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ASYMPTOTIC BEHAVIOR OF NONLINEAR DELAYED DISCRETE SYSTEMS

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Abstract: In this paper, we consider a two-dimensional vector nonlinear difference system with time delay. By using stability theory and optimal control theory, sufficient conditions for the global asymptotic stability of a unique positive equilibrium are obtained. For the associated optimal control problem of maximizing the consumption functional, the existence of optimal solutions is established as well as their stability. Finally, numerical simulation is carried out to illustrate the validity of our results. Our results generalize the corresponding results on a one-dimensional vector nonlinear difference system.

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1 Introduction

In many fields, including mathematical biology, medicine, homeland security and manufacturing industry, economics etc, much of the data are discrete rather than continuous. As well known, difference equation is an effective mathematical tool to study the variation law of discrete variables. A lot of papers and monographs have been written on this subject, see [1–4] and further references therein. Among them, delayed nonlinear equations were investigated by a large number of researchers, see in [1, 5–7]. For such equations, the asymptotic behavior of the solution to difference equations is an important topic arising from a large number of practical problems as well as the optimal control problem, see [8–11] and textbooks [2, 3, 12].

To describe an economic problem, Ivanov considered the following equation in [13]:

$$x_{n+1} = ax_n + u_n f(x_{n-K}) \tag{1.1}$$

where $0 < a < 1$, $u_n \in [0,1]$ is the control coefficient. x_n is the economic output at any time n, the output of the next time interval $n+1$ is composed of two parts, one is ax_n, a certain proportion of the output at time n, the other is the output $f(x_{n-K})$ which results

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Biography: Tian Yanling (1972–), female, born at Nanning, Guangxi, associated professor, major in applied mathematics, E-mail: tianyl@scnu.edu.cn from the production output K steps back, caused by the delay factors due to some economic results. They dealt with two problems in [13]. One is the global asymptotic stability of its unique positive equilibrium for equation (1.1) without control, i.e $u_n \equiv 1$ for any $n \in \mathbb{N}^+$. The other is an optimal control problem, which is associated with an important economic problem, maximization of consumption functional. They found a control sequence $\{u_n\}_{n=1}^{\infty}$ to control the consumption achieve to maximal value, where the value of u_n is in $[0,1]$ but not identically equal to 1.

But one variable or one discrete equation can not describe more complex situation sometimes. In our current paper, we consider a more complex situation where a firm has two factories. The same economic problem leads to the following difference equations with delay

$$\begin{cases} x_{n+1} = a_1 x_n + u_n(f_1(x_{n-K}) + g_1(y_{n-K})), \\ y_{n+1} = a_2 y_n + v_n(f_2(x_{n-K}) + g_2(y_{n-K})), \end{cases} \tag{1.2}$$

where $0 < a_i < 1$, functions $f_i, g_i : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous on $\mathbb{R}^+$ for $i = 1, 2$ and $K \geq 1$ is the integer delay. x_n, y_n are the economic output of the two factories respectively, and can be measured at discrete time intervals (daily, monthly, or yearly). For example, the output of one factory at any time n denoted by x_n, the output of the next time interval $n+1$ is composed of two parts, one is a certain proportion of the output of the company at time n, $a_1 x_n$, and the other is the output $f_1(x_{n-K}) + g_1(y_{n-K})$. The economic significance of the two parts are similar to [13]. Obviously, our model is an generalization of equation (1.1). Moreover, it is interesting to discuss how to assign resources to the two factories to control the consumption achieve to maximal value. But there are many difficulties due to this general setting. First, some more complicated conditions will be proposed to guarantee the existence and the uniqueness of the positive equilibrium by using inverse function method. Second, we need to develop additional techniques to establish the stability of the positive equilibrium. Finally, the most difficult thing is to study the control problem. Not only are there four functions $f_i,\ g_i (i = 1, 2)$ in the system, but also the control sequence $\{u_n,\ v_n\}_{n=1}^{\infty}$ are more complicated than those in [13]. Therefore, we will propose suitable assumptions on those functions and develop additional techniques.

Our paper is organized as follows. We establish the existence and the uniqueness of the positive equilibrium of the system considered in the current paper with $u_n = v_n \equiv 1$ and study the global asymptotic stability of the equilibrium in section 2. Section 3 is concerned with the associated optimal control problem. The properties of the solution to the control equation are obtained. For any one of the optimal equilibria, an associated control sequence is found to make the solution converge to it. Finally, numerical simulation is carried out in section 4 to illustrate the validity of our assumptions and results.

2 Preliminaries and Global Asymptotic Stability

In this section, we put forward two hypotheses, introduce basic definitions and state stability results concerning the difference system (1.2) with $u_n = v_n = 1$, that is the following system:

$$\begin{cases} x_{n+1} = a_1 x_n + (f_1(x_{n-K}) + g_1(y_{n-K})), \\ y_{n+1} = a_2 y_n + (f_2(x_{n-K}) + g_2(y_{n-K})). \end{cases} \tag{2.1}$$

2.1 Positive Equilibrium Analysis of Equation (2.1)

It is difficult to guarantee the existence and the uniqueness of the equilibrium of the system (2.1) since there are four functions $f_i,\ g_i (i = 1, 2)$ in the right of the system. To overcome this difficulty, we introduce inverse function method. Some functions are given first. Define

$$H_1^{(1)}(r) := (1 - a_1)r - f_1(r),\ H_1^{(2)}(r) := (1 - a_2)r - g_2(r).$$

Their inverse functions of $H_1^{(1)}(r)$ and $H_1^{(2)}(r)$ are denoted by $(H_1^{(1)})^{-1}(r),\ (H_1^{(2)})^{-1}(r)$ respectively. Define

$$H_2^{(1)}(r) := (1 - a_2)r - g_2(r) - f_2((H_1^{(1)})^{-1}(g_1(r))),$$
$$H_2^{(2)}(r) := (1 - a_1)r - f_1(r) - g_1((H_1^{(2)})^{-1}(f_2(r))).$$

Two assumptions are given as follows.

(H1) Assume that functions $H_1^{(1)},\ H_1^{(2)}$ have inverse functions for $r > 0$ and $(H_1^{(1)})^{-1}(r) > 0,\ (H_1^{(2)})^{-1}(r) > 0$ if $r > 0$.

(H2) Assume that there is $Y^* > 0$ such that the function $H_2^{(1)}$ satisfies

$$H_2^{(1)}(r) < 0, \text{ if } 0 < r < Y^*;\quad H_2^{(1)}(r) > 0, \text{ if } r > Y^*. \tag{2.2}$$

Let $X^* = (H_1^{(1)})^{-1}(g_1(Y^*))$. Assume that the function $H_2^{(2)}$ satisfies

$$H_2^{(2)}(r) < 0, \text{ if } 0 < r < X^*;\quad H_2^{(2)}(r) > 0, \text{ if } r > X^*. \tag{2.3}$$

Assumptions (H1), (H2) and simple calculations lead to the following lemma.

Lemma 2.1 Assume that assumptions (H1) and (H2) hold. Then difference system (2.1) has a unique positive equilibrium $(X^*,\ Y^*)$. Moreover,

$$(X^*, Y^*) = \left(\frac{f_1(X^*) + g_1(Y^*)}{1 - a_1},\ \frac{f_2(X^*) + g_2(Y^*)}{1 - a_2} \right).$$

2.2 Asymptotic Behavior of the Solution to Equation (2.1)

Before our main results, we introduce some definitions and notations.

