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OSCILLATION OF FRACTIONAL NEUTRAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract: In this paper, we study the oscillation of a fractional neutral partial differential equation subject to Neumann boundary condition. Using an integration average technique and the Laplace transform, we obtain some new oscillation criteria, which are the generalization of some classical results involving partial differential equations. Some examples are given to show the applications of our main results.

Keywords: fractional differential equations; neutral; oscillation; laplace transform

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1 Introduction

Fractional calculus has become of increasing use for analyzing not only stochastic processes driven by fractional Brownian processes [1–3], but also nonrandom fractional phenomena in physics [4–5], like the study of porous systems, for instance, and quantum mechanics [6–10]. Fractional differential equations have gained increasing attention due to their various applications. Many important research results have been obtained for the initial value problem, stability, attractiveness, boundary value problem, bifurcation, etc. The study of the oscillatory problem with a view on fractional differential equation is just being initiated. As a new cross-cutting area, recently some attention has been paid to oscillations of fractional differential equations [11–20]. In[11], Meng et al. studied the oscillation of linear fractional order delay differential equations

$$^{C}D_{-}^{\alpha}x(t)-px(t-\tau)=0,$$

where $^{C}D_{-}^{\alpha}x(t)$ denotes the Caputo fractional derivative for a function $x(t)$, $^{C}D_{-}^{\alpha}x(t)=-\frac{1}{\Gamma(1-\alpha)}\int_t^{\infty}\frac{f'(s)}{(s-t)^{\alpha}}ds, t\in R^{+}$. By the Laplace transform the authors obtained a sufficient and necessary condition.

In[14], Zhou et al. studied oscillatory behavior of the fractional differential equation of the form

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=C(t)\Delta u+\sum_{i=0}^{n}P_i(x)u(x,t-\sigma_i)+R(x,t),$$

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supplemented with the initial condition

$$\left.\frac{\partial^{\alpha-1}u(x,t)}{\partial t^{\alpha-1}}\right|_{t\in[-\sigma,0]}=\varphi(x,t)\ for\ x\in\Omega, where\ \sigma=\max\{\sigma_i,i=1,2,...,n\},$$

and boundary conditions:

$$\begin{aligned}\frac{\partial u(x,t)}{\partial N}&=0,(x,t)\in\partial\Omega\times[0,\infty),\\ u(x,t)&=0,(x,t)\in\partial\Omega\times[0,\infty),\\ \frac{\partial u(x,t)}{\partial N}+uv&=0,(x,t)\in\partial\Omega\times[0,\infty),\end{aligned}$$

where $0<\alpha<1$ is a constant, $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is the Riemann-Liouville fractional derivative of order α with respect to t of a function $u(x,t)$. Using the Laplace transform and the inequality technique, the author established some new oscillation criteria.

Motivated by the analysis above, in this paper, we are concerned with oscillation for a class of fractional differential equations as follows

$$\begin{aligned}&{}_0D_t^\alpha(u(x,t)+\sum_{i=1}^{n}r_iu(x,t-\sigma_i))\\ =&a(t)h(u(x,t))\Delta u(x,t)+\sum_{i=1}^{l}b_i(t)h_i(u(x,t-\zeta_i))\Delta u(x,t-p_i)-\sum_{i=1}^{m}q_i(t)f_i(u(x,t-\tau_i)),\end{aligned}\tag{1.1}$$

where $0<\alpha=\frac{odd\text{integer}}{odd\text{integer}}<1$, ${}_0D_t^\alpha x(t)$ is Riemann-Liouville fractional derivative of order, and $(x,t)\in\Omega\times(0,\infty)=G$. Here $\Omega\subset R^N$ is a bounded domain with boundary $\partial\Omega$ smooth enough. The hypotheses are always true as follows:

(H1) : $r_i,\sigma_i,\zeta_i,p_i,\tau_i\in R^+$; $a(t),b_i(t),q_i(t)\in C(R^+,R^+)$; $q_i=\inf q_i(t)>0$.
(H2) : $f_i(u)\in C(R,R)$, $f_i(u)/u\geq C_i=const>0$, for $u\neq 0$.
(H3) : $h(u)\in C(R,R)$, $uh'(u)\geq 0$.

