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INEQUALITIES FOR EIGENVALUES OF A CLASS OF OPERATORS ON RIEMANNIAN MANIFOLDS

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Abstract: In this paper, we investigate the weighted Dirichlet eigenvalue problem of a class of operators on Riemannian manifolds isometrically immersed into a Euclidean space and Riemannian manifolds admitting some special functions. We establish some universal inequalities for eigenvalues of this problem. Moreover, as applications, we derive some results for the weighted Dirichlet eigenvalue problem of quadratic polynomial operator of the Laplacain.

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1 Introduction

In recent years, there is increasing interest in the research of elliptic operators in divergence form (cf. [1–3]). Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. Let $A:\Omega\to \mathrm{End}\,(T\Omega)$ be a smooth symmetric and positive definite section of the bundle of all endomorphisms of the tangent bundle $T\Omega$. Do Carmo, Wang and Xia [4], Sun and Chen [5] researched the Dirichlet eigenvalue problem of elliptic operator in divergence form $\mathrm{div}\,(A\nabla)$ and gave some universal inequalities for its eigenvalues, where ∇ and div were the gradient operator and the divergence operator on M. The operator $\mathrm{div}\,(A\nabla)$ is an interesting operator. It includes the Laplacian Δ as special case. Moreover, it has connections with second order elliptic operators with variable coefficients on $\Omega\subset\mathbb{R}^n$, the linearized operator L_r of the r-th mean curvature of a hypersurface and so on.

The purpose of this paper is to investigate the following problem

$$\begin{cases}\Delta^2 u - p\mathrm{div}\,(A\nabla u)+qu=\lambda\rho u, & \text{in} \quad \Omega,\\ u|_{\partial\Omega}=\dfrac{\partial u}{\partial \nu}|_{\partial\Omega}=0, & \end{cases}\tag{1.1}$$

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Biography: Fu Ke (1995–), female, born at Hangzhou, Zhejiang, postgraduate, major in: Differential geometry.

Corresponding author: Sun Hejun, E-mail: hejunsun@163.com where ρ is a positive continuous function on Ω, ν denotes the outward unit normal to the boundary $\partial\Omega$ and the constants $p, q \geq 0$. It is well known that problem (1.1) has real and discrete spectrum

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty,$$

where each eigenvalue is repeated according to its multiplicity. This problem has some connections with some other problems. When A is the identity, problem (1.1) becomes the following weighted Dirichlet problem of quadratic polynomial operator of the Laplacain.

$$\begin{cases} \Delta^2 u - p\Delta u + qu = \lambda\rho u, & \text{in} \quad \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial\nu}|_{\partial\Omega} = 0. \end{cases} \tag{1.2}$$

Furthermore, when $p = q = 0$ and $\rho \equiv 1$, problem (1.2) becomes the clamped plate problem

$$\begin{cases} \Delta^2 u = \lambda u, & \text{in} \quad \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial\nu}|_{\partial\Omega} = 0. \end{cases} \tag{1.3}$$

Cheng and Yang [6], Cheng, Ichikawa and Mametsuka [7], Wang and Xia [8] established some universal inequalities for problem (1.3). Sun and Chen [9], Sun and Qi [10] obtained some universal inequalities for problem (1.2). Shi [11] gave some inequalities for lower order eigenvalues of problem (1.1) in the case of $p = 1$ and $\rho \equiv 1$.

One of the main goals of this paper is to establish some inequalities for eigenvalues of problem (1.1) on bounded domains of some Riemannian manifolds. Nash's theorem [14] states that any complete Riemannian manifold M can be isometrically immersed into an Euclidean space. We first give the following result:

Theorem 1.1 Let Ω be a connected bounded domain in an n-dimensional complete Riemannian manifold M. Let $A : \Omega \to \mathrm{End}(T\Omega)$ be a smooth symmetric and positive definite section of the bundle of all endomorphisms of the tangent bundle $T\Omega$. Assume that the eigenvalues of A are bounded below by ξ_1 and that $tr\,(A) \leq n\xi_2$ throughout Ω, where ξ_1 and ξ_2 are two positive constants. Denote by λ_i the i-th eigenvalue of problem (1.1). Set $\rho_1 = \min\limits_{x\in\Omega}\rho(x)$ and $\rho_2 = \max\limits_{x\in\Omega}\rho(x)$. If M is isometrically immersed in $\mathbb{R}^N$ with mean curvature vector $\mathbf{H}$, then we have

$$\begin{aligned} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq & \frac{\rho_2}{n\rho_1^{\frac{1}{2}}} \left\{ \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \left[(2n+4) B_i + \frac{n^2 H_0^2 + pn\xi_2}{\rho_1} \right] \right\}^{\frac{1}{2}} \\ & \times \left[\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(4B_i + \frac{n^2 H_0^2}{\rho_1} \right) \right]^{\frac{1}{2}}, \end{aligned} \tag{1.4}$$

where $H_0 = \max\limits_{x\in\Omega} |\mathbf{H}|\,(x)$ and $B_i = \frac{1}{2\rho_1}\left[-p\xi_1 + \sqrt{p^2\xi_1^2 + 4\rho_1(\lambda_i - \frac{q}{\rho_2})} \right]$.

As we know, $H_0 = 0$ when M is an n-dimensional complete minimal submanifold in an Euclidean space and $H_0 = 1$ when M is an n-dimensional unit sphere. Therefore, from Theorem 1.1, we can get some results for problem (1.1) on these two kinds of manifolds.