Define $\mathbf{F} = (F_1, F_2)^T$, $\mathbf{G} = (G_1, G_2)^T$, where $F_i = \frac{1}{1-a_i} f_i$, $G_i = \frac{1}{1-a_i} g_i,\ i = 1, 2$. Define $\mathbf{T} = (T_1,\ T_2)^T$, where $T_i(x, y) = F_i(x) + G_i(y),\ i = 1, 2$. It is obvious that (X^*, Y^*) is a fixed point of the map $\mathbf{T}$. The fixed point of map $\mathbf{T}$ is called attracting if there exists its open (with respect to $\mathbb{R}^+ \times \mathbb{R}^+$) neighborhood $\mathcal{U}$ such that $\mathbf{T}(\mathcal{U}) \subseteq \mathcal{U}$ and $\lim_{n\to\infty} \mathbf{T}^n(x,y) = (X^*, Y^*)$ for each $(x,y) \in \mathcal{U}$, where $\mathbf{T}^n := \mathbf{T} \circ \mathbf{T}^{n-1} = \mathbf{T}(\mathbf{T}^{n-1})$. The largest interval $\mathcal{U} \subseteq \mathbb{R}^+ \times \mathbb{R}^+$ with the above property is called the domain of immediate attraction of the fixed point (X^*, Y^*). An interval $I \subseteq \mathbb{R}^+ \times \mathbb{R}^+$ is said to be invariant under $\mathbf{T}$ if $(T_1(x,y), T_2(x,y)) \subseteq I$ for all $(x,y) \in I$.

Given an initial string $\mathbf{P}_0 = \{P_{-K},\ P_{-K+1},\ ..., P_0\}$, then system (2.1) has a solution (x_n, y_n) for $n > 0$ by consecutive iterations. Define $P_i := (x_i, y_i),\ i \in \{-K, -K+1, ..., 0\} \cup \mathbb{N}^+$. The segment $\{P_{(j-1)K+j}, \cdots,\ P_{jK+j}\}$ is called the $j-$th string $\mathbf{P}_j$. Give a set $S \in \mathbb{R}^+ \times \mathbb{R}^+$, we say that $\mathbf{P}_j \in S$ if $P_{(j-1)K+j+i} \in S$ for all $i \in \{0, 1, \cdots, K\}$. Clearly, $x_n \geq 0,\ y_n \geq 0$ for all $n \geq 1$ if the initial data for $\mathbf{P}_0$ are all in $\mathbb{R}^+ \times \mathbb{R}^+$.

Difference system (2.1) is equivalent to the following one

$$\begin{cases} \mu_1 \triangle x_n = -x_{n+1} + F_1(x_{n-K}) + G_1(y_{n-K}), \\ \mu_2 \triangle y_n = -y_{n+1} + F_2(x_{n-K}) + G_2(y_{n-K}), \end{cases} \tag{2.4}$$

where $\mu_i = \dfrac{a_i}{1-a_i}$, $\triangle x_n := x_{n+1} - x_n$, $\triangle y_n := y_{n+1} - y_n$. Let $\mathbf{A} := \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ and $\triangle \mathbf{P}_n = (\triangle x_n, \triangle y_n)^T$, then the vector form of (2.4) is

$$\mathbf{A} \triangle \mathbf{P}_n = -\mathbf{P}_{n+1} + \mathbf{F}(x_{n-K}) + \mathbf{G}(y_{n-K}) = -\mathbf{P}_{n+1} + \mathbf{T}(x_{n-K}, y_{n-K}).$$

In this subsection we obtain the global asymptotic stability of the unique positive equilibrium.

Theorem 2.1 Suppose that (X^*, Y^*) is an attracting fixed point of map $\mathbf{T}$ and set J is the associated domain of immediate attraction. Then for every initial string $\mathbf{P}_0 \in J$, the corresponding solution $P_n = P_n(\mathbf{P}_0)$ has the property

$$\lim_{n\to\infty} (x_n, y_n) = (X^*, Y^*). \tag{2.5}$$

Proof By the definition of attracting fixed point of map $\mathbf{T}$, one has

$$\mathbf{T}(X^*,\ Y^*) = (X^*,\ Y^*),\ \lim_{n\to\infty} \mathbf{T}^n(x,y) = (X^*, Y^*),\ \forall (x,y) \in J \ni (X^*, Y^*).$$

Let an initial string $\mathbf{P}_0 = \{(x_{-K}, y_{-K}), (x_{-K+1}, y_{-K+1}), \cdots, (x_0, y_0)\}$ be given such that $\mathbf{P}_0 \in J$, then one can find a closed bounded interval $I_0 := [\alpha_1,\ \beta_1] \times [\alpha_2,\ \beta_2] \subset J$ such that $\mathbf{P}_0 \in I_0$ and $\mathbf{T}(I_0) \subseteq I_0$. Since J is the domain of immediate attraction of (X^*, Y^*), one also has

$$I_0 \supseteq \mathbf{T}(I_0) \supseteq \mathbf{T}^2(I_0) \supseteq \cdots \supseteq \mathbf{T}^i(I_0) \supseteq \cdots \text{ and } \cap_{i\geq 0} \mathbf{T}^i(I_0) = (X^*, Y^*), \tag{2.6}$$

where $\mathbf{T}(I_0) := [\gamma_1,\ \delta_1] \times [\gamma_2,\ \delta_2]$.

Let (x_n, y_n) be the solution of (2.4) with associated initial data $\mathbf{P}_0$. Then the following two cases will occur.

Case I. There is $N \geq 0$ such that $(x_N, y_N) \in \mathbf{T}(I_0)$. Then one can show that $(x_n, y_n) \in \mathbf{T}(I_0)$ for all $n \geq N$. Suppose not, let $\tilde{N} > N$ be the first time when the solution leaves the interval $\mathbf{T}(I_0)$ for $n > N$. Without loss of generality, we may assume that $(x_{\tilde{N}}, y_{\tilde{N}}) \in [\alpha_1, \gamma_1] \times [\alpha_2, \beta_2]$ and $(x_n, y_n) \in \mathbf{T}(I_0) \subseteq I_0$ for all $N < n < \tilde{N}$. Then $\triangle x_{\tilde{N}-1} = x_{\tilde{N}} - x_{\tilde{N}-1} < 0$. On the other hand, since $T_1(x_{\tilde{N}-K}, y_{\tilde{N}-K}) \in T_1(I_0) = [\gamma_1, \delta_1]$, then system (2.4) shows that $\triangle x_{\tilde{N}-1} = \dfrac{1}{\mu_1}[-x_{\tilde{N}} + F_1(x_{\tilde{N}-K}) + G_1(y_{\tilde{N}-K})] > 0$, a contradiction. Then (2.5) is valid from (2.6).

Case II. $(x_n, y_n) \notin \mathbf{T}(I_0)$ for all $n \geq 0$. Then we will show that $\lim_{n\to\infty}(x_n, y_n) = (X^*,\ Y^*)$. Let $A := ([\alpha_1, \gamma_1] \times [\alpha_2, \delta_2]) \cup ([\gamma_1, \delta_1] \times [\alpha_2, \gamma_2])$, then $A \subseteq I_0 \setminus T(I_0)$. We claim that $(x_n, y_n) \in A$ for all $n > 0$ if $(x_0, y_0) \in A$, and $(x_n, y_n) \notin [\gamma_1, \delta_1] \times [\gamma_2, \delta_2]$ for all $n > 0$. Indeed, if on the contrary, $P_1 = P_1(\mathbf{P}_0) \notin A$, then $x_1 > \delta_1$ or $y_1 > \delta_2$. Consider a modified initial string $\widetilde{\mathbf{P}}_0 := \{(x_{-K}, y_{-K}), \cdots, (x_{-1}, y_{-1}), (\delta_1, \delta_2)\}$, in view of $(\delta_1, \delta_2) \in \mathbf{T}(I_0)$, one has $P_n(\widetilde{\mathbf{P}}_0) \in \mathbf{T}(I_0)$ for all $n > 0$. So that $P_1(\widetilde{\mathbf{P}}_0) = (\widetilde{x}_1, \widetilde{y}_1) \leq (\delta_1, \delta_2)$. From system (2.4) we can see that