The initial condition $u(x,t)=\phi(x,t)$, $\phi(x,t)\in C(\Omega\times[-\rho,0],R)$, $(x,t)\in\Omega\times[-\rho,0]$, $\rho=\max\{\sigma_i,\zeta_j,p_j,\tau_k,i=1,2,...,n;j=1,2,...,l;k=1,2,...,m\}$. Consider the boundary conditions as follows :

$$\frac{\partial u(x,t)}{\partial N}=0, on\ (x,t)\in\partial\Omega\times R^+,\tag{1.2}$$

where N is the unit exterior normal vector in $\partial\Omega$.

This paper is organized as follows. In the next section, we introduce some useful preliminaries. In section 3, we present various sufficient conditions for oscillation of all solutions to the system (1.1) by using fractional calculus, Laplace transform and Green's function. Finally, we provide some examples to show applications of our criteria.

2 Preliminaries

In this section, we introduce preliminary facts which are used throughout this paper.

Definition 2.1 [21] The fractional integral of order α with the lower limit zero for a function f is defined as

$$({}_0D_t^{-\alpha}f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}ds, t > 0, 0 < \alpha < 1,$$

provided the right side is pointwise defined on $[0, b]$, where $\Gamma(\boldsymbol{.})$ is the gamma function.

Definition 2.2 [21] Riemann-Liouville derivative of order α with the lower limit zero for a function f can be written as

$$({}_0D_t^{\alpha}f)(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{f(s)}{(t-s)^{\alpha}}ds, t > 0, 0 < \alpha < 1.$$

Definition 2.3 A solution of problem (1.1)-(1.2) is said to be oscillatory in G if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory.

We recall some facts about Laplace transforms. If $X(s)$ is the Laplace transform of $x(t)$,

$$X(s) = L[x(t)](s) = \int_0^{\infty} e^{-st}x(t)dt,$$

then the abscissa of convergence of $X(s)$ is defined by $b = \inf\{\gamma \in R : X(\gamma) exists\}$.

Lemma 2.1 [21] Let $(L_0D_t^{\alpha}x)(s)$ be the Laplace transform of the Riemann-Liouville fractional derivative of order α with the lower limit zero for a function x, and $X(s)$ is the Laplace transform of $x(t)$. Further, for $x \in AC[0, b]$ and for any $b > 0$, $|x(t)| \le Ae^{m_0 t}, t > b > 0$ holds for constant $A > 0$ and $m_0 > 0$. Then the relation

$$(L_0D_t^{\alpha}x)(s) = s^{\alpha}X(s) - ({}_0D_t^{\alpha-1}x)(t)\big|_{t=0}, 0 < \alpha < 1$$

is valid for $\mathrm{Re}(s) > m_0$.

3 Main Results

Lemma 3.1 ${}_0D_t^{\alpha}x$ is the Riemann-liouville derivative of order α with the lower limit zero for a function $x(t)$, $X(s)$ is the Laplace transform of $x(t)$, $\sigma > 0$, $0 < \alpha < 1$, then the following relation holds.

$$L\left[{}_0D_t^{\alpha}x(t-\sigma)\right](s) = s^{\alpha}e^{-s\sigma}X(s) + s^{\alpha}e^{-s\sigma}\int_{-\sigma}^{0} e^{-st}x(t)dt - ({}_0D_t^{\alpha-1}x(t-\sigma))\big|_{t=0}.$$