Corollary 1.1 Let Ω be a connected bounded domain in an n-dimensional complete minimal submanifold in a Euclidean space. Denote by λ_i the i-th eigenvalue of problem (1.1). Then we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{2\rho_2}{n\rho_1^{\frac{1}{2}}}\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left[(2n+4)\,B_i+\frac{pn\xi_2}{\rho_1}\right]\right\}^{\frac{1}{2}}\left[\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)B_i\right]^{\frac{1}{2}}. \tag{1.5}$$

Corollary 1.2 Let Ω be a connected bounded domain in an n-dimensional unit sphere $\mathbb{S}^n(1)$. Denote by λ_i the i-th eigenvalue of problem (1.1). Then we have

$$\begin{aligned}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le & \frac{\rho_2}{n\rho_1^{\frac{1}{2}}}\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left[(2n+4)\,B_i+\frac{n^2+pn\xi_2}{\rho_1}\right]\right\}^{\frac{1}{2}}\\ &\times\left[\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(4B_i+\frac{n^2}{\rho_1}\right)\right]^{\frac{1}{2}}.\end{aligned} \tag{1.6}$$

Moreover, we consider problem (1.1) on Riemannian manifolds admitting some special functions. We obtain the following result:

Theorem 1.2 Let Ω be a connected bounded domain in an n-dimensional complete Riemannian manifold M. Let $A:\Omega\to\mathrm{End}(T\Omega)$ be a smooth symmetric and positive definite section of the bundle of all endomorphisms of the tangent bundle $T\Omega$. Assume that $\xi_1 I\le A\le \xi_2 I$ in the sense that the eigenvalues of A lie in the interval $[\xi_1,\xi_2]$ throughout Ω, where ξ_1 and ξ_2 are two positive constants. Denote by λ_i the i-th eigenvalue of problem (1.1). Set $\rho_1=\min\limits_{x\in\Omega}\rho(x)$, $\rho_2=\max\limits_{x\in\Omega}\rho(x)$ and $B_i=\frac{1}{2\rho_1}\left[-p\xi_1+\sqrt{p^2\xi_1^2+4\rho_1(\lambda_i-\frac{q}{\rho_2})}\right]$.

i) If there exists a function $\psi:\Omega\to\mathbb{R}$ and a constant C_0 such that $|\nabla\psi|=1$ and $|\Delta\psi|\le C_0$ on Ω, then

$$\begin{aligned}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le & \frac{\rho_2}{\rho_1^{\frac{1}{2}}}\left[\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left(6B_i+4C_0\sqrt{\frac{B_i}{\rho_1}}+\frac{C_0^2+p\xi_2}{\rho_1}\right)\right]^{\frac{1}{2}}\\ &\times\left[\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(4B_i+4C_0\sqrt{\frac{B_i}{\rho_1}}+\frac{C_0^2}{\rho_1}\right)\right]^{\frac{1}{2}};\end{aligned} \tag{1.7}$$

ii) If there exists a function $\varphi:\Omega\to\mathbb{R}$ and a constant D_0 such that $|\nabla\varphi|=1$ and $\Delta\varphi=D_0$ on Ω, then

$$\begin{aligned}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le & \frac{\rho_2}{\rho_1^{\frac{1}{2}}}\left[\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left(6B_i-\frac{2D_0^2}{\rho_2}+\frac{D_0^2+p\xi_2}{\rho_1}\right)\right]^{\frac{1}{2}}\\ &\times\left[\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(4B_i-\frac{2D_0^2}{\rho_2}+\frac{D_0^2}{\rho_1}\right)\right]^{\frac{1}{2}};\end{aligned} \tag{1.8}$$

iii) If there exist l functions $\phi_\alpha : \Omega \to \mathbb{R}$ such that $\langle \nabla\phi_\alpha, \nabla\phi_\beta\rangle = \delta_{\alpha\beta}$ and $\Delta\phi_\alpha = 0,\ \alpha,\beta = 1,\cdots,l$ on Ω, then

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{2\rho_2}{l\rho_1^{\frac{1}{2}}}\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left[(2l+4)B_i+\frac{pl\xi_2}{\rho_1}\right]\right\}^{\frac{1}{2}}\left[\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)B_i\right]^{\frac{1}{2}}; \tag{1.9}$$

iv) If Ω admits an eigenmap $f = (f_1, f_2, \cdots, f_{N+1}) : \Omega \to \mathbb{S}^N(1)$ corresponding to an eigenvalue μ, that is $\sum_{\alpha=1}^{N+1} f_\alpha^2 = 1$ and $\Delta f_\alpha = -\mu f_\alpha$, $\alpha = 1, 2, \cdots, N+1$, then

$$\begin{aligned}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le & \frac{\rho_2}{\mu\rho_1^{\frac{1}{2}}}\left[\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left(6\mu B_i+\frac{\mu^2+p\mu\xi_2}{\rho_1}\right)\right]^{\frac{1}{2}}\\ &\times\left[\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(4\mu B_i+\frac{\mu^2}{\rho_1}\right)\right]^{\frac{1}{2}}.\end{aligned} \tag{1.10}$$

As applications of the above results, we can give some results for problem (1.2). For example, when A is the identity, we can obtain the following corollary for problem (1.2) from Theorem 1.1.

Corollary 1.3 Let M be an n-dimensional complete Riemannian manifold and Ω be a connected bounded domain in M. Denote by λ_i the i-th eigenvalue of problem (1.2). Set $\rho_1 = \min\limits_{x\in\Omega}\rho(x)$ and $\rho_2 = \max\limits_{x\in\Omega}\rho(x)$. $E_i = \frac{1}{2\rho_1}\left[-p+\sqrt{p^2+4\rho_1(\lambda_i-\frac{q}{\rho_2})}\right]$. If M is isometrically immersed in $\mathbb{R}^N$ with mean curvature vector $\mathbf{H}$, then

$$\begin{aligned}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le & \frac{\rho_2}{n\rho_1^{\frac{1}{2}}}\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left[(2n+4)E_i+\frac{n^2H_0^2+pn}{\rho_1}\right]\right\}^{\frac{1}{2}}\\ &\times\left[\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(4E_i+\frac{n^2H_0^2}{\rho_1}\right)\right]^{\frac{1}{2}},\end{aligned} \tag{1.11}$$

where $H_0 = \max\limits_{x\in\Omega}|\mathbf{H}|(x)$.

