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OPTIMAL INVESTMENT-REINSURANCE PROBLEM FOR N INSURERS WITH A DEFAULTABLE SECURITY

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Abstract: This paper considers a non-zero-sum stochastic differential investment-reinsurance game between n competitive insurers. Each insurer can purchase proportional reinsurance and trade in a more general financial market consisting of a risk-free asset, a defaultable bond and n risky assets. In particular, the risky asset's price process is modeled by a constant elasticity of variance (CEV) model and the defaultable bond with proportional recovery at default. The objective of each insurer is to maximize the expected exponential utility of the terminal wealth relative to the competitors. By applying stochastic optimal control theory, the explicit expressions of equilibrium strategies and equilibrium value functions are derived, respectively. Numerical examples are given to illustrate the effects of parameters on the equilibrium policies, as well as, we examine how the number of insurers on the optimal equilibrium investment strategies. We find that when the number of insurers increases, each insurer will invest more money in risky assets and defaultable bond.

Keywords: investment-reinsurance; non-zero-sum game; defaultable bond; exponential utility function; CEV model

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1 Introduction

Investment and reinsurance are two significant issues for the insurers. Extensive studies have been conducted to find the optimal investment-reinsurance strategies under different objectives, such as minimizing the probability of ruin (Browne [1], Zhang et al. [2]), maximizing the expected utility of terminal wealth (Chen and Yang [3], Bai and Guo [4]) and the mean-variance (MV) criterion (Shen and Zeng [5], Zeng [6]).

The CEV risky asset model has received extensive attentions due to the empirical advantage and mathematical tractability. The study in this area includes but is not limited to the following work. For example, Gu et al. [7] considered the excess-of-loss reinsurance and investment problem for an insurer. Zheng et al. [8] investigated a robust optimal portfolio and reinsurance problem under a Cramér-Lundberg risk model. Wang et al. [9] derived the

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optimal investment strategies for both insurer and reinsurer. Li et al. [10] focused on a stochastic differential game between two insurers.

Most of the above-mentioned research did not take the default risk into consideration. However, the large market share of defaultable bonds make itself one of the most important investment products. Recently, many authors considered the financial market with a risk-free asset, a risky asset and a defaultable bond (see, e.g., Zhao et al. [11], Sun et al. [12] and Wang et al. [13]).

In reality, there exist many insurers playing the roles of both partners and competitors. The non-zero sum game theory is a kind of idea to reflect the interaction among several insurers. Bensoussan et al. [14] formulated a non-zero-sum stochastic differential game between two insurers and obtained explicit solutions for optimal strategies. Deng et al. [15] discussed this problem between two competitive insurers facing the default risk. Zhang et al. [16] invested the same problem under the CEV model. Yang et al. [17] further investigated the optimal reinsurance-investment strategy for n mean-variance insurers.

In our paper, a non-zero-sum stochastic differential game theory is applied to solve the optimal investment reinsurance problem for n competitive insurers under the exponential utility function. In the financial market, we assume the risky asset price process is governed by the CEV model. As well as the risky assets and the risky-free asset the insurers are allowed to invest in a defaultable bond. The explicit expressions of Nash equilibrium strategies and equilibrium value functions under pre-default case and post-default case are derived, respectively. We also analyze the impact of volatility skew on each insurer's equilibrium strategy through numerical examples. Last but not least, we find an interesting conclusion, the number of insurers is also one of the factors that affect insurer's trading strategy.

The rest of this paper is organized as follows. Assumption and problem formulation are described in Section 2. In Section 3, we obtain the optimal strategies and the corresponding value functions under pre-default case and post-default case, respectively. Section 4 provides some numerical studies to analyze our results. Section 5 concludes this paper.