Proof

$$\begin{aligned}
L\left[{}_0D_t^{\alpha}x(t-\sigma)\right](s) &= L\left[\frac{d({}_0D_t^{\alpha-1}x(t-\sigma))}{dt}\right](s)\\
&= sL\left[{}_0D_t^{\alpha-1}x(t-\sigma)\right](s) - ({}_0D_t^{\alpha-1}x(t-\sigma))\big|_{t=0}\\
&= sL\left[\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}x(s-\sigma)ds\right](s) - ({}_0D_t^{\alpha-1}x(t-\sigma))\big|_{t=0}\\
&= sL\left[\frac{t^{-\alpha}}{\Gamma(1-\alpha)} * x(t-\sigma)\right](s) - ({}_0D_t^{\alpha-1}x(t-\sigma))\big|_{t=0}\\
&= sL\left[\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right]L\left[x(t-\sigma)\right] - ({}_0D_t^{\alpha-1}x(t-\sigma))\big|_{t=0},
\end{aligned}$$

where $L\left[x(t-\sigma)\right]=e^{-s\sigma}X(s)+e^{-s\sigma}\int_{-\sigma}^{0}e^{-st}x(t)dt$. So

$$L\left[{}_0D_t^{\alpha}x(t-\sigma)\right](s)=s^{\alpha}e^{-s\sigma}X(s)+s^{\alpha}e^{-s\sigma}\int_{-\sigma}^{0}e^{-st}x(t)dt-\left({}_0D_t^{\alpha-1}x(t-\sigma)\right)\Big|_{t=0}.$$

The proof is complete.

Theorem 3.1 If the fractional differential inequality

$${}_0D_t^{\alpha}(v(t)+\sum_{i=1}^{n}r_iv(t-\sigma_i))+\sum_{i=1}^{m}q_iC_iv(t-\tau_i)\le 0 \tag{3.1}$$

has no eventually positive solutions and the fractional differential inequality

$${}_0D_t^{\alpha}(v(t)+\sum_{i=1}^{n}r_iv(t-\sigma_i))+\sum_{i=1}^{m}q_iC_iv(t-\tau_i)\ge 0 \tag{3.2}$$

has no eventually negative solutions, then every nontrivial solution of the problems (1.1) and (1.2) is oscillatory.

Proof Assume that (1.1) with the boundary condition (1.2) has no oscillation solution, without loss of generality, we assume that $u(x,t)$ is an eventually positive solution of (1.1) and (1.2) which implies that there exists $T>0$ such that $u(x,t)>0, u(x,t-\sigma_i)>0, u(x,t-\tau_i)>0$ in $\Omega\times[T,\infty)$. Since (H1) and (H2), from (1.1), we can obtain

$$\begin{aligned}&{}_0D_t^{\alpha}(u(x,t)+\sum_{i=1}^{n}r_iu(x,t-\sigma_i))\\ \le& a(t)h(u(x,t))\Delta u(x,t)+\sum_{i=1}^{l}b_i(t)h_i(u(x,t-\zeta_i))\Delta u(x,t-p_i)-\sum_{i=1}^{m}q_iC_iu(x,t-\tau_i).\end{aligned} \tag{3.3}$$

Integrating (3.3) with respect to x over Ω yields

$$\begin{aligned}{}_0D_t^{\alpha}(\int_{\Omega}u(x,t)dx+\sum_{i=1}^{n}r_i\int_{\Omega}u(x,t-\sigma_i)dx)\le a(t)\int_{\Omega}h(u(x,t))\Delta u(x,t)dx\\ +\sum_{i=1}^{l}b_i(t)\int_{\Omega}h_i(u(x,t-\zeta_i))\Delta u(x,t-p_i)dx-\sum_{i=1}^{m}q_iC_i\int_{\Omega}u(x,t-\tau_i)dx.\end{aligned} \tag{3.4}$$