Remark 1.1 Let $\{a_i\}_{i=1}^m$, $\{b_i\}_{i=1}^m$ and $\{c_i\}_{i=1}^m$ be three sequences of non-negative real numbers with $\{a_i\}_{i=1}^m$ decreasing and $\{b_i\}_{i=1}^m$ and $\{c_i\}_{i=1}^m$ increasing. Then it holds [8]

$$\left(a_i^2b_i\right)(a_ic_i) \le \left(a_i^2\right)(a_ib_ic_i). \tag{1.12}$$

Using (1.12), we know that (1.11) becomes (1.11) of Theorem 1.1 in [10].

Moreover, when A is the identity, we can derive the following corollary for problem (1.2) from Theorem 1.2.

Corollary 1.4 Let M be an n-dimensional complete Riemannian manifold and Ω be a connected bounded domain in M. Denote by λ_i the i-th eigenvalue of problem (1.2). Set $\rho_1 = \min\limits_{x\in\Omega}\rho(x)$, $\rho_2 = \max\limits_{x\in\Omega}\rho(x)$ and $E_i = \frac{1}{2\rho_1}\left[-p+\sqrt{p^2+4\rho_1(\lambda_i-\frac{q}{\rho_2})}\right]$.

i) If there exists a function $\psi : \Omega \to \mathbb{R}$ and a constant C_0 such that $|\nabla\psi| = 1$ and $|\Delta\psi| \le C_0$ on Ω, then

$$\begin{aligned}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le & \frac{\rho_2}{\rho_1^{\frac{1}{2}}}\left[\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left(6E_i+4C_0\sqrt{\frac{E_i}{\rho_1}}+\frac{C_0^2+p}{\rho_1}\right)\right]^{\frac{1}{2}} \\ & \times\left[\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(4E_i+4C_0\sqrt{\frac{E_i}{\rho_1}}+\frac{C_0^2}{\rho_1}\right)\right]^{\frac{1}{2}};\end{aligned} \tag{1.13}$$

ii) If there exists a function $\varphi : \Omega \to \mathbb{R}$ and a constant D_0 such that $|\nabla\varphi| = 1$ and $\Delta\varphi = D_0$ on Ω, then

$$\begin{aligned}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le & \frac{\rho_2}{\rho_1^{\frac{1}{2}}}\left[\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left(6E_i-\frac{2D_0^2}{\rho_2}+\frac{D_0^2+p}{\rho_1}\right)\right]^{\frac{1}{2}} \\ & \times\left[\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(4E_i-2\frac{D_0^2}{\rho_2}+\frac{D_0^2}{\rho_1}\right)\right]^{\frac{1}{2}};\end{aligned} \tag{1.14}$$

iii) If there exist l functions $\phi_\alpha : \Omega \to \mathbb{R}$ such that $\langle\nabla\phi_\alpha, \nabla\phi_\beta\rangle = \delta_{\alpha\beta}$ and $\Delta\phi_\alpha = 0$, $\alpha, \beta = 1, \cdots, l$ on Ω, then

$$\begin{aligned}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le & \frac{2\rho_2}{l\rho_1^{\frac{1}{2}}}\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left[(2l+4)\rho_1E_i+pl\right]\right\}^{\frac{1}{2}} \\ & \times\left[\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)E_i\right]^{\frac{1}{2}};\end{aligned} \tag{1.15}$$

iv) If Ω admits an eigenmap $f = (f_1, f_2, \cdots, f_{N+1}) : \Omega \to \mathbb{S}^N(1)$ corresponding to an eigenvalue μ, that is $\sum_{\alpha=1}^{N+1} f_\alpha^2 = 1$ and $\Delta f_\alpha = -\mu f_\alpha$, $\alpha = 1, 2, \cdots, N+1$, then

$$\begin{aligned}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le & \frac{\rho_2}{\mu\rho_1^{\frac{1}{2}}}\left[\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left(6\mu E_i+\frac{\mu^2+p\mu}{\rho_1}\right)\right]^{\frac{1}{2}} \\ & \times\left[\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(4\mu E_i+\frac{\mu^2}{\rho_1}\right)\right]^{\frac{1}{2}}.\end{aligned} \tag{1.16}$$

Remark 1.2 It is easy to find that when $p = q = 0$ and $\rho \equiv 1$, (1.13-1.16) for problem (1.2) become the results of Theorem 1.1 for problem (1.3) in [8].

2 Proof of the Main Results

In this section, we first prove a general inequality which plays the key role in the proofs of Theorems 1.1 and 1.2.

Lemma 2.1 Let Ω be a bounded domain in an n-dimensional compact Riemannian manifold M. Assume that $A:\Omega\rightarrow \operatorname{End}(T\Omega)$ is a smooth symmetric and positive definite section of the bundle of all endomorphisms of the tangent bundle $T\Omega$. Denote by λ_i the i-th eigenvalue of problem (1.1) and u_i the i-th weighted orthonormal eigenfunctions. For any function $h\in C^4\left(\bar{\Omega}\right)$ and any integer k, we have

$$\begin{aligned}&\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\int_\Omega u_i^2\left|\nabla h\right|^2\\ \leq&\frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_\Omega\frac{1}{\rho}\left[\left\langle\nabla h,\nabla u_i\right\rangle^2+u_i\Delta h\left\langle\nabla h,\nabla u_i\right\rangle+\frac{u_i^2\left(\Delta h\right)^2}{4}\right]\\ &+\delta\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\left(-2\int_\Omega u_i\Delta u_i\left|\nabla h\right|^2+4\int_\Omega u_i\Delta h\left\langle\nabla h,\nabla u_i\right\rangle\right.\\ &\left.+4\int_\Omega\left\langle\nabla h,\nabla u_i\right\rangle^2+\int_\Omega u_i^2\left(\Delta h\right)^2+p\int_\Omega u_i^2\left\langle\nabla h,A\nabla h\right\rangle\right),\end{aligned}\tag{2.1}$$

where δ is any positive constant.