2 Model Setup

Let $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P\big)$ is a complete probability space, where the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is right-continuous and P-complete; $[0,T]$ is a fixed and finite time horizon. $\mathcal{F}_t$ stands for the information available until time t, which is generated by $2n$ mutually independent standard Brownian motions $\{W_{1_0}(t)\}$, $\{W_{2_0}(t)\}$, $\cdots$, $\{W_{n_0}(t)\}$, $\{W_1(t)\}$, $\{W_2(t)\}$, $\cdots$, $\{W_n(t)\}$.

Suppose that there are n insurers in the insurance market, named insurer $1, 2, \cdots, n$. The ith insurer's claim process $\{Z_i(t), t \geq 0\}$ is described by the drifted Brownian motion

$$dZ_i(t) = a_i dt - \sigma_{i_0} dW_{i_0}(t),$$

where a_i is the expectation of claim, σ_{i_0} is the volatility. According to the expected value premium principle, the ith insurer's premium rate is $c_i = (1+\eta_{i_1})a_i$, where η_{i_1} is the ith insurer's relative safety loading.

The insurer can buy proportional reinsurance or acquire new business to reduce the claim risk. Let $q_i(t)(0 \leq q_i(t) \leq 1)$ be the proportional reinsurance level, that is, when a claim occurs, the reinsurer pays $100(1-q_i(t))\%$ of the claim while the insurer pays $100q_i(t)\%$. In this case, the insurer should distribute part of the premium to the reinsurer at the rate of $(1+\eta_{i_2})a_i(1-q_i(t))$, where $\eta_{i_2}(>\eta_{i_1})$ refers to the safety loading of the reinsurer. It is worth to notice that $q_i(t) \in (1,+\infty)$ means acquiring new business. With the reinsurance strategy $q_i(t)$, $t \in [0,T]$, the ith insurer's surplus process can be expressed by

$$dR_i(t) = a_i(\theta_i + \eta_{i_2} q_i(t))dt + \sigma_{i_0} q_i(t) dW_{i_0}(t),$$

where $\theta_i = \eta_{i_1} - \eta_{i_2}$. We assume that the financial market contains a risk-free asset, n risky assets and a defaultable bond. The risk-free asset's price process $B(t)$ is described by $dB(t) = rB(t)dt$, $B(0) = b_0 > 0$, where $r > 0$ is the risk-free asset interest rate. The price process of the jth risky asset $S_j(t)$ follows the CEV model $dS_j(t) = S_j(t)[\mu_j dt + \sigma_j S_j^{\beta}(t) dW_j(t)]$, $S_j(0) = s_{j_0}$, where μ_j is the expected instantaneous return rate of the jth risky asset; σ_j is a positive constant; β is called the elasticity parameter; $\sigma_j S_j(t)^{\beta}$ is the instantaneous volatility. As usual, we assume that $\mu_j > r$ and β satisfies the general condition $\beta \geq 0$. Let $T_1 > T$ denotes the maturity time of the defaultable bond, and a nonnegative random variable τ represents the default time of the corporate issuing this bond. The dynamics of the defaultable bond under measure P can be described as

$$dp(t,T_1) = p(t-,T_1)[rdt + (1-H(t))(1-\Delta)\delta dt - (1-H(t-))\zeta dM^p(t)],$$

where, the default process $H(t)$ has a constant default intensity h^p, $1/\Delta \geq 1$ is the default risk premium, $\zeta \in [0,1]$ denotes the constant loss rate when a default happens, 1-ζ is the default recovery rate and δ represents the risk neutral credit spread. Let $\mathcal{G}_t = \mathcal{F}_t \vee \sigma\{H(s) : 0 \leq s \leq t\}$ such that $\mathbb{G} := \{\mathcal{G}_t\}_{t\in[0,T]}$ is the smallest filtration. The martingale jump process $M^p(t)$ is given by $M^p(t) := H(t) - \int_0^t (1-H(u-))h^p du$, which is a $(\mathbb{G}, P)$-martingale.

