

SELF-ADJOINT AND CO-ISOMETRIC COMPOSITION OPERATOR ON QUATERNIONIC FOCK SPACE

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Abstract: In this paper, the properties of self-adjoint and co-isometric composition operators on the quaternionic Fock space are studied. In combination with the density of reproducing kernels, we obtain the characterizations of the necessary and sufficient conditions for self-adjoint and co-isometric composition operators, and we get the co-isometric composition operator is unitary on the quaternionic Fock space.

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1 Introduction

The study of slice regular functions of a quaternionic variable starts from the important works by Gentili and Struppa [1, 2], which generalize the classical theory of holomorphic functions of one complex variable. Later on, various kinds of slice regular function spaces appear, such as Fock space ([3, 4]), Bloch, Besov and Dirichlet spaces ([5]) and Bergman space ([6]) in the slice regular settings. At the same time, the characterizations for some classical linear operators acting on the above quaternionic spaces are also in demand. Naturally, there are some fundamental questions about the properties of composition operator on the above quaternionic spaces. Considering the importance of quaternionic Fock space in quantum mechanics and quaternionic formulation (see e.g. [3, 7]), we will concentrate on characterizing the composition operators acting on quaternionic Fock space.

Given a linear operator T acting between two normed linear spaces X and Y , T is said to be isometric if $\|Tf\|_Y = \|f\|_X$ for all $f \in X$. For a Hilbert space \mathcal{H} , the linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ is isometric if $T^*T = I$, where I is the identity operator. Further, if the isometric operator T is also onto, it is called a unitary operator defined as $TT^* = T^*T = I$. Particularly, $T : \mathcal{H} \rightarrow \mathcal{H}$ is co-isometric if $TT^* = I$, meaning that T^* is isometric. Hereafter, we recall $T : \mathcal{H} \rightarrow \mathcal{H}$ is self-adjoint if $T^* = T$. For composition operator acting on quaternionic Fock space, its boundedness and compactness are investigated in [8, 9] and the references

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therein. Following this line, we continue to deal with the self-adjointness and co-isometry of composition operators on the quaternionic Fock space in Section 3 and Section 4, respectively.

2 Preliminaries

2.1 Quaternions and Slice Regular Functions

Let \mathbb{H} be the noncommutative, associative, real algebra of quaternions with standard basis $\{1, i, j, k\}$, subject to $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. So the symbol \mathbb{H} is the set of the quaternions $q = x_0 + x_1i + x_2j + x_3k = \text{Re}(q) + \text{Im}(q)$ with $\text{Re}(q) = x_0$ and $\text{Im}(q) = x_1i + x_2j + x_3k$, where $x_j \in \mathbb{R}$ for $j = 0, 1, 2, 3$. Besides, the conjugate of q is defined by $\bar{q} = \text{Re}(q) - \text{Im}(q) = x_0 - (x_1i + x_2j + x_3k)$. And it holds that $\overline{pq} = \bar{q}\bar{p}$ for all $p, q \in \mathbb{H}$. The Euclidean norm of a quaternion q then gives

$$|q| = \sqrt{q\bar{q}} = \sqrt{\bar{q}q} = \sqrt{\sum_{i=0}^3 x_i^2}$$

for $x_i \in \mathbb{R}$.

We use the symbol \mathbb{S} to stand for the two-dimensional unit sphere of purely imaginary quaternion, i.e. $\mathbb{S} = \{q = x_1i + x_2j + x_3k : x_1^2 + x_2^2 + x_3^2 = 1\}$. Note that $I^2 = -1$ for each $I \in \mathbb{S}$. For any fixed $I \in \mathbb{S}$ we set $\mathbb{C}_I := \{x + Iy : x, y \in \mathbb{R}\}$, which can be considered as a complex plane in \mathbb{H} passing through 0, 1 and I . An element in $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$ (also called slices) is denoted by $x + Iy$. Furthermore, it holds that

$$\mathbb{H} = \bigcup_{I \in \mathbb{S}} \mathbb{C}_I.$$

Interestingly, the real axis belongs to \mathbb{C}_I for every $I \in \mathbb{S}$. And any non real quaternion q is uniquely associated to the element $I_q \in \mathbb{S}$ given by $I_q := (ix_1 + jx_2 + kx_3)/|ix_1 + jx_2 + kx_3|$, so q belongs to the complex plane \mathbb{C}_{I_q} . Now we present the definition of slice regular function.