Proof For each $i=1,\cdots,k$, we define $\varphi_i:\Omega\rightarrow\mathbb{R}$ by $\varphi_i=hu_i-\sum_{j=1}^{k}a_{ij}u_j$, where $a_{ij}=\displaystyle\int_\Omega\rho hu_iu_j=a_{ji}$. Since $\displaystyle\int_\Omega\rho\varphi_iu_j=0$, $\forall i,j=1,\cdots,k$, we know from Rayleigh-Ritz inequality that

$$\begin{aligned}\lambda_{k+1}\int_\Omega\rho\varphi_i^2\leq&\int_\Omega\varphi_i\left[\Delta^2\varphi_i-p\operatorname{div}\left(A\nabla\varphi_i\right)+q\varphi_i\right]\\ =&\int_\Omega\varphi_i\left[\Delta^2\left(hu_i\right)-p\operatorname{div}\left(A\nabla\left(hu_i\right)\right)+qhu_i\right].\end{aligned}\tag{2.2}$$

By direct computation, we have

$$-\operatorname{div}\left(A\nabla\left(hu_i\right)\right)=-h\operatorname{div}\left(A\nabla u_i\right)-2\left\langle\nabla h,A\nabla u_i\right\rangle-u_i\operatorname{div}\left(A\nabla h\right)\tag{2.3}$$

and

$$\begin{aligned}\Delta^2\left(hu_i\right)=&\Delta\left(h\Delta u_i+2\left\langle\nabla h,\nabla u_i\right\rangle+u_i\Delta h\right)\\ =&h\Delta^2u_i+2\left\langle\nabla h,\nabla\Delta u_i\right\rangle+2\Delta\left\langle\nabla h,\nabla u_i\right\rangle+u_i\Delta^2h\\ &+2\left\langle\nabla u_i,\nabla\Delta h\right\rangle+2\Delta u_i\Delta h.\end{aligned}\tag{2.4}$$

Substituting (2.3) and (2.4) into (2.2), we get

$$\left(\lambda_{k+1}-\lambda_i\right)\int_\Omega\rho\varphi_i^2\leq\int_\Omega hp_iu_i-\sum_{j=1}^{k}a_{ij}\int_\Omega p_iu_j,\tag{2.5}$$

where $p_i=\Delta h\Delta u_i+2\left\langle\nabla h,\nabla\Delta u_i\right\rangle+2\Delta\left\langle\nabla h,\nabla u_i\right\rangle+\Delta\left(u_i\Delta h\right)-pu_i\operatorname{div}\left(A\nabla h\right)-2p\left\langle\nabla h,A\nabla u_i\right\rangle.$

Using the divergence theorem, we deduce

$$\begin{aligned}\int_\Omega\Delta u_j\left\langle\nabla h,\nabla u_i\right\rangle=&-\int_\Omega h\operatorname{div}\left(\Delta u_j\nabla u_i\right)=-\int_\Omega h\left\langle\nabla\Delta u_j,\nabla u_i\right\rangle-\int_\Omega h\Delta u_i\Delta u_j\\ =&\int_\Omega u_i\operatorname{div}\left(h\nabla\Delta u_j\right)-\int_\Omega h\Delta u_i\Delta u_j\end{aligned}\tag{2.6}$$

and

$$\begin{aligned}\int_\Omega u_i \mathrm{div}\left(h\nabla\Delta u_j\right) &= \int_\Omega hu_i\Delta^2 u_j + \int_\Omega u_i\left\langle \nabla h, \nabla\Delta u_j\right\rangle \\ &= \int_\Omega hu_i\Delta^2 u_j - \int_\Omega \Delta u_j \mathrm{div}\left(u_i\nabla h\right) \\ &= \int_\Omega hu_i\Delta^2 u_j - \int_\Omega \Delta u_j\left\langle \nabla h, \nabla u_i\right\rangle - \int_\Omega u_i\Delta u_j\Delta h.\end{aligned}\tag{2.7}$$

Using (2.6) and (2.7), we derive

$$\begin{aligned}&\int_\Omega \Delta u_j\left\langle \nabla h, \nabla u_i\right\rangle - \int_\Omega \Delta u_i\left\langle \nabla h, \nabla u_j\right\rangle \\ =&\int_\Omega u_i\mathrm{div}\left(h\nabla\Delta u_j\right) - \int_\Omega u_j\mathrm{div}\left(h\nabla\Delta u_i\right) \\ =&\int_\Omega hu_i\Delta^2 u_j - \int_\Omega \Delta u_j\left\langle \nabla h, \nabla u_i\right\rangle - \int_\Omega u_i\Delta u_j\Delta h - \int_\Omega hu_j\Delta^2 u_i \\ &+\int_\Omega \Delta u_i\left\langle \nabla h, \nabla u_j\right\rangle + \int_\Omega u_j\Delta u_i\Delta h.\end{aligned}\tag{2.8}$$

It yields

$$\begin{aligned}&2\int_\Omega \Delta u_j\left\langle \nabla h, \nabla u_i\right\rangle - 2\int_\Omega \Delta u_i\left\langle \nabla h, \nabla u_j\right\rangle \\ =&\int_\Omega hu_i\Delta^2 u_j - \int_\Omega u_i\Delta u_j\Delta h - \int_\Omega hu_j\Delta^2 u_i + \int_\Omega u_j\Delta u_i\Delta h.\end{aligned}\tag{2.9}$$