Suppose that each insurer is allowed to invest his surplus in financial market described above. Let $\pi_{i1}(t)$, $\pi_{i2}(t)$, $\cdots$, $\pi_{in}(t)$ and $\pi_{ip}(t)$ represent the money amounts that the ith insurer invests in risky assets $1, 2, \cdots, n$ and the defaultable bond at time t, the rest of the surplus is then invested in the risk-free asset. We denote the whole reinsurance-investment strategy by $\pi_i(t) = (q_i(t), \pi_{i1}(t), \pi_{i2}(t), \cdots, \pi_{in}(t), \pi_{ip}(t))$. Based on the trading strategy $\{\pi_i(t), t \in [0,T]\}$, the ith insurer's wealth process $X_i^{\pi_i}(t)$ is presented by

$$\begin{aligned} dX_i^{\pi_i}(t) =& dR_i(t) + \frac{X_i^{\pi_i}(t) - \sum\limits_{j=1}^{n} \pi_{ij}(t) - \pi_{ip}(t)}{B(t)} dB(t) + \sum_{j=1}^{n} \frac{\pi_{ij}(t)}{S_j(t)} dS_j(t) + \frac{\pi_{ip}(t)}{p(t-,T_1)} dp(t-,T_1) \\ =& [rX_i^{\pi_i}(t) + a_i(\theta_i + \eta_{i_2} q_i(t)) + \sum_{j=1}^{n} (\mu_j - r)\pi_{ij}(t) + (1-H(t))(1-\Delta)\delta \pi_{ip}(t)]dt \\ &+ \sum_{j=1}^{n} \pi_{ij}(t)\sigma_j S_j^{\beta}(t) dW_j(t) + \sigma_{i_0} q_i(t) dW_{i_0}(t) - (1-H(t-))\zeta \pi_{ip}(t) dM^p(t), \quad (2.1) \end{aligned}$$

where $X_i^{\pi_i}(0) = x_{i_0}$ is the ith insurer's initial wealth.

Definition 2.1 For any fixed $t \in [0,T]$, a trading strategy $\pi_i(t) = (q_i(t), \pi_{i1}(t), \pi_{i2}(t), \cdots, \pi_{in}(t), \pi_{ip}(t))$ is said to be admissible if it satisfies

(1) $\pi_i(t)$ is $\mathcal{G}$-progressively measurable;

(2) $\forall t \in [0,T], q_i(t) \in [0,+\infty)$ and $E_P\{\int_0^T [\sum\limits_{j=1}^n \pi_{ij}^2(t) S_j^{2\beta}(t) + \pi_{ip}^2(t) + q_i^2(t)]dt\} < \infty$;

(3) For $\forall x_{i_0} \in \mathbb{R}$, the stochastic differential equation (2.1) with respect to $\pi_i(t)$ has a pathwise unique solution $X_i^{\pi_i}(t)$ satisfying $\mathrm{E}_P[\exp(-\gamma_i X_i^{\pi_i}(t))] < \infty$.

Let Π_i denote the set of all admissible strategies of the ith insurer. In a competitive market, each insurer considers the relative performance compared with the others. Following Espinosa and Touzi [18], we define the relative performance for $n > 2$ insurers as the difference between the performance of an insurer and the average performance of his competitors. For the ith insurer, the average performance of his competitors is defined as

$$\bar{X}_i^{(\pi_m)_{m\neq i}}(t) := \frac{1}{n-1} \sum_{m=1, m\neq i}^{n} X_m^{\pi_m}(t) = \frac{1}{n-1} \sum_{m\neq i} X_m^{\pi_m}(t),$$

here $(\pi_m)_{m\neq i} := (\pi_1(t), \pi_2(t), \cdots, \pi_{i-1}(t), \pi_{i+1}(t), \cdots, \pi_n(t))$. We use $\hat{X}_i^{\pi_i,(\pi_m)_{m\neq i}}(t)$ to represent the ith insurer's relative wealth process, which is denoted by