Definition 2.1 [10, Definition 2.1.1] Let U be an open set in \mathbb{H} and a function $f : U \rightarrow \mathbb{H}$ be real differentiable. The function f is called (left) slice regular if, for every $I \in \mathbb{S}$, its restriction $f_I(x + Iy) = f(x + Iy)$ is holomorphic, i.e. It has continuous partial derivatives and satisfies

$$\bar{\partial}_I f(x + yI) := \frac{1}{2} \left(\frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + yI) = 0$$

for all $x + yI \in U \cap \mathbb{C}_I$. The class of all slice regular functions on U is denoted by $\mathcal{R}(U)$. In addition, $\mathcal{R}(\mathbb{H})$ will be called as the space of entire slice regular functions if $U = \mathbb{H}$.

Let $I, J \in \mathbb{S}$ be such that I and J are orthogonal, so that I, J, IJ is an orthogonal basis of \mathbb{H} and write the restriction f_I as the function $f = f_0 + If_1 + Jf_2 + IJf_3$ on the complex plane \mathbb{C}_I . It can further be written as $f_I = F + GJ$ with $F = f_0 + If_1$ and $G = f_2 + If_3$. Let

U be an open set in \mathbb{H} and the following splitting lemma holds, which relates slice regularity with classical holomorphy.

Lemma 2.2 [10, Lemma 2.1.4] (Splitting Lemma) If f is a slice regular function on U , then for every $I \in \mathbb{S}$, and every $J \in \mathbb{S}$, perpendicular to I , there are two holomorphic functions $F, G : U_I = U \cap \mathbb{C}_I \rightarrow \mathbb{C}_I$ such that for any $z = x + yI \in U_I$, it is $f_I(z) = F(z) + G(z)J$.

It is proved that slice regular functions possess good properties on specific open sets called axially symmetric slice domains.

Definition 2.3 [10, Definition 2.2.1] Let $U \subset \mathbb{H}$ be a domain.

- (1) U is called a slice domain (or s -domain for short) if it intersects the real axis and if, for any $I \in \mathbb{S}$, $U_I := \mathbb{C}_I \cap U$ is a domain in \mathbb{C}_I .
- (2) U is axially symmetric if for every $x + yI \in U$ with $x, y \in \mathbb{R}$ and $I \in \mathbb{S}$, all the elements $x + y\mathbb{S} = \{x + yJ : J \in \mathbb{S}\}$ are contained in U .

The representation formula of a slice regular function on an axially symmetric domain allows to recover all its values from its values on a single slice \mathbb{C}_I .

Proposition 2.4 [10, Theorem 2.2.4] (Representation Formula) Let f be a slice regular function on an axially symmetric s -domain $U \subset \mathbb{H}$. Let $J \in \mathbb{S}$ and let $x \pm yJ \in U \cap \mathbb{C}_J$, then the following equality holds for all $q = x + yI \in U$,

$$f(x + yI) = \frac{1}{2} [(1 + IJ)f(x - yJ) + (1 - IJ)f(x + yJ)].$$

2.2 Quaternionic Fock Space

We first recall the definition of quaternionic Fock space.

Definition 2.5 [3, Definition 3.6] Let I be any elements in \mathbb{S} and $p|_{\mathbb{C}_I} = z$, consider the set

$$\mathcal{F}^2(\mathbb{H}) = \{f \in \mathcal{R}(\mathbb{H}) : \|f\| = \int_{\mathbb{C}_I} e^{-|z|^2} |f_I(z)|^2 d\sigma(x, y) < \infty\},$$

where $d\sigma(x, y) := \frac{1}{\pi} dx dy$. $\mathcal{F}^2(\mathbb{H})$ is called the (slice regular) quaternionic Fock space.