$$x_1(\mathbf{P}_0) = a_1 x_0 + f_1(x_{-K}) + g_1(y_{-K}) < a_1\delta_1 + f_1(x_{-K}) + g_1(y_{-K}) = \widetilde{x}_1(\widetilde{\mathbf{P}}_0),$$

$$y_1(\mathbf{P}_0) = a_2 y_0 + f_2(x_{-K}) + g_2(y_{-K}) < a_2\delta_2 + f_2(x_{-K}) + g_2(y_{-K}) = \widetilde{y}_1(\widetilde{\mathbf{P}}_0),$$

a contradiction with $P_1(\mathbf{P}_0) \notin A$. Therefore, we obtain the conclusion that $(x_n, y_n) \in A$ for all $n > 0$. That means the solution (x_n, y_n) is bounded from above. Note that $F_i,\ G_i (i = 1, 2)$ are continuous over I_0, then $\lim_{n\to\infty}(x_n, y_n) = (\overline{x}, \overline{y}) \in A$. Since if $\alpha_1 \leq \overline{x} < \gamma_1$, then $\dfrac{1}{\mu_1}[-\overline{x} + F_1(\overline{x}) + G_1(\overline{y})] > 0$ and if $\alpha_2 \leq \overline{y} < \gamma_2$, then $\dfrac{1}{\mu_2}[-\overline{y} + F_2(\overline{x}) + G_2(\overline{y})] > 0$, then $\overline{x} = \gamma_1,\ \overline{y} = \gamma_2$. On the other hand, since $\lim_{n\to\infty}(\triangle x_n, \triangle y_n) = (0, 0)$, system (2.4) implies that $\overline{x} = F_1(\overline{x}) + G_1(\overline{y}),\ \overline{y} = F_2(\overline{x}) + G_2(\overline{y})$, which implies $(\overline{x},\ \overline{y}) = (X^*,\ Y^*)$. Thus, from (2.4), one can conclude that (2.5) is valid. The theorem is proved.

The following corollary gives a sufficient condition to guarantee that the fixed point (X^*, Y^*) is an attracting fixed point of map $\mathbf{T}$.

Corollary 2.1 Suppose that

$$\begin{gathered}\max_{(s,r)\in[0,X^*]\times[0,Y^*]}(F_1(r) + G_1(s)) \leq X^*, \quad \max_{(s,r)\in[0,X^*]\times[0,Y^*]}(F_2(r) + G_2(s)) \leq Y^*,\\ \min_{(s,r)\in[X^*,+\infty)\times[Y^*,+\infty)}(F_1(r) + G_1(s)) \geq X^*, \quad \min_{(s,r)\in[X^*,+\infty)\times[Y^*,+\infty)}(F_2(r) + G_2(s)) \geq Y^*.\end{gathered} \tag{2.7}$$

If there are increasing functions $\omega_i^{\pm} (i = 1, 2)$ with

$$\begin{gathered}\omega_1^-(r) > r \text{ for } 0 < r < X^*,\ \omega_1^+(r) < r \text{ for } r > X^*,\\ \omega_2^-(s) > s \text{ for } 0 < s < Y^*,\ \omega_2^+(s) < s \text{ for } s > Y^*,\end{gathered} \tag{2.8}$$

such that

$$\begin{gathered}F_1(r) + \min_{s\in[0,Y^*]} G_1(s) \geq \omega_1^-(r),\ F_1(r) + \max_{s\geq Y^*} G_1(s) \leq \omega_1^+(r),\\ \min_{r\in[0,X^*]} F_2(r) + G_2(s) \geq \omega_2^-(s),\ \max_{r\geq X^*} F_2(r) + G_2(s) \leq \omega_2^+(s),\end{gathered} \tag{2.9}$$

then

$$\lim_{n\to\infty}(x_n, y_n) = (X^*, Y^*)$$

provided $\mathbf{P}_0 \in I_0 := ([0, X^*] \times [0, Y^*]) \cup ([X^*, +\infty) \times [Y^*, +\infty))$.

Proof It is easy to see that (2.7) implies that I_0 is the invariant domain of operator $\mathbf{T}$. Moreover, (2.8) and (2.9) imply that $\cap_{i\geq 0}\mathbf{T}^i(I_0) = (X^*, Y^*)$. Therefore, (X^*, Y^*) is the the attracting fixed point of map $\mathbf{T}$ and the associated domain of immediate attraction is I_0. Hence, one can obtain (2.5) when the initial string $\mathbf{P}_0 \in I_0$. The corollary is proved.

3 Convergence of Solutions in Optimal Control Problems

Consider the following difference system

$$\begin{cases} x_{n+1} = a_1x_n + u_n(f_1(x_{n-K}) + g_1(y_{n-K})), \\ y_{n+1} = a_2y_n + v_n(f_2(x_{n-K}) + g_2(y_{n-K})),\ n = 0, 1, 2, \cdots, \end{cases} \tag{3.1}$$

where the parameters $a_i (i = 1, 2)$ and the functions $f_i, g_i (i = 1, 2)$ are as same as those of system (2.1), and u_n, v_n are control sequences with values in $[0, 1]$ which are motivated by practical applications.

Obviously, the general setting leads to additional difficulties and much more technicality. How to handle with the four functions and the six solution subsequences in (3.1)? To this end, the functions $f_i, g_i, (i = 1, 2)$ are assumed to satisfy the following conditions.

(H3) Let $1 - a = \min\{(1 - a_1), (1 - a_2)\}$. For a given bounded region $[0, M] \times [0, M]$, there exist positive constants $h_i \in (0, 1],\ i = 1, 2$ with $(1 - a_1)h_1 + (1 - a_2)h_2 \leq 1 - a$ such that for $\forall (x, y) \in [0, M] \times [0, M]$,

$$\Sigma_{i=1}^2(f_i(x) + g_i(y)) \leq \Sigma_{i=1}^2(f_i(h_1(x + y)) + g_i(h_2(x + y))). \tag{3.2}$$

(H4) Let $\tilde{F}(r) = f_1(r) + f_2(r) - (1 - a_1)r,\ \tilde{G}(s) = g_1(s) + g_2(s) - (1 - a_2)s$. Define $\mathcal{A} = \{\bar{s} | \tilde{G}(\bar{s}) = \max_{0\leq s\leq Y^*} \tilde{G}(s)\},\ \mathcal{B} = \{\bar{r} | \tilde{F}(\bar{r}) = \max_{0\leq r\leq X^*} \tilde{F}(r)\}$. Then

(1) $\tilde{F}(r) < 0$ for $r > X^*$ and $\tilde{G}(s) < 0$ for $s > Y^*$;

(2) $f_1(r) + g_1(s') \geq (1 - a_1)r$ if $s' \in \mathcal{A}$ for $0 < r < X^*$ and $f_2(r') + g_2(s) \geq (1 - a_2)s$ if $r' \in \mathcal{B}$ for $0 < s < Y^*$;

(3) there is $\tilde{M} > 0$ such that $f_1(r) + f_2(r) < (1 - b)r,\ g_1(r) + g_2(r) < (1 - b)r$ for $r > \tilde{M}$, where $b > a$.

Remark The condition (H3) plays an important role in our next discussion and it is reasonable. For example, it is easy to prove that (H3) is satisfied when $f_i(x) = \sqrt{x}+x,\ g_i(y) = \sqrt{y}+y,\ i = 1, 2$ when $a_1 = a_2$. Let $a_1 \neq a_2$, choose $\lambda < \frac{1}{2}$ such that $\lambda(1-a_1)+\lambda(1-a_2) \leq 1-a$.