$$\hat{X}_i^{\pi_i,(\pi_m)_{m\neq i}}(t) = (1-\tau_i) X_i^{\pi_i}(t) + \tau_i (X_i^{\pi_i}(t) - \bar{X}_i^{(\pi_m)_{m\neq i}}(t)), \tag{2.2}$$

here the constant $\tau_i \in [0,1)$ measures the ith insurer's sensitivity to the average performance of his competitors and reflects the degree of competition among n insurers. Combining equations (2.1) and (2.2), the ith insurer's relative wealth process can be written as

$$\begin{aligned}
& d\hat{X}_i^{\pi_i,(\pi_m)_{m\neq i}}(t) \\
=& \big[r\hat{X}_i^{\pi_i,(\pi_m)_{m\neq i}}(t) + a_i(\theta_i + \eta_{i_2} q_i(t)) - \frac{\tau_i}{n-1} \sum_{m\neq i} a_m(\theta_m + \eta_{m_2} q_m(t)) \\
& + \sum_{j=1}^n (\mu_j - r) f(\pi_{ij}, \pi_{mj})(t) + (1 - H(t-))(1-\Delta)\delta f(\pi_{ip}, \pi_{mp})(t) \big] dt + \sigma_{i_0} q_i(t) dW_{i_0}(t) \\
& + \sum_{j=1}^n f(\pi_{ij}, \pi_{mj})(t) \sigma_j S_j^{\beta}(t) dW_j(t) - \frac{\tau_i}{n-1} \sum_{m\neq i} \sigma_{m_0} q_m(t) dW_{m_0}(t) \\
& - (1 - H(t-))\zeta f(\pi_{ip}, \pi_{mp})(t) dM^p(t),
\end{aligned} \tag{2.3}$$

here $f(\pi_{ij}, \pi_{mj})(t) := \pi_{ij}(t) - \frac{\tau_i}{n-1} \sum\limits_{m\neq i} \pi_{mj}(t)$, $f(\pi_{ip}, \pi_{mp})(t) := \pi_{ip}(t) - \frac{\tau_i}{n-1} \sum\limits_{m\neq i} \pi_{mp}(t)$.

In this paper, we assume the ith insurer has an exponential utility function, i.e.,

$$U_i(x) = -\frac{1}{\gamma_i} e^{-\gamma_i x}, \tag{2.4}$$

with the constant risk aversion parament $\gamma_i > 0$. The ith insurer looks for the optimal reinsurance and investment strategy to maximize the objective function

$$J_i^{\pi_i,(\pi_m)_{m\neq i}}(t, \hat{x}_i, \bar{s}, z) = \mathrm{E}_{t,\hat{x}_i,\bar{s},z} \left[U_i \left(\hat{X}_i^{\pi_i,(\pi_m)_{m\neq i}}(T) \right) \right] \tag{2.5}$$

where $\mathrm{E}_{t,\hat{x}_i,\bar{s},z}[\cdot] = \mathrm{E}_{t,\hat{x}_i,\bar{s},z}[\cdot|\hat{X}_i^{\pi_i,(\pi_m)_{m\neq i}}(t) = \hat{x}_i, \bar{S}(t) = \bar{s}, H(t) = z]$. $\bar{s} = (s_1, s_2, \cdots, s_n)$ and $\bar{S}(t) = (S_1(t), S_2(t), \cdots, S_n(t))$ are two row vectors.