It has been proved the definition of quaternionic Fock space does not depend on the imaginary unit $I \in \mathbb{S}$. Furthermore, $\mathcal{F}^2(\mathbb{H})$ is a Hilbert space endowed with the inner product

$$\langle f, g \rangle = \int_{\mathbb{C}_I} e^{-|z|^2} \overline{g_I(z)} f_I(z) d\sigma(x, y).$$

Besides, the monomials p^n ($n \in \mathbb{N}$) form an orthogonal basis in $\mathcal{F}^2(\mathbb{H})$ satisfying $\langle p^n, p^n \rangle = n!$. Given a variable $p \in \mathbb{H}$ and a parameter $q \in \mathbb{H}$, setting

$$e_*^{pq} = \sum_{n=0}^{+\infty} \frac{p^n q^n}{n!},$$

it is immediate that e_*^{pq} is a function slice regular in p . Denote

$$K_q(p) := e_*^{p\bar{q}} = \sum_{n=0}^{+\infty} \frac{p^n \bar{q}^n}{n!},$$

and it holds that $\langle f, K_q \rangle = f(q)$ for any $f \in \mathcal{F}^2(\mathbb{H})$ (see, e.g. [3, Theorem 3.10]). This means $K_q(p)$ is the reproducing kernel of $\mathcal{F}^2(\mathbb{H})$ and it is true that $\|K_q\|^2 = \langle K_q, K_q \rangle = K_q(q) = e^{|q|^2}$. Furthermore, let \mathcal{M} be the set of all functions behaving as

$$f(p) = \sum_{k=1}^n K_{q_k}(p) b_k, \quad \forall p \in \mathbb{H},$$

where $b_k, q_k \in \mathbb{H}$ for $k = 1, \dots, n$. Letting $\alpha = 1$ and $\beta = 2$ in the lemma below, it yields \mathcal{M} is dense in $\mathcal{F}^2(\mathbb{H})$.

Lemma 2.6 [4, Theorem 4.10] Let $\alpha > 0$ and $0 < \beta < \infty$. The set \mathcal{M} is dense in the quaternionic Fock spaces $\mathcal{F}_{\text{Slice}}^{\alpha, \beta}(\mathbb{H})$.

2.3 Composition Operator

Let $\varphi : \mathbb{H} \rightarrow \mathbb{H}$ be a slice regular function such that $\varphi(\mathbb{C}_I) \subset \mathbb{C}_I$ for some $I \in \mathbb{S}$. The composition operator $C_\varphi : \mathcal{F}^2(\mathbb{H}) \rightarrow \mathcal{F}^2(\mathbb{H})$ is defined on \mathbb{C}_I by

$$(C_\varphi f)_I(z) = (f \circ \varphi_I)(z) = F \circ \varphi_I(z) + G \circ \varphi_I(z)J$$

for all $f \in \mathcal{F}^2(\mathbb{H})$ with $f_I(z) = F(z) + G(z)J$. And then we can extend $C_\varphi f$ to the whole \mathbb{H} by Proposition 2.4. For the adjoint operator of C_φ , it holds that

$$\langle C_\varphi^* K_p, K_q \rangle = \langle K_p, C_\varphi K_q \rangle = \overline{\langle C_\varphi K_q, K_p \rangle},$$