If $f_1(x)+f_2(x)=\lambda h(x),\ g_1(y)+g_2(y)=(1-\lambda)h(y)$, where h is convex and decreasing, then one has

$$\begin{aligned}\Sigma_{i=1}^2(f_i(x)+g_i(y)) &\le \lambda h(\lambda x+(1-\lambda)y)+(1-\lambda)h(\lambda x+(1-\lambda)y)\\ &\le \lambda h(\lambda(x+y))+(1-\lambda)h(\lambda(x+y))=\Sigma_{i=1}^2(f_i(\lambda(x+y))+g_i(\lambda(x+y))).\end{aligned}$$

Thus, (H3) is satisfied.

Next, we give the notations about the consumption. Let $\mathbf{P}_0$ be any fixed initial string. For arbitrary control $\mathbf{U}=(u_n,v_n)_{n=0}^{\infty}$ system (3.1) defines a unique solution $\mathbf{P}=\mathbf{P}(\mathbf{P}_0,\mathbf{U})=(P_n)_{n=1}^{\infty}=(x_n,y_n)_{n=1}^{\infty}$, which is found by consecutive iterations for all $n>0$. Similar to [13], the consumption is denoted by $[(1-u_n)(f_1(x_{n-K})+g_1(y_{n-K}))+(1-v_n)(f_2(x_{n-K})+g_2(y_{n-K}))]$. The following consumption functional

$$C(\mathbf{P}_0,\mathbf{U})=\liminf_{n\to\infty}[(1-u_n)(f_1(x_{n-K})+g_1(y_{n-K}))+(1-v_n)(f_2(x_{n-K})+g_2(y_{n-K}))] \quad (3.3)$$

and its maximization problem over the solutions to system (3.1)

$$\text{Maximize}: C(\mathbf{P}_0,\mathbf{U}),\ \text{subject to (3.1)} \quad (3.4)$$

are used to denote the minimal level of consumption for sufficiently large time periods. A positive constant c^* is used to describe the maximum steady consumption that could be achieved in problem (3.4) and defined as

$$c^*=\max\{\mathrm{L}_1(x,y)+\mathrm{L}_2(x,y),\ (x,y)\in\mathbb{R}^+\times\mathbb{R}^+\},$$

where $\mathrm{L}_1(x,y)=f_1(x)+g_1(y)-(1-a_1)x,\ \mathrm{L}_2(x,y)=f_2(x)+g_2(y)-(1-a_2)y,\ (x,y)\in\mathbb{R}^+\times\mathbb{R}^+$. And set $\tau,\ \tau'$ are defined as follows:

$$\begin{aligned}\tau=\{(x,y)\in[0,X^*]\times[0,Y^*]:\mathrm{L}_1(x,y)+\mathrm{L}_2(x,y)=c^*,\ \text{there are }\ \overline{u}_1,\overline{v}_1\in[0,1]\ \text{such that}\\ x=a_1x+\overline{u}_1(f_1(x)+g_1(y)),\ y=a_2y+\overline{v}_1(f_2(x)+g_2(y))\}\end{aligned} \quad (3.5)$$

and

$$\tau'=\{\overline{p}:\text{there are }\ \overline{x},\ \overline{y}\ \text{such that}\ \overline{x}+\overline{y}=\overline{p}\ \text{and}\ (\overline{x},\overline{y})\in\tau\},$$

where set τ contains the steady state guaranteing this consumption c^*, the element of τ' is the sum of each component of the element in set τ. First, set τ is non-empty evidently. Actually, by assumption (H4), there is $(x',y')\in[0,X^*]\times[0,Y^*]$ such that $\mathrm{L}_1(x',y')+\mathrm{L}_2(x',y')=c^*$, and $(x',y')\in\tau$ since

$$\bar{u}_1=\frac{(1-a_1)x'}{f_1(x')+g_1(y')}\le 1,\ \bar{v}_1=\frac{(1-a_2)x'}{f_2(x')+g_2(y')}\le 1.$$

It follows that set τ' is non-empty. Second, τ is a closed set and it may contain more than one point. One would like to find a control sequence $(u_n,v_n),n\ge 0$ that maximizes the minimal level of consumption for sufficiently large time periods for any given initial string $\mathbf{P}_0$, that means the solution to (3.1) will be convergent to the optimal steady state in set τ under the control sequence $(u_n,\ v_n)$.

In what follows, we present three theorems to describe the properties of the solutions to equation (3.1) as well as the relation between $C(\mathbf{P}_0, \mathbf{U})$ and c^* over the solutions to (3.1) for any given initial string $\mathbf{P}_0$. Theorem 3.1 shows $C(\mathbf{P}_0, \mathbf{U}) \le c^*$ for all $\mathbf{P}_0$ and $\mathbf{U}$; Theorem 3.2 shows that there exists an optimal control $\mathbf{U}_{\mathbf{P}_0}$ to problem (3.4) such that functional (3.3) achieves its maximum possible value; that is, $C(\mathbf{P}_0, \mathbf{U}) = c^*$; Theorem 3.3 shows how to construct a control sequence to make the solution to (3.1) converge to an optimal control equilibrium in set τ or set τ'.

Theorem 3.1 The functional (3.3) is bounded above over the solutions to (3.1), that is, for all $\mathbf{P}_0$ and $\mathbf{U}$, there exists the inequality

$$C(\mathbf{P}_0, \mathbf{U}) \le c^*. \tag{3.6}$$

Proof Step 1. We claim that the solution $\mathbf{P}$ to (3.1) is bounded; that is, there exists $\bar{M}$ such that

$$\limsup_{n\to\infty}(x_n + y_n) \le \bar{M},\ \forall \mathbf{P}.$$

Consider solution $\mathbf{P} = \mathbf{P}(\mathbf{P}_0, \mathbf{U}) = (x_n, y_n)_{n=0}^{\infty}$ corresponding to the initial string $\mathbf{P}_0$ and control $\mathbf{U} = (u_n, v_n)_{n=0}^{\infty}$. Let $S_n = x_n + y_n,\ r^* = \max\{X^*,\ Y^*\}$, $L = \max\{f_1(x) + f_2(x),\ x \in [0, \tilde{M}]\} + \max\{g_1(y) + g_2(y),\ y \in [0, \tilde{M}]\}$, $\bar{M} = \max\{S_{-k}, ..., S_{-1}, S_0;\ \frac{1}{1-a}L,\ \bar{M}_1\}$, where $\bar{M}_1 \ge \frac{L}{b-a}$. Let m be the value such that $S_k \le \bar{M}$ for $k \le m$ and $S_{m+1} > \bar{M}$.

(1) $x_{m-K} < \tilde{M},\ y_{m-K} < \tilde{M}$.

$$\begin{aligned}0 < S_{m+1} - S_m &\le -(1-a_1)x_m + f_1(x_{m-K}) + g_1(y_{m-K}) - (1-a_2)y_m + f_2(x_{m-K}) + g_2(y_{m-K})\\ &\le -(1-a)S_m + L \le -(1-a)S_m + (1-a)\bar{M} \le -(1-a)S_m + (1-a)S_{m+1}.\end{aligned}$$

(2) $x_{m-K} \le \tilde{M},\ \tilde{M} \le y_{m-K} < \bar{M}$.

$$\begin{aligned}0 < S_{m+1} - S_m &\le -(1-a_1)x_m + f_1(x_{m-K}) + g_1(y_{m-K}) - (1-a_2)y_m + f_2(x_{m-K}) + g_2(y_{m-K})\\ &\le -(1-a)S_m + L + (1-b)\bar{M} \le -(1-a)S_m + (1-a)\bar{M} \le -(1-a)S_m + (1-a)S_{m+1}.\end{aligned}$$

(3) $x_{m-K} > \tilde{M},\ y_{m-K} > \tilde{M}$.

$$\begin{aligned}0 < S_{m+1} - S_m &\le -(1-a_1)x_m + f_1(x_{m-K}) + g_1(y_{m-K}) - (1-a_2)y_m + f_2(x_{m-K}) + g_2(y_{m-K})\\ &\le -(1-a)S_m + (1-a)S_{m-K} \le -(1-a)S_m + (1-a)S_{m+1}.\end{aligned}$$

Thus, $0 < S_{m+1} - S_m < (1-a)(S_{m+1} - S_m)$, a contradiction is obtained since $0 < a < 1$.