Problem 1 The classical non-zero-sum stochastic differential game between n competing insurers is to find a Nash equilibrium $(\pi_1^*, \pi_2^*, \cdots, \pi_n^*) \in \Pi_1 \times \Pi_2 \times \cdots \times \Pi_n$ such that for any $(\pi_1, \pi_2, \cdots, \pi_n) \in \Pi_1 \times \Pi_2 \times \cdots \times \Pi_n$, we have

$$\begin{aligned}
&J_1^{\pi_1^*,\pi_2^*,\ldots,\pi_n^*}(t, \hat{x}_1, \bar{s}, z) \geq J_1^{\pi_1,\pi_2^*,\ldots,\pi_n^*}(t, \hat{x}_1, \bar{s}, z),\\
&J_2^{\pi_1^*,\pi_2^*,\ldots,\pi_n^*}(t, \hat{x}_2, \bar{s}, z) \geq J_2^{\pi_1^*,\pi_2,\ldots,\pi_n^*}(t, \hat{x}_2, \bar{s}, z),\\
&\qquad\qquad\vdots\\
&J_n^{\pi_1^*,\pi_2^*,\ldots,\pi_n^*}(t, \hat{x}_n, \bar{s}, z) \geq J_1^{\pi_1^*,\pi_2^*,\ldots,\pi_n}(t, \hat{x}_n, \bar{s}, z).
\end{aligned} \tag{2.6}$$

3 Solusion to the Model

In this section, we derive the equilibrium strategies and corresponding equilibrium value functions in the post-default case($z = 1$) and the pre-default case($z = 0$), respectively.

By the stochastic control theory, we define the ith insurer's value function $V^i(t, \hat{x}_i, \bar{s}, z)$ by

$$V^i(t, \hat{x}_i, \bar{s}, z) = \sup_{\pi_i\in\Pi_i} \mathrm{E}_{t,\hat{x}_i,\bar{s},z}\left[U\left(\hat{X}_i^{\pi_i,(\pi_m^*)_{m\neq i}}(T)\right)\right] \tag{3.1}$$

with $V^i(T, \hat{x}_i, \bar{s}, z) = U_i(\hat{x}_i)$.

Before giving the Hamilton-Jacobi-Bellman(HJB) equation, we first denote $C^{1,2,2,\cdots,2}([0, T] \times \mathbb{R} \times \mathbb{R}^+ \times \cdots \times \mathbb{R}^+ \times \{0, 1\}) := \big\{\omega(t, x, s_1, \cdots, s_n, z)|\omega(t, x, s_1, \cdots, s_n, z)$ is once continuously differentiable in t, and twice continuously differentiable in s_1, s_2, $\cdots$, s_n and $x\big\}$. In addition, we omit the function parameters for notational simplicity in the following paragraph.

For any functions $V^i(t, \hat{x}_i, s_1, \cdots, s_n, 1)$, $V^i(t, \hat{x}_i, s_1, \cdots, s_n, 0) \in C^{1,2,2,\cdots,2}([0, T] \times \mathbb{R} \times \mathbb{R}^+ \times \cdots \times \mathbb{R}^+ \times \{0, 1\})$, denote the variational operator $\mathcal{A}^{\pi_i,(\pi_m)_{m\neq i}}V^i$ as follows:

$$\begin{aligned}
\mathcal{A}^{\pi_i,(\pi_m)_{m\neq i}}V^i =&V_t^i + [r\hat{x}_i + a_i(\theta_i + \eta_{i_2}q_i) - \frac{\tau_i}{n-1}\sum_{m\neq i} a_m(\theta_m + \eta_{m_2}q_m)\\
&+ \sum_{j=1}^{n}(\mu_j - r)f(\pi_{ij}, \pi_{mj}) + (1 - z)\delta f(\pi_{ip}, \pi_{mp})]V_{\hat{x}_i}^i + \sum_{j=1}^{n}\mu_j s_j V_{s_j}^i\\
&+ \frac{1}{2}[\sum_{j=1}^{n} f^2(\pi_{ij}, \pi_{mj})\sigma_j^2 s_j^{2\beta} + \sigma_{i_0}^2 q_i^2 + \frac{\tau_i^2}{(n-1)^2}\sum_{m\neq i}\sigma_{m_0}^2 q_m^2]V_{\hat{x}_i\hat{x}_i}^i\\
&+ \sum_{j=1}^{n} f(\pi_{ij}, \pi_{mj})\sigma_j^2 s_j^{2\beta+1}V_{\hat{x}_i s_j}^i + \frac{1}{2}\sum_{j=1}^{n}\sigma_j^2 s_j^{2\beta+2}V_{s_j s_j}^i\\
&+ h^p[V^i(t, \bar{s}, \hat{x}_i - \zeta f(\pi_{ip}, \pi_{mp}), 1) - V^i(t, \bar{s}, \hat{x}_i, 0)](1 - z),
\end{aligned} \tag{3.2}$$