implying

$$(C_\varphi^* K_p)(q) = \overline{(C_\varphi K_q)(p)}. \quad (2.1)$$

3 self-adjoint composition operator on $\mathcal{F}^2(\mathbb{H})$

Very recently, Lian and Liang creatively investigated the boundedness and compactness of weighted composition operators on quaternionic Fock spaces in [8, Theorem 3.1]. Particularly, let $f = 1$, namely, $F(w) = 1$ and $G(w) = 0$ in Theorem 3.1, it follows the composition operator $C_\varphi : \mathcal{F}^2(\mathbb{H}) \rightarrow \mathcal{F}^2(\mathbb{H})$ is bounded if and only if $\varphi(p) = p\lambda + \varphi(0)$ with $|\lambda| \leq 1$, $\lambda \in \mathbb{C}_I$ and $\sup_{w \in \mathbb{C}_I} e^{|\varphi(w)|^2 - |w|^2} < \infty$. In the sequel, we will always suppose $\varphi \in \mathcal{R}(\mathbb{H})$ and have the form $\varphi(p) = p\mu + c$, with $|\mu| \leq 1$ and $\mu \in \mathbb{C}_I$ such that C_φ is bounded on $\mathcal{F}^2(\mathbb{H})$. Firstly, the equivalent condition for the adjoint of a bounded composition operator to be some composition operator on $\mathcal{F}^2(\mathbb{H})$ is stated as follows.

Theorem 3.7 Let $\varphi_i \in \mathcal{R}(\mathbb{H})$ such that $\varphi_i(\mathbb{C}_I) \subset \mathbb{C}_I$ for some $I \in \mathbb{S}$ and $i = 1, 2$. Denote $\varphi_1(p) = p\mu + c$, $\varphi_2(p) = p\nu + d$, where $|\mu| \leq 1$, $|\nu| \leq 1$ and $\mu, \nu \in \mathbb{C}_I$ such that $C_{\varphi_i} : \mathcal{F}^2(\mathbb{H}) \rightarrow \mathcal{F}^2(\mathbb{H})$ is bounded for $i = 1, 2$. Then $C_{\varphi_1}^* = C_{\varphi_2}$ if and only if $c = 0$, $d = 0$ and $\bar{\mu} = \nu$.

Proof Necessity Assume that $C_{\varphi_1}^* = C_{\varphi_2}$, by (2.1), it follows that $(C_{\varphi_1}^* K_p)(q) = \overline{(C_{\varphi_1} K_q)(p)} = (C_{\varphi_2} K_p)(q)$, implying

$$\sum_{n=0}^{\infty} \frac{q^n \left(\overline{\varphi_1(p)}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{(\varphi_2(q))^n \bar{p}^n}{n!}, \quad \forall p, q \in \mathbb{H}. \tag{3.1}$$

Letting $p = 0$ or $q = 0$ respectively in (3.1), we see that

$$\sum_{n=0}^{\infty} \frac{q^n \bar{c}^n}{n!} = 1, \quad \forall q \in \mathbb{H}, \text{ or } \sum_{n=0}^{\infty} \frac{d^n \bar{p}^n}{n!} = 1, \quad \forall p \in \mathbb{H}.$$

This yields $c = 0$ and $d = 0$, respectively. Hence, $\varphi_1(p) = p\mu$, $\varphi_2(p) = p\nu$. Then (3.1) becomes into

$$\sum_{n=0}^{\infty} \frac{q^n (\bar{p}\mu)^n}{n!} = \sum_{n=0}^{\infty} \frac{(q\nu)^n \bar{p}^n}{n!}, \quad \forall p, q \in \mathbb{H}. \tag{3.2}$$

Denote $p|_{\mathbb{C}_I} = z, q|_{\mathbb{C}_I} = w$ and restrict (3.2) on \mathbb{C}_I , it yields that

$$\sum_{n=0}^{\infty} \frac{w^n \bar{\mu}^n \bar{z}^n}{n!} = \sum_{n=0}^{\infty} \frac{w^n \nu^n \bar{z}^n}{n!}, \quad \forall z, w \in \mathbb{C}_I.$$

Comparing coefficients on both sides it follows that $\bar{\mu} = \nu$.