Step 2. We claim that for every solution $\mathbf{P} = \mathbf{P}(\mathbf{P}_0, \mathbf{U})$ to (3.1) the inequality (3.6) holds.

Recall that $C(\mathbf{P}_0, \mathbf{U}) = \liminf_{n\to\infty} c_n$, where

$$c_n = (1-u_n)(f_1(x_{n-K}) + g_1(y_{n-K})) + (1-v_n)(f_2(x_{n-K}) + g_2(y_{n-K})). \tag{3.7}$$

From (3.1) and (3.7) it follows that

$$c_n = f_1(x_{n-K}) + g_1(y_{n-K}) - x_{n+1} + a_1 x_n + f_2(x_{n-K}) + g_2(y_{n-K}) - y_{n+1} + a_2 y_n. \tag{3.8}$$

Denote $p_1 := \limsup_{n\to\infty} x_n < \infty$, then there is $k_m \to \infty$ such that $x_{k_m+1} \to p_1$; let $p_2 := \limsup_{k_m\to\infty} y_{k_m+1} < \infty$, then there is $\{k'_m\} \subset \{k_m\}$ such that $y_{k'_m+1} \to p_2$ as $k'_m \to \infty$. Without loss of generality, we have

$$\lim_{k'_m\to\infty} x_{k'_m-K} = x'', \ \lim_{k'_m\to\infty} x_{k'_m} = x';$$
$$\lim_{k'_m\to\infty} y_{k'_m-K} = y'', \ \lim_{k'_m\to\infty} y_{k'_m} = y'.$$

From (3.8) we have

$$\begin{aligned}\lim_{k'_m\to\infty} c_{k'_m} &= f_1(x'') + g_1(y'') - p_1 + a_1 x' + f_2(x'') + g_2(y'') - p_2 + a_2 y' \\ &\le f_1(x'') + g_1(y'') - (1-a_1)p_1 + f_2(x'') + g_2(y'') - (1-a_2)p_2 \\ &\le f_1(x'') + g_1(y'') - (1-a_1)x'' + f_2(x'') + g_2(y'') - (1-a_2)y''.\end{aligned}$$

By the definition of c^* we have

$$f_1(x'') + g_1(y'') - (1-a)x'' + f_2(x'') + g_2(y'') - (1-a)y'' \le c^*.$$

Therefore, $C(\mathbf{P}_0, \mathbf{U}) = \liminf_{n\to\infty} c_n \le \lim_{n\to\infty} c_{k'_m} \le c^*$. The theorem is proved.

Theorem 3.2 For any given initial string $\mathbf{P}_0$, there exists an optimal control $\mathbf{U}_{\mathbf{P}_0}$ to problem (3.4) such that functional (3.3) achieves its maximum possible value, that is,

$$C(\mathbf{P}_0, \mathbf{U}) = c^*.$$

In addition, if the set of optimal equilibrium points defined by (3.5) has an empty interior $int\tau' = \emptyset$, then

$$\lim_{n\to\infty} (x_n + y_n) \in \tau' \tag{3.9}$$

for all optimal solution $\mathbf{P}$.

Proof We first claim that if $\mathbf{P} = \mathbf{P}(\mathbf{P}_0, \mathbf{U}) = (x_n, y_n)_{n=0}^{\infty}$ is a solution to (3.1) such that $C(\mathbf{P}_0, \mathbf{U}) = c^*$, then

$$(1-a_1)x_{n-K} + (1-a_2)y_{n-K} + a_1 x_n + a_2 y_n - (x_{n+1} + y_{n+1}) \ge -\xi_n, \ \forall n,$$

where $\xi_n > 0$ and $\xi_n \to 0$ as $n \to \infty$. Moreover,

$$\limsup_{n\to\infty} (x_n + y_n) \in \tau'. \tag{3.10}$$

Since $\liminf_{n\to\infty} c_n = c^*$, then there is a sequence of positive numbers $\xi_n \to 0$ such that $c_n \geq c^* - \xi_n$. Then from (3.8)

$$f_1(x_{n-K}) + g_1(y_{n-K}) - x_{n+1} + a_1x_n + f_2(x_{n-K}) + g_2(y_{n-K}) - y_{n+1} + a_2y_n \geq c^* - \xi_n,\ \forall n.$$

Then

$$\begin{aligned}&f_1(h_1(x_{n-K}+y_{n-K})) + g_1(h_2(x_{n-K}+y_{n-K})) + f_2(h_1(x_{n-K}+y_{n-K}))\\ &+ g_2(h_2(x_{n-K}+y_{n-K})) - (x_{n+1}+y_{n+1}) + a_1x_n + a_2y_n \geq c^* - \xi_n,\ \forall n.\end{aligned}$$

On the other hand, by definition of c^* it follows that

$$\begin{aligned}&f_1(h_1(x_{n-K}+y_{n-K})) + g_1(h_2(x_{n-K}+y_{n-K})) + f_2(h_1(x_{n-K}+y_{n-K}))\\ &+ g_2(h_2(x_{n-K}+y_{n-K})) - (1-a_1)x_{n-K} - (1-a_2)y_{n-K}\\ \leq &f_1(h_1(x_{n-K}+y_{n-K})) + g_1(h_2(x_{n-K}+y_{n-K})) + f_2(h_1(x_{n-K}+y_{n-K}))\\ &+ g_2(h_2(x_{n-K}+y_{n-K})) - (1-a_1)h_1(x_{n-K}+y_{n-K}) - (1-a_2)h_2(x_{n-K}+y_{n-K}) \leq c^*.\end{aligned}$$

Therefore, from the last two inequalities we have

$$(1-a_1)x_{n-K} + (1-a_2)y_{n-K} + a_1x_n + a_2y_n - (x_{n+1}+y_{n+1}) \geq -\xi_n,\ \forall\ n.$$

Now we prove (3.10). Denote $p := \limsup_{n\to\infty}(x_n+y_n) < \infty$. By the definition of p there is a subsequence $k_m \to \infty$ satisfying the following conditions:

$$x_{k_m+1} + y_{k_m+1} \to p;\ x_{k_m} + y_{k_m} \to p' \leq p;\ x_{k_m-K} + y_{k_m-K} \to p'' \leq p.$$

From (3.2) and (3.8), together with the fact $a < 1$ and $p'' \leq p$, we have

$$\begin{aligned}c^* =&\liminf_{n\to\infty} c_n \leq \lim_{k_m\to\infty} c_{k_m}\\ &= \lim_{k_m\to\infty}(f_1(x_{k_m-K}) + g_1(y_{k_m-K}) - x_{k_m+1} + a_1x_{k_m} + f_2(x_{k_m-K}) + g_2(y_{k_m-K}) - y_{k_m+1} + a_2y_{k_m})\\ &\leq \lim_{k_m\to\infty}[f_1(h_1(x_{k_m-K}+y_{k_m-K})) + g_1(h_2(x_{k_m-K}+y_{k_m-K})) + f_2(h_1(x_{k_m-K}+y_{k_m-K}))\\ &\qquad + g_2(h_2(x_{k_m-K}+y_{k_m-K})) - (x_{k_m+1}+y_{k_m+1}) + a_1x_{k_m} + a_2y_{k_m}]\\ &\leq f_1(h_1p'') + g_1(h_2p'') + f_2(h_1p'') + g_2(h_2p'') - (1-a)p\\ &\leq f_1(h_1p'') + g_1(h_2p'') + f_2(h_1p'') + g_2(h_2p'') - (1-a_1)h_1p - (1-a_2)h_2p\\ &\leq f_1(h_1p'') + g_1(h_2p'') - (1-a_1)h_1p'' + f_2(h_1p'') + g_2(h_2p'') - h_2(1-a_2)p''.\end{aligned}\tag{3.11}$$