By Green's formula and the boundary condition (1.2), we have

$$\begin{aligned}&\int_{\Omega}h(u)\Delta u(x,t)dx=\int_{\partial\Omega}h(u)\frac{\partial u(x,t)}{\partial N}dS-\int_{\Omega}h'(u)|grad\ u|^2dx\\ =&-\int_{\Omega}h'(u)|grad\ u|^2dx\le 0,\\ &\int_{\Omega}h_i(u(x,t-\zeta_i))\Delta u(x,t-p_i)dx\le 0.\end{aligned} \tag{3.5}$$

Let $v(t)=\int_\Omega u(x,t)dx$, from (3.4) and (3.5), we get

$$ {}_0D_t^\alpha(v(t)+\sum_{i=1}^n r_i v(t-\sigma_i))+\sum_{i=1}^m q_iC_i v(t-\tau_i)\le 0. \tag{3.6}$$

That is, there exists eventually positive solution for the inequality (3.1) which contradicts the conditions of theorem.

Secondly, if $u(x,t)$ is an eventually negative solution of the problem (1.1) and (1.2), then using above procedure, we can easily show that $v(t)=\int_\Omega u(x,t)dx$ is an eventually negative solution of the fractional differential inequality (3.2) which again contradicts the conditions of theorem. This completes the proof.

Theorem 3.2 Assume that $\tau>\sigma$, $\tau=\min\{\tau_i, i=1,2,...,m\}$, $\sigma=\max\{\sigma_i, i=1,2,...,n\}$, if

$$p(\lambda)=\lambda^\alpha+\lambda^\alpha\sum_{i=1}^n r_i e^{-\lambda\sigma_i}+\sum_{i=1}^m q_iC_i e^{-\lambda\tau_i}=0 \tag{3.7}$$

has no real roots, then every solution of (1.1) and (1.2) is oscillatory.

Proof Suppose that $u(x,t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $u(x,t)$ is an eventually positive solution of (1.1). We proceed as in the proof of theorem 3.1 to get that (3.6) holds. Taking Laplace transform of both sides of (3.6), we obtain

$$\begin{gathered} s^\alpha V(s)-A+V(s)\sum_{i=1}^n r_i s^\alpha e^{-s\sigma_i}+\sum_{i=1}^n r_i s^\alpha e^{-s\sigma_i}\int_{-\sigma_i}^0 e^{-st}v(t)dt-\sum_{i=1}^n B_i r_i+ \\ V(s)\sum_{i=1}^m q_iC_i e^{-s\tau_i}+\sum_{i=1}^m q_iC_ie^{-s\tau_i}\int_{-\tau_i}^0 e^{-st}v(t)dt\le 0, \end{gathered}$$

where $V(s)=\int_0^\infty e^{-st}v(t)dt$, $A=({}_0D_t^{\alpha-1}v(t))\big|_{t=0}$, $B_i=({}_0D_t^{\alpha-1}v(t-\sigma_i))\big|_{t=0}$. Hence

$$\begin{aligned} &V(s)(s^\alpha+\sum_{i=1}^n r_i s^\alpha e^{-s\sigma_i}+\sum_{i=1}^m q_iC_ie^{-s\tau_i}) \\ \le& A+\sum_{i=1}^n B_ir_i-\sum_{i=1}^n r_i s^\alpha e^{-s\sigma_i}\int_{-\sigma_i}^0 e^{-st}v(t)dt-\sum_{i=1}^m q_iC_ie^{-s\tau_i}\int_{-\tau_i}^0 e^{-st}v(t)dt. \end{aligned} \tag{3.8}$$