Sufficiency Suppose $\varphi_1(p) = p\mu$ and $\varphi_2(p) = p\bar{\mu}$ with $|\mu| \leq 1$. Still denote $p|_{\mathbb{C}_I} = z$, and $q|_{\mathbb{C}_I} = w$. By (2.1), it yields that

$$(C_{\varphi_1}^* K_z)(w) = \sum_{n=0}^{\infty} \frac{w^n \bar{\mu}^n \bar{z}^n}{n!} = (C_{\varphi_2} K_z)(w), \quad \forall z, w \in \mathbb{C}_I.$$

Next we extend the above equation onto $\mathcal{F}^2(\mathbb{H})$ by Proposition 2.4. For any $q = x + yJ \in \mathbb{H}$ and denoting $w = x + yI$, we get

$$\begin{aligned} (C_{\varphi_2} K_z)(q) &= K_z(\varphi_2(q)) \\ &= \frac{1}{2}(1 + JI)(C_{\varphi_2} K_z)(\bar{w}) + \frac{1}{2}(1 - JI)(C_{\varphi_2} K_z)(w) \\ &= \frac{1}{2}(1 + JI)(C_{\varphi_1}^* K_z)(\bar{w}) + \frac{1}{2}(1 - JI)(C_{\varphi_1}^* K_z)(w) \\ &= (C_{\varphi_1}^* K_z)(q). \end{aligned}$$

Taking the conjugate on both sides, we deduce that $\overline{(C_{\varphi_2} K_z)(q)} = \overline{(C_{\varphi_1}^* K_z)(q)} = (C_{\varphi_1} K_q)(z)$. Further, for $p = x + yJ \in \mathbb{H}$, denoting $z = x + yI$ and using Proposition 2.4, we have that

$$\begin{aligned} (C_{\varphi_1} K_q)(p) &= \frac{1}{2}(1 + JI)(C_{\varphi_1} K_q)(\bar{z}) + \frac{1}{2}(1 - JI)(C_{\varphi_1} K_q)(z) \\ &= \frac{1}{2}(1 + JI)\overline{(C_{\varphi_2} K_z)(q)} + \frac{1}{2}(1 - JI)\overline{(C_{\varphi_2} K_z)(q)} \\ &= K_{\varphi_2(q)}(p) = \overline{(C_{\varphi_2} K_p)(q)}. \end{aligned}$$

This means $(C_{\varphi_1}^* K_p)(q) = (C_{\varphi_2} K_p)(q), \quad \forall p, q \in \mathbb{H}$. Since Lemma 2.6 implies the set \mathcal{M} is dense in $\mathcal{F}^2(\mathbb{H})$, so $C_{\varphi_1}^* f = C_{\varphi_2} f$ holds for all $f \in \mathcal{F}^2(\mathbb{H})$, and then the proof is completed.

Theorem 3.7 with $c = d$ and $\mu = \nu$ singles out the self-adjoint composition operators on $\mathcal{F}^2(\mathbb{H})$.

Corollary 3.8 Let $\varphi \in \mathcal{R}(\mathbb{H})$ such that $\varphi(\mathbb{C}_I) \subset \mathbb{C}_I$ for some $I \in \mathbb{S}$. Denote $\varphi(p) = p\mu + c$ with $|\mu| \leq 1$ and $\mu \in \mathbb{C}_I$ such that C_φ is bounded on $\mathcal{F}^2(\mathbb{H})$. Then C_φ is self-adjoint if and only if $c = 0$ and $\mu \in \mathbb{R}$.

4 Co-isometric Composition Operator on $\mathcal{F}^2(\mathbb{H})$

Motivated by the relations among unitary, isometric and co-isometric operators, we continue to characterize the co-isometric composition operator on the quaternionic Fock space.

Theorem 4.9 Let $\varphi \in \mathcal{R}(\mathbb{H})$ such that $\varphi(\mathbb{C}_I) \subset \mathbb{C}_I$ for some $I \in \mathbb{S}$. Denote $\varphi(p) = p\mu - c$ with $|\mu| \leq 1$ and $\mu \in \mathbb{C}_I$ such that C_φ is bounded on $\mathcal{F}^2(\mathbb{H})$. Then C_φ is co-isometric if and only if $c = 0$ and $|\mu| = 1$.