This means that $(h_1p'', h_2p'') \in \tau$, from (3.6). Therefore,

$$c^* = f_1(h_1p'') + g_1(h_2p'') - (1-a_1)h_1p'' + f_2(h_1p'') + g_2(h_2p'') - (1-a_2)h_2p''.$$

Now, if $p'' < p$, then going back to (3.11) we obtain a contradiction in the form

$$\begin{aligned}c^* &\leq f_1(h_1p'') + g_1(h_2p'') + f_2(h_1p'') + g_2(h_2p'') - (1-a_1)h_1p - (1-a_2)h_2p\\ &< f_1(h_1p'') + g_1(h_2p'') + f_2(h_1p'') + g_2(h_2p'') - (1-a_1)h_1p'' - (1-a_2)h_2p''\\ &\leq f_1(h_1p'') + g_1(h_2p'') - (1-a_1)h_1p'' + f_2(h_1p'') + g_2(h_2p'') - (1-a_2)h_2p'' = c^*.\end{aligned}\tag{3.12}$$

Therefore, $p = p'' = \limsup_{n\to\infty}(x_n + y_n) \in \tau'$, which implies that (3.10) is true.

Finally, we prove (3.9), where is similar to [13]. Denote $q := \liminf_{n\to\infty}(x_n + y_n) < \infty$, $p := \limsup_{n\to\infty}(x_n + y_n) < \infty$ and $p \in \tau'$. Take any positive number $\eta \in (0, \overline{\eta}]$, where $\overline{\eta}$ is defined as $\overline{\eta} := \min\{p - \overline{q},\ \frac{p-q}{2}\} > 0$. Here $\overline{q} := a^K q + (1 - a^K)p$ and K is the delay in system (3.1). Denote $\widetilde{p} := p - \frac{1}{2}\eta\ < p$. Similar to the proof of Proposition 3.4 in [13], we can construct a sequence $k_m \to \infty$ satisfying

$$x_{k_m+1} + y_{k_m+1} \to p \geq \widetilde{p};\ x_{k_m} + y_{k_m} \to p' \leq \widetilde{p};\ x_{k_m-K} + y_{k_m-K} \to p'' \leq \widetilde{p}. \tag{3.13}$$

Therefore, similar to the calculation in (3.11), there is

$$c^* \leq f_1(h_1p'') + g_1(h_2p'') - (1 - a_1)h_1p'' + f_2(h_1p'') + g_2(h_2p'') - (1 - a_2)h_2p''.$$

In view of the definition of c^* and (3.13), similar to (3.12), we obtain $p'' = \widetilde{p} = p - \frac{1}{2}\eta \in \tau'$, $\forall \eta \in (0, \overline{\eta}]$ follows. This contradicts the assumption that τ has an empty interior and (3.9) is proved.

Theorem 3.3 Suppose that functions f_1, g_2, $H_1^{(i)} (i = 1, 2)$ are increasing, while functions f_2 and g_1 are monotone and possess the same monotonicity. For any given initial string $\mathbf{P}_0$ and any optimal equilibrium $(\tilde{x}, \tilde{y}) \in \tau$, $\tilde{p} = \tilde{x} + \tilde{y} \in \tau'$, there is a control $\mathbf{U}$ such that the corresponding solution $\mathbf{P} = \mathbf{P}(\mathbf{P}_0, \mathbf{U}) = (x_n, y_n)_{n=0}^{\infty}$ to (3.1) converges to that equilibrium $(\tilde{x}, \tilde{y})$; that is

$$\lim_{n\to\infty}(x_n, y_n) = (\tilde{x}, \tilde{y}) \in \tau.$$

Proof Take any $\tilde{p} \in \tau'$ and $(\tilde{x}, \tilde{y}) \in \tau$. From (2.2), (2.3) and (3.5), at least one of the two statements $\tilde{x} < X^*$ and $\tilde{y} < Y^*$ holds.

If at least one of $\tilde{x}$ and $\tilde{y}$ is vanish, then one can use the method as the same as that in [13] to find the control sequence, we omit it here.

Next, we consider the case $\tilde{x} > 0$, $\tilde{y} > 0$. Without loss of generality, we assume that $\tilde{y} < Y^*$, then from assumption (H2), there holds

$$f_2((H_1^{(1)})^{-1}(g_1(y))) + g_2(y) > (1 - a_2)y > 0,\ \forall y \in (0, \tilde{y}]. \tag{3.14}$$

For any given any initial string $\mathbf{P}_0$, since $\tilde{x} > 0$, $\tilde{y} > 0$, then there is a number n_1 such that

$$x_{n_1-j} + y_{n_1-j} \in (0, \widetilde{p}],\quad \forall j = 0, 1, \cdots, K.$$

For $n \geq n_1$ we define the sequences $\tilde{u}_n$, $\tilde{v}_n$ as follows:

$$\tilde{u}_n = \min\{1, \frac{\tilde{x} - a_1x_n}{f_1(x_{n-K}) + g_1(y_{n-K})}\},\ \tilde{v}_n = \min\{1, \frac{\tilde{y} - a_2y_n}{f_2(x_{n-K}) + g_2(y_{n-K})}\}, n \geq n_1.$$

Clearly, as long as $\tilde{x} > a_1x_n, \tilde{y} > a_2y_n$, the values of $\tilde{u}_n, \tilde{v}_n$ stay in the interval $[0, 1]$; that is, $\tilde{u}_n, \tilde{v}_n$ can be used as two control parameters. Consider a sequence (x_n, y_n) defined by

$$\begin{aligned} x_{n+1} &= a_1x_n + \tilde{u}_n(f_1(x_{n-K}) + g_1(y_{n-K})), \\ y_{n+1} &= a_2y_n + \tilde{v}_n(f_2(x_{n-K}) + g_2(y_{n-K})), \end{aligned} \tag{3.15}$$

then

$$x_{n+1}+y_{n+1}=a_1x_n+\tilde{u}_n(f_1(x_{n-K})+g_1(y_{n-K}))+a_2y_n+\tilde{v}_n(f_2(x_{n-K})+g_2(y_{n-K})).$$

First we show that this sequence (x_y, y_n) is a solution to (3.1); that is, $\tilde{u}_n, \tilde{v}_n \in [0,1]$ for all $n \geq n_1$. For $n = n_1$, if $\tilde{x} - a_1x_{n_1} \leq f_1(x_{n_1-K}) + g_1(y_{n_1-K})$, then $\tilde{u}_{n_1} = \dfrac{\tilde{x}-a_1x_{n_1}}{f_1(x_{n_1-K})+g_1(y_{n_1-K})}$, and $x_{n_1+1} = \tilde{x}$; else if $\tilde{x} - a_1x_{n_1} > f_1(x_{n_1-K}) + g_1(y_{n_1-K})$, then $\tilde{u}_{n_1} = 1$ and $x_{n_1+1} < \tilde{x}$, therefore $x_{n_1+1} \leq \tilde{x}$. Similarly, $y_{n_1+1} \leq \tilde{y}$. Therefore,

$$x_{n_1+1} \leq \tilde{x},\ y_{n_1+1} \leq \tilde{y},\ x_{n_1+1}+y_{n_1+1} \leq \tilde{p}.$$

Moreover, one has

$$x_{n_1+1}+y_{n_1+1}=a_1x_{n_1}+f_1(x_{n_1-K})+g_1(y_{n_1-K})+a_2y_{n_1}+f_2(x_{n_1-K})+g_2(y_{n_1-K}) \geq \delta,$$

where

$$\delta := \min_{j=0,1,\cdots,K}\{a_1x_{n_1-j}+a_2y_{n_1-j}, f_1(x_{n_1-j})+g_1(y_{n_1-j}), f_2(x_{n_1-j})+g_2(y_{n_1-j})\} > 0.$$

Therefore, $x_{n_1+1}+y_{n_1+1} \in [\delta,\ \tilde{p}]$. This in particular means that $\tilde{p} - a_1x_{n_1+1} - a_2y_{n_1+1} > 0$ or $\tilde{u}_{n_1+1} \in (0,1], \tilde{v}_{n_1+1} \in (0,1]$.