where $V_t^i, V_{\hat{x}_i}^i, V_{s_i}^i, V_{\hat{x}_i\hat{x}_i}^i, V_{\hat{x}_i s_j}^i$ and $V_{s_j s_j}^i$ are the partial derivative of function $V^i(t, \hat{x}_i, s_1, \cdots, s_n, z)$.

According to the dynamic programming principle, the HJB equation can be derived as

$$\sup_{\pi_i\in\Pi_i}\{\mathcal{A}^{\pi_i,(\pi_m^*)_{m\neq i}}V^i(t, \hat{x}_i, \bar{s}, z)\} = 0, \quad 0 \leq t < T \tag{3.3}$$

with the boundary condition $V^i(T,\hat{x}_i,\bar{s},z)=U_i(\hat{x}_i)$.

3.1 Post–default Case: z=1

In the post-default case, we have $p(t,T_1)=0,\ \tau\le t\le T$. Thus $\pi^*_{ip}(t)=0$ for $i\in 1,2,\cdots,n$. Form the definition of $\mathcal{A}^{\pi_i,(\pi_m)_{m\neq i}}V^i$, Eq. (3.3) is expressed as

$$\begin{aligned}&\sup_{\pi_i\in\Pi_i}\Big\{V^i_t+[r\hat{x}_i+a_i(\theta_i+\eta_{i_2}q_i)-\frac{\tau_i}{n-1}\sum_{m\neq i}a_m(\theta_m+\eta_{m_2}q^*_m)\\&+\sum_{j=1}^n(\mu_j-r)f(\pi_{ij},\pi^*_{mj})]V^i_{\hat{x}_i}+\sum_{j=1}^n\mu_js_jV^i_{s_j}+\frac{1}{2}[\sum_{j=1}^nf^2(\pi_{ij},\pi^*_{mj})\sigma_j^2s_j^{2\beta}+\sigma_{i_0}^2q_i^2\\&+\frac{\tau_i^2}{(n-1)^2}\sum_{m\neq i}\sigma^2_{m_0}q^{*2}_m]V^i_{\hat{x}_i\hat{x}_i}+\sum_{j=1}^nf(\pi_{ij},\pi^*_{mj})\sigma_j^2s_j^{2\beta+1}V^i_{\hat{x}_is_j}+\frac{1}{2}\sum_{j=1}^n\sigma_j^2s_j^{2\beta+2}V^i_{s_js_j}\Big\}=0.\end{aligned}\tag{3.4}$$

We try to infer that the solution of Eq. (3.4) satisfies the following functional form

$$V^i(t,\hat{x}_i,\bar{s},1)=V^i(t,\hat{x}_i,s_1,s_2,\cdots,s_n,1)=-\frac{1}{\gamma_i}e^{-\gamma_i[e^{r(T-t)}\hat{x}_i+\sum\limits_{j=1}^nG_{ij}(t,s_j)-F_{i1}(t)]},\tag{3.5}$$

with the boundary conditions $F_{i1}(T)=0$ and $G_{ij}(T,s_j)=0,\ j=1,2,\cdots,n$. Substituting (3.5) into Eq. (3.4), we can obtain