Let

$$\begin{aligned} p(s)&=s^\alpha+\sum_{i=1}^n r_i s^\alpha e^{-s\sigma_i}+\sum_{i=1}^m q_iC_ie^{-s\tau_i}, \\ \Phi(s)&=A+\sum_{i=1}^n B_ir_i-\sum_{i=1}^n r_i s^\alpha e^{-s\sigma_i}\int_{-\sigma_i}^0 e^{-st}v(t)dt-\sum_{i=1}^m q_iC_ie^{-s\tau_i}\int_{-\tau_i}^0 e^{-st}v(t)dt, \end{aligned}$$

then from (3.8) we get

$$V(s)p(s) \leq \Phi(s). \tag{3.9}$$

Since $p(s)=0$ has no real roots and $p(0)>0$, $p(s)>0$. By positivity of $v(t)$ in $[-\rho,0]$, there exists a constant $z>0$ such that $z<v(t)$. Since $r_i>0$, $\tau>\sigma$ then

$$\begin{aligned}
\frac{\sum\limits_{i=1}^{m} q_iC_ie^{-s\tau_i}\int_{-\tau_i}^{0} e^{-st}v(t)dt}{-\sum\limits_{i=1}^{n} r_is^{\alpha}e^{-s\sigma_i}\int_{-\sigma_i}^{0} e^{-st}v(t)dt} &\geq \frac{\sum\limits_{i=1}^{m} q_iC_ie^{-s\tau_i}\int_{-\tau}^{0} e^{-st}v(t)dt}{-\sum\limits_{i=1}^{n} r_is^{\alpha}e^{-s\sigma_i}\int_{-\sigma}^{0} e^{-st}v(t)dt} \\
&\geq \frac{\sum\limits_{i=1}^{m} q_iC_ie^{-s\tau_i}}{-\sum\limits_{i=1}^{n} r_is^{\alpha}e^{-s\sigma_i}} \geq \frac{\sum\limits_{i=1}^{m} q_iC_ie^{-s\tau}}{-\sum\limits_{i=1}^{n} r_is^{\alpha}e^{-s\sigma}} \\
&= \frac{\sum\limits_{i=1}^{m} q_iC_i}{\sum\limits_{i=1}^{n} r_i}\frac{e^{-s(\tau-\sigma)}}{(-s)^{\alpha}} \to +\infty \ (s\to-\infty).
\end{aligned}$$

Thus there exists a constant $k<0$ such that $s<k$,

$$\frac{\sum\limits_{i=1}^{m} q_iC_ie^{-s\tau_i}\int_{-\tau_i}^{0} e^{-st}v(t)dt}{-\sum\limits_{i=1}^{n} r_is^{\alpha}e^{-s\sigma_i}\int_{-\sigma_i}^{0} e^{-st}v(t)dt} \geq 2.$$

Then,

$$\begin{aligned}
&-\sum_{i=1}^{n} r_is^{\alpha}e^{-s\sigma_i}\int_{-\sigma_i}^{0} e^{-st}v(t)dt - \sum_{i=1}^{m} q_iC_ie^{-s\tau_i}\int_{-\tau_i}^{0} e^{-st}v(t)dt \\
&= -\sum_{i=1}^{n} r_is^{\alpha}e^{-s\sigma_i}\int_{-\sigma_i}^{0} e^{-st}v(t)dt\Bigg(1-\frac{\sum\limits_{i=1}^{m} q_iC_ie^{-s\tau_i}\int_{-\tau_i}^{0} e^{-st}v(t)dt}{-\sum\limits_{i=1}^{n} r_is^{\alpha}e^{-s\sigma_i}\int_{-\sigma_i}^{0} e^{-st}v(t)dt}\Bigg) \\
&\leq \sum_{i=1}^{n} r_is^{\alpha}e^{-s\sigma_i}\int_{-\sigma_i}^{0} e^{-st}v(t)dt \leq \sum_{i=1}^{n} r_is^{\alpha}e^{-s\sigma_i}z\int_{-\sigma_i}^{0} e^{-st}dt \\
&= \sum_{i=1}^{n} r_is^{\alpha}z\frac{1}{s}\frac{e^{s\sigma_i}-1}{e^{s\sigma_i}} = \sum_{i=1}^{n} r_iz\frac{1-e^{-s\sigma_i}}{(-s)^{1-\alpha}} \to -\infty \ (s\to-\infty).
\end{aligned}$$

Thus we conclude that $\Phi(s)\to-\infty(s\to-\infty)$, but $p(s)$ and $V(s)$ are positive. Hence, (3.9) leads to a contradiction. The proof is complete.