Proof Necessity Suppose C_φ is co-isometric on $\mathcal{F}^2(\mathbb{H})$ and denote $p|_{\mathbb{C}_I} = z, q|_{\mathbb{C}_I} = w$ for $I \in \mathbb{S}$. And then it follows that $(C_\varphi C_\varphi^* K_z)(w) = K_z(w), \quad \forall z, w \in \mathbb{C}_I$. According to (2.1), it follows that

$$K_{\varphi(z)}(\varphi(w)) = K_z(w), \quad \forall z, w \in \mathbb{C}_I. \tag{4.1}$$

If $\mu = 0, \varphi(p) \equiv -c$, and (4.1) becomes into $K_{-c}(-c) = K_z(w)$, implying C_φ cannot be co-isometric. So $\mu \neq 0$ and then taking $w = \frac{c}{\mu}$, we deduce that

$$1 = \sum_{n=0}^{\infty} \frac{\bar{z}^n c^n}{n! \mu^n}.$$

Comparing coefficients on both sides we obtain $c = 0$. Further, let $z = w$ in (4.1), it turns out $K_{\varphi(z)}(\varphi(z)) = K_z(z)$, implying

$$\sum_{n=0}^{\infty} \frac{|\mu|^{2n} |z|^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!}, \quad \forall z \in \mathbb{C}_I.$$

It follows that $|\mu| = 1$.

Sufficiency Assume that $\varphi(z) = z\mu$ with $|\mu| = 1$, it follows that

$$(C_\varphi C_\varphi^* K_z)(w) = \sum_{n=0}^{\infty} \frac{(\bar{z}\bar{\mu})^n (w\mu)^n}{n!} = \sum_{n=0}^{\infty} \frac{\bar{z}^n w^n}{n!} = K_z(w),$$

for any $z, w \in \mathbb{C}_I$. And then we extend the above equation onto $\mathcal{F}^2(\mathbb{H})$ by Proposition 2.4.

For any $q = x + yJ \in \mathbb{H}$ and denoting $w = x + yI$, we obtain

$$\begin{aligned} (C_\varphi C_\varphi^* K_z)(q) &= \frac{1}{2}(1 + JI)(C_\varphi C_\varphi^* K_z)(\bar{w}) + \frac{1}{2}(1 - JI)(C_\varphi C_\varphi^* K_z)(w) \\ &= \frac{1}{2}(1 + JI)(K_z)(\bar{w}) + \frac{1}{2}(1 - JI)(K_z)(w) \\ &= K_z(q). \end{aligned}$$

Taking the conjugate on both sides, we deduce that

$$\overline{(C_\varphi C_\varphi^* K_z)(q)} = (C_\varphi C_\varphi^* K_q)(z) = K_q(z).$$

And then, for $p = x + yJ \in \mathbb{H}$, denoting $z = x + yI$ and using Proposition 2.4, we get

$$\begin{aligned} (C_\varphi C_\varphi^* K_q)(p) &= \frac{1}{2}(1 + JI)(C_\varphi C_\varphi^* K_q)(\bar{z}) + \frac{1}{2}(1 - JI)(C_\varphi C_\varphi^* K_q)(z) \\ &= \frac{1}{2}(1 + JI)K_q(\bar{z}) + \frac{1}{2}(1 - JI)K_q(z) \\ &= K_q(p). \end{aligned}$$

This means $(C_\varphi C_\varphi^* K_p)(q) = K_p(q)$, $\forall p, q \in \mathbb{H}$. Employing the set \mathcal{M} is dense in $\mathcal{F}^2(\mathbb{H})$, it yields that $(C_\varphi C_\varphi^*)f = f$ for all $f \in \mathcal{F}^2(\mathbb{H})$. This completes the proof.

Interestingly, we can further prove the co-isometric composition operator is unitary on $\mathcal{F}^2(\mathbb{H})$.

Corollary 4.10 Let $\varphi \in \mathcal{R}(\mathbb{H})$ such that $\varphi(\mathbb{C}_I) \subset \mathbb{C}_I$ for some $I \in \mathbb{S}$. Denote $\varphi(p) = p\mu + c$ with $|\mu| \leq 1$ and $\mu \in \mathbb{C}_I$ such that C_φ is bounded on $\mathcal{F}^2(\mathbb{H})$. Then C_φ is co-isometric if and only if C_φ is unitary.