Moreover, we have

$$\begin{aligned}\int_\Omega u_iu_j\mathrm{div}\left(A\nabla h\right) + \int_\Omega 2u_j\left\langle \nabla h, A\nabla u_i\right\rangle &= \int_\Omega h\mathrm{div}\left(A\nabla\left(u_iu_j\right)\right) - \int_\Omega 2h\mathrm{div}\left(u_jA\nabla u_i\right) \\ &= \int_\Omega hu_i\mathrm{div}\left(A\nabla u_j\right) - \int_\Omega hu_j\mathrm{div}\left(A\nabla u_i\right)\end{aligned}\tag{2.10}$$

and

$$\int_\Omega u_j\left\langle \nabla h, \nabla\Delta u_i\right\rangle = -\int_\Omega \Delta u_i\mathrm{div}\left(u_j\nabla h\right) = -\int_\Omega u_j\Delta u_i\Delta h - \int_\Omega \Delta u_i\left\langle \nabla h, \nabla u_j\right\rangle.\tag{2.11}$$

Using (2.9)-(2.11), we obtain

$$\begin{aligned}\int_\Omega p_iu_j =&\int_\Omega \left[u_j\Delta u_i\Delta h + 2u_j\left\langle \nabla h, \nabla\Delta u_i\right\rangle + 2u_j\Delta\left\langle \nabla h, \nabla u_i\right\rangle + u_j\Delta\left(u_i\Delta h\right)\right. \\ &\left.-pu_iu_j\mathrm{div}\left(A\nabla h\right) - 2pu_j\left\langle \nabla h, A\nabla u_i\right\rangle\right] \\ =&\int_\Omega \left[-u_j\Delta u_i\Delta h + hu_i\Delta^2 u_j - u_i\Delta u_j\Delta h - hu_j\Delta^2 u_i + u_j\Delta u_i\Delta h\right. \\ &\left.+u_i\Delta u_j\Delta h - phu_i\mathrm{div}\left(A\nabla u_j\right) + phu_j\mathrm{div}\left(A\nabla u_i\right)\right] \\ =&\int_\Omega hu_i\left[\Delta^2 u_j - p\mathrm{div}\left(A\nabla u_j\right) + qu_j\right] - \int_\Omega hu_j\left[\Delta^2 u_i - p\mathrm{div}\left(A\nabla u_i\right) + qu_i\right] \\ =&\left(\lambda_j - \lambda_i\right)a_{ij}.\end{aligned}\tag{2.12}$$

Substituting (2.12) into (2.5), we get

$$\left(\lambda_{k+1}-\lambda_i\right)\int_\Omega \rho\varphi_i^2 \leq \int_\Omega h u_i p_i+\sum_{j=1}^k\left(\lambda_i-\lambda_j\right) a_{ij}^2. \tag{2.13}$$

Set $b_{ij}=\displaystyle\int_\Omega u_j\left(\langle\nabla h, \nabla u_i\rangle+\frac{u_i \Delta h}{2}\right)$. Then one gets from the divergence theorem that

$$\begin{aligned} b_{ij}+b_{ji} &=\int_\Omega\left(u_j\left\langle\nabla h, \nabla u_i\right\rangle+u_i\left\langle\nabla h, \nabla u_j\right\rangle+u_i u_j \Delta h\right) \\ &=\int_\Omega\left\langle\nabla h, \nabla\left(u_i u_j\right)\right\rangle+\int_\Omega u_i u_j \Delta h \\ &=0. \end{aligned} \tag{2.14}$$

Moreover, since

$$-\int_\Omega h u_i^2 \Delta h=\int_\Omega\left\langle\nabla\left(h u_i^2\right), \nabla h\right\rangle=2 \int_\Omega h u_i\left\langle\nabla h, \nabla u_i\right\rangle+\int_\Omega u_i^2|\nabla h|^2,$$

it holds

$$-2 \int_\Omega \varphi_i\left(\left\langle\nabla h, \nabla u_i\right\rangle+\frac{u_i \Delta h}{2}\right)=\int_\Omega u_i^2|\nabla h|^2+2 \sum_{j=1}^k a_{ij} b_{ij}. \tag{2.15}$$

Multiplying both sides of (2.15) by $\left(\lambda_{k+1}-\lambda_i\right)^2$, using the Schwarz inequality, we get

$$\begin{aligned} & \left(\lambda_{k+1}-\lambda_i\right)^2\left(\int_\Omega u_i^2|\nabla h|^2+2 \sum_{j=1}^k a_{ij} b_{ij}\right) \\ \leq & \frac{\lambda_{k+1}-\lambda_i}{\delta} \int_\Omega\left[\frac{1}{\sqrt{\rho}}\left(\left\langle\nabla h, \nabla u_i\right\rangle+\frac{u_i \Delta h}{2}\right)-\sum_{j=1}^k b_{ij} \sqrt{\rho} u_j\right]^2+\delta\left(\lambda_{k+1}-\lambda_i\right)^3 \int_\Omega \rho \varphi_i^2. \end{aligned} \tag{2.16}$$

Substituting (2.13) into (2.16), we obtain

$$\begin{aligned} & \left(\lambda_{k+1}-\lambda_i\right)^2\left(\int_\Omega u_i^2|\nabla h|^2+2 \sum_{j=1}^k a_{ij} b_{ij}\right) \\ \leq & \frac{\lambda_{k+1}-\lambda_i}{\delta}\left[\int_\Omega \frac{1}{\rho}\left(\left\langle\nabla h, \nabla u_i\right\rangle+\frac{u_i \Delta h}{2}\right)^2-\sum_{j=1}^k b_{ij}^2\right] \\ & +\delta\left(\lambda_{k+1}-\lambda_i\right)^2\left[\int_\Omega h u_i p_i+\sum_{j=i}^k\left(\lambda_i-\lambda_j\right) a_{ij}^2\right]. \end{aligned} \tag{2.17}$$