$$\begin{aligned}&\sup_{\pi_i\in\Pi_i}\Big\{-\sum_{j=1}^n\frac{\partial G_{ij}}{\partial t}+F'_{i1}(t)-[a_i(\theta_i+\eta_{i_2}q_i)-\frac{\tau_i}{n-1}\sum_{m\neq i}a_m(\theta_m+\eta_{m_2}q^*_m)\\&+\sum_{j=1}^n(\mu_j-r)f(\pi_{ij},\pi^*_{mj})]e^{r(T-t)}-\sum_{j=1}^n\mu_js_j\frac{\partial G_{ij}}{\partial s_j}+\frac{\gamma_i}{2}[\sum_{j=1}^nf^2(\pi_{ij},\pi^*_{mj})\sigma_j^2s_j^{2\beta}\\&+\sigma^2_{i_0}q_i^2+\frac{\tau_i^2}{(n-1)^2}\sum_{m\neq i}\sigma^2_{m_0}q^{*2}_m]e^{2r(T-t)}+\gamma_ie^{r(T-t)}\sum_{j=1}^nf(\pi_{ij},\pi^*_{mj})\sigma_j^2s_j^{2\beta+1}\frac{\partial G_{ij}}{\partial s_j}\\&+\frac{\gamma_i}{2}\sum_{j=1}^n\sigma_j^2s_j^{2\beta+2}(\frac{\partial G_{ij}}{\partial s_j})^2-\frac{1}{2}\sum_{j=1}^n\sigma_j^2s_j^{2\beta+2}\frac{\partial^2G_{ij}}{\partial s_j^2}\Big\}=0.\end{aligned}\tag{3.6}$$

By the first order condition of Eq.(3.6), we have:

$$q^*_i(t)=\frac{a_i\eta_{i_2}}{\gamma_i\sigma^2_{i_0}e^{r(T-t)}},\tag{3.7}$$

$$\pi^*_{ij}(t)=\frac{\mu_j-r}{\gamma_i\sigma_j^2s_j^{2\beta}e^{r(T-t)}}-\frac{s_j}{e^{r(T-t)}}\frac{\partial G_{ij}}{\partial s_j}+\frac{\tau_i}{n-1}\sum_{m\neq i}\pi^*_{mj}(t).\tag{3.8}$$

Inserting Eqs.(3.7) and (3.8) into Eq.(3.6), we have:

$$\begin{aligned}&-\sum_{j=1}^n[\frac{\partial G_{ij}}{\partial t}+rs_j\frac{\partial G_{ij}}{\partial s_j}+\frac{1}{2}\sigma_j^2s_j^{2\beta+2}\frac{\partial^2G_{ij}}{\partial s_j^2}+\frac{(\mu_j-r)^2}{2\gamma_i\sigma_j^2s_j^{2\beta}}-\frac{\tau_ia_j^2\eta^2_{j_2}}{(n-1)\gamma_j\sigma^2_{j_0}}-\frac{\gamma_i\tau_i^2a_j^2\eta^2_{j_2}}{2(n-1)^2\gamma_j^2\sigma^2_{j_0}}]\\&+F'_{i1}(t)-[a_i\theta_i-\frac{\tau_i}{n-1}\sum_{m\neq i}a_m\theta_m]e^{r(T-t)}-(1+\frac{\tau_i}{n-1})^2\frac{a_i^2\eta^2_{i_2}}{2\gamma_i\sigma^2_{i_0}}=0.\end{aligned}\tag{3.9}$$

Eq.(3.9) can be decomposed into two equations

$$\frac{\partial G_{ij}}{\partial t}+rs_j\frac{\partial G_{ij}}{\partial s_j}+\frac{1}{2}\sigma_j^2 s_j^{2\beta+2}\frac{\partial^2 G_{ij}}{\partial s_j^2}+\frac{(\mu_j-r)^2}{2\gamma_i\sigma_j^2 s_j^{2\beta}}-\frac{\tau_i a_j^2\eta_{j_2}^2}{(n-1)\gamma_j\sigma_{j_0}^2}-\frac{\gamma_i\tau_i^2 a_j^2\eta_{j_2}^2}{2(n-1)^2\gamma_j^2\sigma_{j_0}^2}=0, \tag{3.10}$$

$$F_{i1}'(t)-[a_i(\theta_i-\frac{\tau_i}{n-1}\sum_{m\neq i}a_m\theta_m]e^{r(T-t)}-(1+\frac{\tau_i}{n-1})^2\frac{a_i^2\eta_{i_2}^2}{2\gamma_i\sigma_{i_0}^2}=0. \tag{3.11}$$