Continuing this process we obtain sequences $\tilde{u}_n, \tilde{v}_n$ and $x_n, y_n, x_n + y_n$ such that

$$\tilde{u}_n \in (0,1],\ \tilde{v}_n \in (0,1],\ \text{and } 0 < x_{n+1} \leq \tilde{x},\ 0 < y_{n+1} \leq \tilde{y},\ x_{n+1}+y_{n+1} \in [\delta, \widetilde{p}], \quad \forall n \geq n_1.$$

Moreover,

$$\tilde{u}_n = 1 \text{ if } x_n < \tilde{x}, \quad \tilde{v}_n = 1 \text{ if } y_n < \tilde{y} \ . \tag{3.16}$$

Thus, (3.15) defines a solution to (3.1).

Now we show that $\lim_{n\to\infty}(x_n+y_n) \in \tau'$. We first claim that if there is $\bar{x} > 0$ such that $\lim_{n\to\infty} x_n = \bar{x}$, then there is $\bar{y} > 0$ such that $\lim_{n\to\infty} y_n = \bar{y}$ and $\bar{y}+\bar{x} = \bar{p} \in \tau'$. If the claim is not true, then let $\liminf_{n\to\infty} y_n = y^-$, $\limsup_{n\to\infty} y_n = y^+$. Since $u_n \in [0,1]$, by taking the limit of (3.1) we have

$$y^+ \leq a_2y^+ + f_2(\bar{x}) + g_2(y^+).$$

On the other hand, for $y^- < \tilde{y}$, then $v_n = 1$, take the limit of (3.1) we have

$$y^- \geq a_2y^- + f_2(\bar{x}) + g_2(y^-).$$

Thus, we obtain

$$y^- - a_2y^- - g_2(y^-) \geq y^+ - a_2y^+ - g_2(y^+).$$

Note that $H_1^{(2)}(r) = (1-a_2)r - g_2(r)$ is increasing and $y^- \leq y^+$, we have $y^- = y^+$. That is, $\{y_n\}$ is also convergent. Set $y' := \lim_{n\to\infty} y_n$, $\bar{y} = \bar{p} - \bar{x}$, $\bar{p} \in \tau'$. We claim that $y' = \bar{y}$. By the definition of τ',

$$\bar{y} = a_2\bar{y} + \bar{v}_1(f_2(\bar{x}) + g_2(\bar{y})) \leq a_2\bar{y} + f_2(\bar{x}) + g_2(\bar{y}),$$

together with $y' = a_2 y' + f_2(\bar{x}) + g_2(y')$, we obtain

$$\bar{y} - a_2\bar{y} - g_2(\bar{y})) \le f_2(\bar{x}) = y' - a_2 y' - g_2(y'),$$

thus our claim is true. Similarly, if there is $\bar{y} > 0$ such that $\lim_{n\to\infty} y_n = \bar{y}$, then there is $\bar{x} > 0$ such that $\lim_{n\to\infty} x_n = \bar{x}$ and $\bar{x} + \bar{y} = \bar{p} \in \tau'$.

Denote $q := \liminf_{n\to\infty}(x_n + y_n)$, $q_x := \limsup_{n\to\infty} x_n$, $q_y := \liminf_{n\to\infty} y_n$, on the contrary assume, that is $q < \tilde{p}$. From $x_n + y_n \in (\delta, \tilde{p}]$, $\forall n \ge n_1$, we know that $q \ge \delta > 0$. Consider sequences $k_m \to \infty$ and $k'_m \to \infty$ such that

$$\begin{aligned}
&x_{k_m+1} \to q_x; \quad y_{k'_m+1} \to q_y;\\
&x_{k_m} \to q'_x \le q_x; \quad y_{k'_m} \to q'_y \ge q_y;\\
&x_{k_m-K} \to q''_x \le q_x; \quad y_{k'_m-K} \to q'_y \ge q_y.
\end{aligned}$$

Since for sufficiently large k'_m the inequalities $y_{k'_m+1} < \tilde{y}$ hold, from (3.16) we have $\tilde{u}_{k_m} = 1$, $\tilde{v}_{k'_m} = 1$. Then

$$\begin{aligned}
x_{k_m+1} &\le a_1 x_{k_m} + f_1(x_{k_m-K}) + g_1(y_{k_m-K}),\\
y_{k'_m+1} &= a_2 y_{k'_m} + f_2(x_{k'_m-K}) + g_2(y_{k'_m-K}).
\end{aligned} \tag{3.17}$$

By taking the limit of the first equation of (3.17) and since f_1, g_2 are increasing and f_2, g_1 are decreasing, we have

$$\begin{aligned}
q_x &\le a_1 q'_x + f_1(q''_x) + g_1(\lim_{k_m\to\infty} y_{k_m-K})\\
&\le a_1 q_x + f_1(q_x) + g_1(q_y),
\end{aligned}$$

that is, $g_1(q_y) \ge H_1^{(1)}(q_x)$. Since $H_1^{(1)}$ is continuous and increasing, then $q_x \le (H_1^{(1)})^{-1}(g_1(q_y))$. On the other hand, take the limit of the second equation of (3.17) we also obtain

$$\begin{aligned}
q_y &= a_2 q'_y + f_2(\lim_{k_m\to\infty} x_{k'_m-K}) + g_2(q''_y)\\
&\ge a_2 q_y + f_2(q_x) + g_2(q_y) \ge a_2 q_y + f_2((H_1^{(1)})^{-1}(g_1(q_y))) + g_2(q_y),
\end{aligned}$$

which is a contradiction with (3.14). If f_2, g_1 are increasing, let $q_x := \liminf_{n\to\infty} x_n$, by taking the limit of the first equation of (3.17) we have

$$q_x = a_1 q'_x + f_1(q''_x) + g_1(\lim_{k_m\to\infty} y_{k_m-K}) \ge a_1 q_x + f_1(q_x) + g_1(q_y),$$

that is, $g_1(q_y) \le H_1^{(1)}(q_x)$. Thus we also obtain the similar contradiction. Therefore, $\lim_{n\to\infty}(x_n + y_n) = \tilde{p} \in \tau'$. Together with $0 < x_{n+1} \le \tilde{x}$, $0 < y_{n+1} \le \tilde{y}$, one has $\lim_{n\to\infty}(x_n, y_n) = (\tilde{x}, \tilde{y}) \in \tau$. Therefore, $\lim_{n\to\infty}(x_n + y_n) = \tilde{p} \in \tau'$. Together with $0 < x_{n+1} \le \tilde{x}$, $0 < y_{n+1} \le \tilde{y}$, one has $\lim_{n\to\infty}(x_n, y_n) = (\tilde{x}, \tilde{y}) \in \tau$.