Proof We only need to prove C_φ is unitary when it is co-isometric. At this time, Theorem 4.9 implies $\varphi(p) = p\mu$ with $|\mu| = 1$. For any two functions $f, g \in \mathcal{F}^2(\mathbb{H})$ writing as

$$\begin{aligned} f(p) &= \sum_{n=0}^{\infty} p^n a_n, \quad \text{where } a_n \in \mathbb{H} \text{ and } \sum_{n=0}^{\infty} |a_n|^2 n! < \infty, \\ g(p) &= \sum_{n=0}^{\infty} p^n b_n, \quad \text{where } b_n \in \mathbb{H} \text{ and } \sum_{n=0}^{\infty} |b_n|^2 n! < \infty, \end{aligned}$$

it turns out that

$$\begin{aligned} \langle C_\varphi f, C_\varphi g \rangle &= \int_{\mathbb{C}_I} e^{-|z|^2} \overline{(C_\varphi g)_I(z)} (C_\varphi f)_I(z) d\sigma(x, y) \\ &= \int_{\mathbb{C}_I} e^{-|z|^2} \overline{g_I(\varphi_I(z))} f_I(\varphi_I(z)) d\sigma(x, y) \\ &= \int_{\mathbb{C}_I} e^{-|z|^2} \overline{\left(\sum_{n=0}^{\infty} (z\mu)^n b_n \right)} \left(\sum_{n=0}^{\infty} (z\mu)^n a_n \right) d\sigma(x, y) \\ &= \sum_{n=0}^{\infty} n! \bar{b}_n a_n = \langle f, g \rangle, \end{aligned}$$

implying C_φ is isometric. So we conclude C_φ is unitary, ending the proof.

References

- [1] Gentili G, Struppa D C. A new approach to Cullen-regular functions of a quaternionic variable[J]. C. R. Math., 2006, 342(10): 741–744.
- [2] Gentili G, Struppa D C. A new theory of regular functions of a quaternionic variable[J]. Adv. Math., 2007, 216(1): 279–301.
- [3] Alpay D, Colombo F, Sabadini I, Salomon G. The Fock space in the slice hyperholomorphic Settings[A]. Bernstein S, Kähler U, Sabadini I, Sommen F. Hypercomplex Analysis: New Perspectives and Applications[C]. Switzerland: Birkhäuser, Cham, 2014: 43–59.
- [4] Diki K, Gal S G, Sabadini I. Polynomial approximation in slice regular Fock spaces[J]. Complex Anal. Oper. Theory., 2019, 13(6): 2729–2746.
- [5] Villalba C, Colombo F, Gantner J, et al. Bloch, Besov and Dirichlet spaces of slice hyperholomorphic functions[J]. Complex Anal. Oper. Theory., 2015, 9(2): 479–517.
- [6] Colombo F, González-Cervantes J O, Maria E L. On two approaches to the Bergman theory for slice regular functions[A]. Graziano G, Advances in Hypercomplex Analysis[C]. Berlin: Springer Milano, 2015.
- [7] Colombo F, Sabadini I, Struppa D C. Noncommutative functional calculus[M]. Berlin: Springer Basel, 2011.
- [8] Lian P, Liang Y. Weighted composition operator on quaternionic Fock space[J]. Banach J. Math. Anal., 2021, 15(1): <https://doi.org/10.1007/s43037-020-00087-6>.
- [9] Liang Y. A left linear weighted composition operator on quaternionic Fock space[J]. Results in Mathematics, 2019, 74(1): <https://doi.org/10.1007/s00025-018-0948-9>.
- [10] Colombo F, Sabadini I, Struppa D C. Entire slice regular functions[M]. Switzerland: Springer, Cham, 2016.

四元数福克空间上的自伴与余等距复合算子

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摘要: 本文研究了四元数福克空间上自伴与余等距复合算子的性质. 利用再生核的稠密性, 获得了自伴与余等距复合算子充要条件的刻画, 并得到了四元数福克空间上余等距复合算子是酉的.

关键词: 复合算子; 四元数福克空间; 再生核; 自伴性; 余等距

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