Theorem 3.3 Assume that $\tau>\sigma$, $\tau=\min\{\tau_i, i=1,2,...,m\}$, $\sigma=\max\{\sigma_i, i=$

$1, 2, ..., n\}$, if

$$\left(\frac{1}{m}\sum_{i=1}^{m}\tau_i - \frac{1}{n}\sum_{i=1}^{n}\sigma_i\right)\left(\frac{m\left(\prod\limits_{i=1}^{m} q_iC_i\right)^{\frac{1}{m}}}{1+n\left(\prod\limits_{i=1}^{n} r_i\right)^{\frac{1}{n}}}\right)^{\frac{1}{\alpha}} > \frac{1}{e}, \tag{3.10}$$

then every solution of (1.1) and (1.2) is oscillatory.

Proof By using the arithmetic-geometric mean inequality $\sum\limits_{i=1}^{m} a_i \geq m\left(\prod\limits_{i=1}^{m} a_i\right)^{\frac{1}{m}}$, for $\lambda < 0$, we find

$$\begin{aligned}\lambda^\alpha + \lambda^\alpha\sum_{i=1}^{n} r_ie^{-\lambda\sigma_i} + \sum_{i=1}^{m} q_iC_ie^{-\lambda\tau_i} &\geq \lambda^\alpha + \lambda^\alpha n\left(\prod_{i=1}^{n} r_ie^{-\lambda\sigma_i}\right)^{\frac{1}{n}} + m\left(\prod_{i=1}^{m} q_iC_ie^{-\lambda\tau_i}\right)^{\frac{1}{m}}\\ &= \lambda^\alpha + \lambda^\alpha n\left(\prod_{i=1}^{n} r_i\right)^{\frac{1}{n}} e^{-\lambda\frac{\sum\limits_{i=1}^{n}\sigma_i}{n}} + m\left(\prod_{i=1}^{m} q_iC_i\right)^{\frac{1}{m}} e^{-\lambda\frac{\sum\limits_{i=1}^{m}\tau_i}{m}}\\ &= \lambda^\alpha + \lambda^\alpha Ae^{-\lambda B} + Ce^{-\lambda D}.\end{aligned}$$

Let

$$f(\lambda) = \lambda^\alpha + \lambda^\alpha Ae^{-\lambda B} + Ce^{-\lambda D} \tag{3.11}$$

where $A = n\left(\prod\limits_{i=1}^{n} r_i\right)^{\frac{1}{n}}, B = \frac{1}{n}\sum\limits_{i=1}^{n}\sigma_i, C = m\left(\prod\limits_{i=1}^{m} q_iC_i\right)^{\frac{1}{m}}, D = \frac{1}{m}\sum\limits_{i=1}^{m}\tau_i$ and $D > B$.

Assume that Eq. (3.11) has a real roots λ_1, if $\lambda_1 \geq 0$, then $f(\lambda_1) > 0$, it is impossible. Thus we conclude that $\lambda_1 < 0$. Since α is the ratio of two odd integers, it follows from (3.11) that

$$\begin{aligned}&\lambda_1^\alpha + \lambda_1^\alpha Ae^{-\lambda_1 B} + Ce^{-\lambda_1 D} = 0\\ &\lambda_1^\alpha = -\frac{Ce^{-\lambda_1 D}}{1+Ae^{-\lambda_1 B}} = -\frac{C}{e^{\lambda_1 D} + Ae^{\lambda_1(D-B)}} \leq -\frac{C}{1+A}.\end{aligned}$$