Taking the sum on i from 1 to k in (2.17) and noticing

$$a_{ij}=a_{ji}, \quad b_{ij}=-b_{ji},$$

$$\sum_{i,j=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2 a_{ij}b_{ij}=-\sum_{i,j=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)\left(\lambda_i-\lambda_j\right)a_{ij}b_{ij}$$

and

$$\sum_{i,j=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\left(\lambda_i-\lambda_j\right)a_{ij}^2=-\sum_{i,j=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)\left(\lambda_i-\lambda_j\right)^2 a_{ij}^2,$$

we induce

$$\begin{aligned}\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\int_\Omega u_i^2\left|\nabla h\right|^2\leq&\frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_\Omega\frac{1}{\rho}\left(\langle\nabla h,\nabla u_i\rangle+\frac{u_i\Delta h}{2}\right)^2\\&+\delta\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\int_\Omega hu_ip_i.\end{aligned}\tag{2.18}$$

Since

$$\begin{aligned}\int_\Omega hu_i\left\langle\nabla h,\nabla\Delta u_i\right\rangle=&-\int_\Omega\Delta u_i\operatorname{div}\left(hu_i\nabla h\right)\\=&-\int_\Omega hu_i\Delta u_i\Delta h-\int_\Omega h\Delta u_i\left\langle\nabla u_i,\nabla h\right\rangle-\int_\Omega u_i\Delta u_i\left|\nabla h\right|^2,\end{aligned}\tag{2.19}$$

$$\int_\Omega hu_i\Delta\left\langle\nabla h,\nabla u_i\right\rangle=\int_\Omega h\Delta u_i\left\langle\nabla h,\nabla u_i\right\rangle+\int_\Omega u_i\Delta h\left\langle\nabla h,\nabla u_i\right\rangle+2\int_\Omega\left\langle\nabla h,\nabla u_i\right\rangle^2,\tag{2.20}$$

$$\int_\Omega hu_i\Delta\left(u_i\Delta h\right)=\int_\Omega u_i^2\left(\Delta h\right)^2+\int_\Omega hu_i\Delta u_i\Delta h+2\int_\Omega u_i\Delta h\left\langle\nabla h,\nabla u_i\right\rangle\tag{2.21}$$

and

$$\begin{aligned}\int_\Omega 2hu_i\left\langle\nabla h,A\nabla u_i\right\rangle=&-\frac{1}{2}\int_\Omega u_i^2\operatorname{div}\left(A\nabla h^2\right)\\=&-\int_\Omega hu_i^2\operatorname{div}\left(A\nabla h\right)-\int_\Omega u_i^2\left\langle\nabla h,A\nabla h\right\rangle,\end{aligned}\tag{2.22}$$

we have

$$\begin{aligned}\int_\Omega hp_iu_i=&-2\int_\Omega u_i\Delta u_i\left|\nabla h\right|^2+4\int_\Omega u_i\Delta h\left\langle\nabla h,\nabla u_i\right\rangle+4\int_\Omega\left\langle\nabla h,\nabla u_i\right\rangle^2\\&+\int_\Omega u_i^2\left(\Delta h\right)^2+p\int_\Omega u_i^2\left\langle\nabla h,A\nabla h\right\rangle.\end{aligned}\tag{2.23}$$

Substituting (2.23) into (2.18), we obtain (2.1). This completes the proof of Lemma 2.1.

Now we give the proof of Theorems 1.1 and 1.2 by using Lemma 2.1.

Proof of Theorem 1.1 Let $x_1,x_2,\ldots,x_N$ be the standard coordinate functions of $\mathbb{R}^N$. Then it holds

$$\sum_{\alpha=1}^{N}\left|\nabla x_\alpha\right|^2=n,\tag{2.24}$$

$$\Delta\left(x_1,x_2,\ldots,x_N\right)=n\mathbf{H},\tag{2.25}$$

$$\sum_{\alpha=1}^{N}\left\langle\nabla u_i,\nabla x_\alpha\right\rangle^2=\left|\nabla u_i\right|^2 \tag{2.26}$$

and

$$\sum_{\alpha=1}^{N}\Delta x_\alpha\left\langle\nabla u_i,\nabla x_\alpha\right\rangle=\sum_{\alpha=1}^{N}\Delta x_\alpha\nabla u_i\left(x_\alpha\right)=\left\langle n\mathbf{H},\nabla u_i\right\rangle=0. \tag{2.27}$$

Taking $h=x_\alpha$ in (2.1), taking the sum on α from 1 to N , and using (2.24–2.27), we get

$$\begin{aligned}\frac{n}{\rho_2}\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\leq&\delta\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\left[(2n+4)\int_\Omega\left|\nabla u_i\right|^2+\int_\Omega u_i^2\left(n\mathbf{H}\right)^2+p\int_\Omega u_i^2tr(A)\right]\\&+\frac{1}{\rho_1\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_\Omega\left(\left|\nabla u_i\right|^2+\frac{\left(n\mathbf{H}\right)^2}{4}u_i^2\right).\end{aligned} \tag{2.28}$$

From

$$\lambda_i=\int_\Omega u_i\left[\Delta^2u_i-p\mathrm{div}\left(A\nabla u_i\right)+qu_i\right]=\int_\Omega\left(\Delta u_i\right)^2+p\int_\Omega\left\langle\nabla u_i,A\nabla u_i\right\rangle+q\int_\Omega u_i^2$$

and

$$\int_\Omega\left|\nabla u_i\right|^2=-\int_\Omega u_1\Delta u_i\leq\left(\int_\Omega u_i^2\int_\Omega\left(\Delta u_i\right)^2\right)^{\frac{1}{2}},$$

we have

$$\rho_1\rho_2\left(\int_\Omega\left|\nabla u_i\right|^2\right)^2+p\xi_1\rho_2\int_\Omega\left|\nabla u_i\right|^2+q-\rho_2\lambda_i\leq0. \tag{2.29}$$