4 Numerical Results

In this section, we will give an example. Given that $a_1 = 0.51, a_2 = 0.65$ and the initial string

$$\begin{aligned} P_0 &= \{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5), (x_6, y_6)\} \\ &= \{(24.8, 22.8), (20.4, 22), (23.6, 25), (21.2, 23.2), (22.6, 21.7)\}. \end{aligned}$$

$$\mathbf{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} x^{0.78} \\ x^{0.65} \end{pmatrix}, \qquad \mathbf{G} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 0.06 * \ln x \\ 0.08 * \ln x \end{pmatrix}.$$

Under this circumstance, we obtain the equilibrium (X^*, Y^*) of system (1) is $(27.3499, 25.2828)$

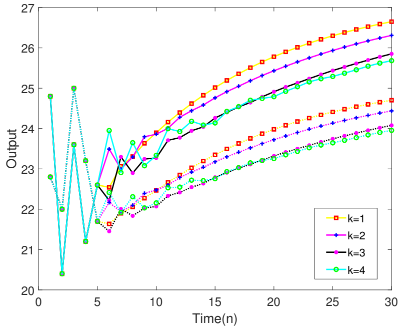


Figure 1: Numerical simulation of Equation (2.1) with different time delay K, where the solid line is x_n and the dotted line is y_n

From Figure1, we can see that the higher value of time delay K is, the lower output of factory is. Moreover, whatever the value of K is, the yield of x_n and y_n will approach to equilibrium $(27.3499, 25.2828)$ when n is sufficiently large.

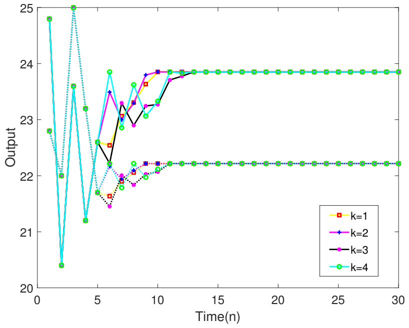
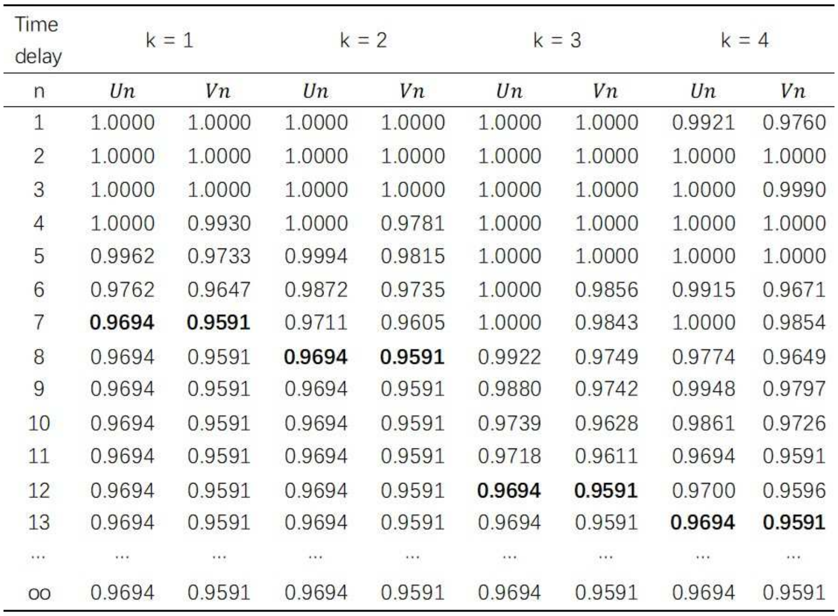


Figure 2: Numerical simulation of Equation (2.1) with different time delay K, where the solid line is x_n and the dotted line is y_n

Table 1: The convergence of control function u_n and v_n with different time delay K



Time delay	k = 1		k = 2		k = 3		k = 4	
n	*Un*	*Vn*	*Un*	*Vn*	*Un*	*Vn*	*Un*	*Vn*
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9921	0.9760
2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9990
4	1.0000	0.9930	1.0000	0.9781	1.0000	1.0000	1.0000	1.0000
5	0.9962	0.9733	0.9994	0.9815	1.0000	1.0000	1.0000	1.0000
6	0.9762	0.9647	0.9872	0.9735	1.0000	0.9856	0.9915	0.9671
7	**0.9694**	**0.9591**	0.9711	0.9605	1.0000	0.9843	1.0000	0.9854
8	0.9694	0.9591	**0.9694**	**0.9591**	0.9922	0.9749	0.9774	0.9649
9	0.9694	0.9591	0.9694	0.9591	0.9880	0.9742	0.9948	0.9797
10	0.9694	0.9591	0.9694	0.9591	0.9739	0.9628	0.9861	0.9726
11	0.9694	0.9591	0.9694	0.9591	0.9718	0.9611	0.9694	0.9591
12	0.9694	0.9591	0.9694	0.9591	**0.9694**	**0.9591**	0.9700	0.9596
13	0.9694	0.9591	0.9694	0.9591	0.9694	0.9591	**0.9694**	**0.9591**
...	...	...	...	...	...	...	...	...
oo	0.9694	0.9591	0.9694	0.9591	0.9694	0.9591	0.9694	0.9591

By Theorem 3.3 with optimal equilibrium $(\tilde{x}, \tilde{y}) = (23.852, 22.217)$, we obtain the numerical results of equation (3.1) as Figure2 shown. The convergence of control sequences u_n and v_n could be seen in Table 1. In this example, we find that (u_n, v_n) converges to the same result $(0.9694, 0.9591)$ with different time delay and the rate of convergence is barely affected by value K.

References

[1] Cooke K, Ivanov A. On the discretization of a delay differential equation[J]. Journal of Difference Equations and Applications, 2000, 6(1): 105–119.

[2] Hirsch M. Optimization and cooperative control strategies: proceedings of the 8th international conference on cooperative control and optimization[M]. New York: Springer Science and Business Media, 2009.

[3] Mamedov M A. Asymptotical stability of optimal paths in nonconvex problems[M]. New York: Springer, 2009: 95–134.

[4] Richtmyer R D, Morton K W. Difference methods for initial-value problems[J]. Physics Today, 1994, 12(4).

[5] Braverman E, Liz E. Global stabilization of periodic orbits using a proportional feedback control with pulses[J]. Nonlinear Dynamics, 2012, 67(4): 2467–2475.

[6] El-Morshedy H A, Liz E. Globally attracting fixed points in higher order discrete population models[J]. Journal of Mathematical Biology, 2006, 53(3): 365–384.

[7] Kuang Yang. Delay differential equations: with applications in population dynamics[M]. New York: Academic press, 1993.

[8] Ivanov A F, Trofimchuk S I. Periodic solutions of a discretized differential delay equation[J]. Journal of Difference Equations and Applications, 2010, 16(2–3): 157–171.

[9] Gwinner J. On optimality conditions for infinite programs[M]. Berlin: Springer, 1981: 21–27.

[10] Mammadov M A. Turnpike theory: stability of optimal trajectories[M]. Boston: Springer, 2009: 3948–3955.

[11] Zaslavski A. Turnpike properties in the calculus of variations and optimal control[M]. New York: Springer Science and Business Media, 2005.

[12] Pardalos P M, Yatsenko V A. Optimization and control of bilinear systems: theory, algorithms, and applications[M]. New York: Springer Science and Business Media, 2010.

[13] Ivanov A F, Mammadov M A, Trofimchuk S I. Global stabilization in nonlinear discrete systems with time-delay[J]. Journal of Global Optimization, 2013, 56(2): 251–263.

一类非线性时滞离散系统的渐近行为

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摘要： 本文研究了一类二维时滞非线性差分系统. 利用稳定性理论和最优控制理论等方法, 本文首先获得了保证正平衡点的存在唯一以及全局渐近稳定的充分条件; 然后对消费函数的最大化的最优控制问题进行讨论, 获得了最优解的存在性与稳定性条件; 最后, 通过数值模拟验证了结果的有效性. 本文推广了一维时滞非线性差分系统中给出的相关结论.

关键词： 时滞差分系统; 全局渐近稳定; 最优控制方法

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