Then

$$(-\lambda_1)^\alpha \geq \frac{C}{1+A}, \quad (-\lambda_1)^{1-\alpha} \geq \left(\frac{C}{1+A}\right)^{\frac{1-\alpha}{\alpha}}. \tag{3.12}$$

By (3.12) and the inequality $e^x \geq ex$ for $x \geq 0$, we get

$$\begin{aligned}(-\lambda_1)^\alpha &= \frac{Ce^{-\lambda_1 D}}{1+Ae^{-\lambda_1 B}} = \frac{Ce^{\lambda_1(B-D)}}{e^{\lambda_1 B} + A} \geq \frac{Ce\lambda_1(B-D)}{1+A} = \frac{Ce(D-B)}{1+A}(-\lambda_1)\\ &= \frac{Ce(D-B)}{1+A}(-\lambda_1)^\alpha(-\lambda_1)^{1-\alpha} \geq \frac{Ce(D-B)}{1+A}(-\lambda_1)^\alpha\left(\frac{C}{1+A}\right)^{\frac{1-\alpha}{\alpha}},\end{aligned}$$

which implies that $1 \geq e(D-B)\left(\frac{C}{1+A}\right)^{\frac{1}{\alpha}}$, which contradicts the conditions (3.10). The proof is complete.

4 Example

In this section we give examples to illustrate our results.

Example 4.1 Consider the following fractional differential equation

$$\begin{aligned}&{}_0D_t^{1/5}(u(x,t)+\frac{1}{4}u(x,t-\frac{1}{3})+u(x,t-\frac{1}{2}))\\&=e^tu^2\Delta u(x,t)+u^4(x,t-3)\Delta u(x,t-\frac{1}{2})+t^2u^6(x,t-5)\Delta u(x,t-\frac{4}{5})-\\&[(t+\frac{1}{t})u(x,t-\frac{3}{2})+\sin(u(x,t-1))+2u(x,t-1)]\end{aligned}\tag{4.1}$$

with the boundary conditions

$$\frac{\partial u(0,t)}{\partial x}=\frac{\partial u(5,t)}{\partial x}=0,(x,t)\in(0,5)\times(0,\infty).$$

Notice $\alpha=\frac{1}{5}$, $r_1=\frac{1}{4}$, $r_2=1$, $\sigma_1=\frac{1}{3}$, $\sigma_2=\frac{1}{2}$, $a(t)=e^t$, $h(u)=u^2$,$h_1(u)=u^4$, $h_2(u)=u^6$, $b_1(t)=1$, $b_2(t)=t^2$, $\zeta_1=3$, $\zeta_2=5$, $p_1=\frac{1}{2}$, $p_2=\frac{4}{5}$, $q_1(t)=t+\frac{1}{t}$, $q_2(t)=1$, $f_1(u)=u$, $f_2(u)=\sin(u)+2u$, $\tau_1=\frac{3}{2}$, $\tau_2=1$, then it is easy to find $q_1=2,\ q_2=1, C_1=1,\ C_2=1$. Therefore, $(\frac{1}{m}\sum\limits_{i=1}^{m}\tau_i-\frac{1}{n}\sum\limits_{i=1}^{n}\sigma_i)\left(\frac{m\left(\prod\limits_{i=1}^{m}q_iC_i\right)^{\frac{1}{m}}}{1+n\left(\prod\limits_{i=1}^{n}r_i\right)^{\frac{1}{n}}}\right)^{\frac{1}{\alpha}}=\frac{10\sqrt{2}}{3}>\frac{1}{e}$, then (4.1) is oscillatory by Theorem 3.3.

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中立型分数阶偏微分方程的振动性

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摘要: 本文研究了一类中立型分数阶偏微分方程的振动性, 利用积分平均值方法和拉普拉斯变换, 得到了方程振动新的准则, 推广了中立型偏微分方程振动的一些经典结论.

关键词: 分数阶微分方程; 中立型; 振动; 拉普拉斯变换

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