This is a quadratic inequality of $\int_\Omega\left|\nabla u_i\right|^2$. Thus we obtain

$$\int_\Omega\left|\nabla u_i\right|^2\leq B_i. \tag{2.30}$$

Introducing (2.30) into (2.28), we infer

$$\begin{aligned}\frac{n}{\rho_2}\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\leq\delta\sum_{i=1}^{k}&\left[(\lambda_{k+1}-\lambda_i\right)^2\left[(2n+4)B_i+\frac{n^2H_0^2+pn\xi_2}{\rho_1}\right]\\&+\frac{1}{\rho_1\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(B_i+\frac{n^2H_0^2}{4\rho_1}\right).\end{aligned} \tag{2.31}$$

Taking

$$\delta=\left\{\frac{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(B_i+\frac{n^2H_0^2}{4\rho_1}\right)}{\rho_1\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\left[(2n+4)B_i+\frac{n^2H_0^2+pn\xi_2}{\rho_1}\right]}\right\}^{\frac{1}{2}}$$

in (2.31), we can obtain (1.4). This finishes the proof of Theorem 1.1.

Proof of Theorem 1.2 i) Taking $h=\psi$ in (2.1), we get

$$\begin{aligned}
&\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\int_\Omega u_i^2\left|\nabla\psi\right|^2\\
\leq&\frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_\Omega\frac{1}{\rho}\left[\left\langle\nabla\psi,\nabla u_i\right\rangle^2+u_i\Delta\psi\left\langle\nabla\psi,\nabla u_i\right\rangle+\frac{u_i^2\left(\Delta\psi\right)^2}{4}\right]\\
&+\delta\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\left(-2\int_\Omega u_i\Delta u_i\left|\nabla\psi\right|^2+4\int_\Omega u_i\Delta\psi\left\langle\nabla\psi,\nabla u_i\right\rangle\right.\\
&\left.+4\int_\Omega\left\langle\nabla\psi,\nabla u_i\right\rangle^2+\int_\Omega u_i^2\left(\Delta\psi\right)^2+p\int_\Omega u_i^2\left\langle\nabla\psi,A\nabla\psi\right\rangle\right).
\end{aligned}\tag{2.32}$$

By using (2.30), we have

$$\int_\Omega u_i\Delta\psi\left\langle\nabla\psi,\nabla u_i\right\rangle\leq C_0\left(\int_\Omega u_i^2\int_\Omega\left|\nabla u_i\right|^2\right)^{\frac{1}{2}}\leq C_0\sqrt{\frac{B_i}{\rho_1}}.\tag{2.33}$$

Substituting (2.33) into (2.32), we deduce

$$\begin{aligned}
\frac{1}{\rho_2}\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\leq\delta\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\left(6B_i+4C_0\sqrt{\frac{B_i}{\rho_1}}+\frac{C_0^2+p\xi_2}{\rho_1}\right)&\\
+\frac{1}{\rho_1\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(B_i+C_0\sqrt{\frac{B_i}{\rho_1}}+\frac{C_0^2}{4\rho_1}\right)&.
\end{aligned}\tag{2.34}$$

Taking

$$\delta=\left[\frac{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(B_i+C_0\sqrt{\frac{B_i}{\rho_1}}+\frac{C_0^2}{4\rho_1}\right)}{\rho_1\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\left(6B_i+4C_0\sqrt{\frac{B_i}{\rho_1}}+\frac{C_0^2+p\xi_2}{\rho_1}\right)}\right]^{\frac{1}{2}}$$

in (2.34), we get (1.7).

ii) Taking $h=\varphi$ in (2.1), we have

$$\begin{aligned}
&\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\int_\Omega u_i^2\left|\nabla\varphi\right|^2\\
\leq&\frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_\Omega\frac{1}{\rho}\left[\left\langle\nabla\varphi,\nabla u_i\right\rangle^2+u_i\Delta\varphi\left\langle\nabla\varphi,\nabla u_i\right\rangle+\frac{u_i^2\left(\Delta\varphi\right)^2}{4}\right]\\
&+\delta\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\left(-2\int_\Omega u_i\Delta u_i\left|\nabla\varphi\right|^2+4\int_\Omega u_i\Delta\varphi\left\langle\nabla\varphi,\nabla u_i\right\rangle\right.\\
&\left.+4\int_\Omega\left\langle\nabla\varphi,\nabla u_i\right\rangle^2+\int_\Omega u_i^2\left(\Delta\varphi\right)^2+p\int_\Omega u_i^2\left\langle\nabla\varphi,A\nabla\varphi\right\rangle\right).
\end{aligned}\tag{2.35}$$

Since

$$\int_\Omega u_i\Delta\varphi\left\langle\nabla\varphi,\nabla u_i\right\rangle=\frac{D_0}{2}\int_\Omega\left\langle\nabla\varphi,\nabla u_i^2\right\rangle=-\frac{D_0}{2}\int_\Omega u_i^2\Delta\varphi\leq-\frac{D_0^2}{2\rho_2},$$

we get

$$\begin{aligned}\frac{1}{\rho_2}\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2 \le \delta\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\left(6B_i-\frac{2D_0^2}{\rho_2}+\frac{D_0^2+p\xi_2}{\rho_1}\right)\\ +\frac{1}{\rho_1\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(B_i-\frac{D_0^2}{2\rho_2}+\frac{D_0^2}{4\rho_1}\right).\end{aligned}\tag{2.36}$$

