/3.\end{aligned} \tag{4.15}$$

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具有非标准增长条件的微分形式的**Caccioppoli** 不等式及其高阶可积性

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摘 要: 本文主要研究了微分形式中的相关不等式. 利用A-调和方程的性质及与该方程相关的弱逆Holder 不等式和一类满足非标准增长条件的Young 函数的性质, 获得了一类特殊的微分形式(即非齐次A-调和张量)在该类Young 函数作用下的Caccoppoli 不等式及其高阶可积性. 该结论将微分形式中Caccoppoli 不等式由Lp 空间推广到了由该类Young 函数构成的Orlicz 空间, 同时验证了该Caccoppoli 不等式可以用于微分形式的定量估计和定性分析.

关键词: Caccioppoli不等式; 高阶可积性; 微分形式; 非标准增长条件; A-调和方程

MR(2010)**主题分类号**: 58A10; 42B37 **中图分类号**: O186.15