Taking

$$\delta=\left[\frac{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(B_i-\frac{D_0^2}{2\rho_2}+\frac{D_0^2}{4\rho_1}\right)}{\rho_1\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\left(6B_i-\frac{2D_0^2}{\rho_2}+\frac{D_0^2+p\xi_2}{\rho_1}\right)}\right]^{\frac{1}{2}}$$

in (2.36), we obtain (1.8).

iii) Because $\langle\nabla\phi_\alpha,\nabla\phi_\beta\rangle=\delta_{\alpha\beta}$ and $\Delta\phi_\alpha=0$, $\alpha,\beta=1,\cdots,l$, we know

$$\sum_{\alpha=1}^{l}\left\langle\nabla u_i,\nabla\phi_\alpha\right\rangle^2\le\left|\nabla u_i\right|^2.$$

Thus, taking $h=\phi_\alpha$ in (2.1), we obtain by summing over α that

$$\begin{aligned}&\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\int_\Omega u_i^2\sum_{\alpha=1}^{l}\left|\nabla\phi_\alpha\right|^2\\ \le&\delta\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\left(-2\int_\Omega u_i\Delta u_i\sum_{\alpha=1}^{l}\left|\nabla\phi_\alpha\right|^2+4\int_\Omega\left|\nabla u_i\right|^2+p\xi_2\int_\Omega u_i^2\sum_{\alpha=1}^{l}\left|\nabla\phi_\alpha\right|^2\right)\\ &+\frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_\Omega\frac{1}{\rho}\left|\nabla u_i\right|^2.\end{aligned}\tag{2.37}$$

It implies

$$\frac{l}{\rho_2}\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\le\delta\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\left[(2l+4)B_i+\frac{pl\xi_2}{\rho_1}\right]+\frac{1}{\rho_1\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)B_i.\tag{2.38}$$

Taking

$$\delta=\left\{\frac{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)B_i}{\rho_1\sum_{i=1}^{k}\left(\lambda_{k+1}-\lambda_i\right)^2\left[(2l+4)\,B_i+\frac{pl\xi_2}{\rho_1}\right]}\right\}^{\frac{1}{2}}$$

in (2.38), we derive (1.9).

iv) Taking the Laplacian of $\sum_{\alpha=1}^{N+1}f_\alpha^2=1$ and using the fact that $\Delta f_\alpha=-\mu f_\alpha$, $\alpha=1,2,\cdots,N+1$, we have

$$\sum_{\alpha=1}^{N+1}\left|\nabla f_\alpha\right|^2=\mu\tag{2.39}$$

and

$$\sum_{\alpha=1}^{N+1}\int_\Omega u_i\Delta f_\alpha\left\langle\nabla f_\alpha,\nabla u_i\right\rangle=-\frac{\mu}{4}\sum_{\alpha=1}^{N+1}\int_\Omega\left\langle\nabla f_\alpha^2,\nabla u_i^2\right\rangle=\frac{\mu}{4}\sum_{\alpha=1}^{N+1}\int_\Omega u_i^2\Delta f_\alpha^2=0.\tag{2.40}$$

It then follows by taking $h = f_\alpha$ in (2.1) and summing over α that

$$\begin{aligned}
&\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \sum_{\alpha=1}^{N+1} \int_\Omega u_i^2 \left|\nabla f_\alpha\right|^2 \\
\leq&\delta \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \left[-2 \int_\Omega u_i \Delta u_i \left(\sum_{\alpha=1}^{N+1} \left|\nabla f_\alpha\right|^2 \right) + 4 \sum_{\alpha=1}^{N+1} \int_\Omega u_i \Delta f_\alpha \left\langle \nabla f_\alpha, \nabla u_i \right\rangle \right. \\
&\left. +4 \sum_{\alpha=1}^{N+1} \int_\Omega \left\langle \nabla f_\alpha, \nabla u_i \right\rangle^2 + \int_\Omega u_i^2 \sum_{\alpha=1}^{N+1} (-\mu f_\alpha)^2 + p \int_\Omega u_i^2 \sum_{\alpha=1}^{N+1} \left\langle \nabla f_\alpha, A \nabla f_\alpha \right\rangle \right] \\
&+ \frac{1}{\delta} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \sum_{\alpha=1}^{N+1} \int_\Omega \frac{1}{\rho} \left[\left\langle \nabla f_\alpha, \nabla u_i \right\rangle^2 + u_i \Delta f_\alpha \left\langle \nabla f_\alpha, \nabla u_i \right\rangle + \frac{u_i^2 (-\mu f_\alpha)^2}{4} \right].
\end{aligned} \tag{2.41}$$

It implies

$$\begin{aligned}
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \frac{\mu}{\rho_2} \leq& \delta \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \left(6\mu B_i + \frac{\mu^2 + p\mu\xi_2}{\rho_1} \right) \\
&+ \frac{1}{\rho_1 \delta} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(\mu B_i + \frac{\mu^2}{4\rho_1} \right).
\end{aligned} \tag{2.42}$$

Taking

$$\delta = \left[\frac{\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(\mu B_i + \frac{\mu^2}{4\rho_1} \right)}{\rho_1 \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \left(6\mu B_i + \frac{\mu^2 + p\mu\xi_2}{\rho_1} \right)} \right]^{\frac{1}{2}}$$

in (2.42), we obtain (1.10). This ends the proof of Theorem 1.2.

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黎曼流形上一类算子的特征值不等式

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摘 要: 本文研究了等距浸入欧氏空间的黎曼流形、容许特殊函数的黎曼流形上的一类椭圆算子的加权狄利克雷特征值问题. 我们建立了该问题的一些万有特征值不等式. 同时, 作为应用, 我们获得了拉普拉斯算子的二次多项式算子的加权狄利克雷问题的一些结果.

关键词: 特征值; 不等式; 椭圆算子; 黎曼流形

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