In order to solve Eq.(3.10), we define

$$G_{ij}(t,s_j)=A_{ij}(t)+B_{ij}(t)s_j^{-2\beta}, \tag{3.12}$$

and the boundary conditions are $A_{ij}(T)=0$ and $B_{ij}(T)=0$. Plugging Eq.(3.12) into (3.10), we derive

$$\begin{aligned} A_{ij}'(t) &+ \beta(2\beta+1)\sigma_j^2 B_{ij}(t)-\frac{\tau_i a_j^2\eta_{j_2}^2}{(n-1)\gamma_j\sigma_{j_0}^2}-\frac{\gamma_i\tau_i^2 a_j^2\eta_{j_2}^2}{2(n-1)^2\gamma_j^2\sigma_{j_0}^2} \\ &+ [B_{ij}'(t)-2\beta rB_{ij}(t)+\frac{(\mu_j-r)^2}{2\gamma_i\sigma_j^2}]s_j^{-2\beta}=0. \end{aligned} \tag{3.13}$$

By matching coefficients, we obtain two differential equations. Considering the boundary conditions $A_{ij}(T)=0$ and $B_{ij}(T)=0$, we have

$$\begin{aligned} &A_{ij}(t)=\beta(2\beta+1)\sigma_j^2\int_t^T B_{ij}(u)du-\frac{\tau_i}{n-1}[1-\frac{\tau_i\gamma_i}{2(n-1)\gamma_j}]\frac{a_j^2\eta_{j_2}^2}{\gamma_j\sigma_{j_0}^2}(T-t), \\ &B_{ij}(t)=\frac{(\mu_j-r)^2}{4\beta r\gamma_i\sigma_j^2}[1-e^{-2\beta r(T-t)}]. \end{aligned} \tag{3.14}$$

Finally, the solution of Eq. (3.11) is

$$F_{i1}(t)=-[a_i\theta_i-\frac{\tau_i}{n-1}\sum_{m\neq i}a_m\theta_m]\int_t^T e^{r(T-u)}du-(1+\frac{\tau_i}{n-1})^2\frac{a_i^2\eta_{i_2}^2}{2\gamma_i\sigma_{i_0}^2}(T-t). \tag{3.15}$$

Lemma 3.1 Eqs.(3.8) has a unique root $\pi_{ij}^*(t)$ is given by

$$\pi_{ij}^*(t)=\frac{\phi_{ij}(t)}{1+\frac{\tau_i}{n-1}}e^{-r(T-t)}+\frac{\frac{\tau_i}{n-1}}{1+\frac{\tau_i}{n-1}}\frac{1}{1-\psi_n}\sum_{i=1}^n\frac{\phi_{ij}(t)}{1+\frac{\tau_i}{n-1}}e^{-r(T-t)} \quad i,j=1,2,\cdots,n,$$

where

$$\phi_{ij}(t)=\frac{\mu_j-r}{\gamma_j\sigma_j^2 s_j^{2\beta}}+2\beta B_{ij}(t)s_j^{-2\beta}, \qquad \psi_n=\sum_{i=1}^n\frac{\frac{\tau_i}{n-1}}{1+\frac{\tau_i}{n-1}}. \tag{3.16}$$

Proof Similar to the one of Lemma 4.2 in [17], and we omit it.

3.2 Pre–default Case: z=0

In the pre-default case, Eq.(3.3) can be rewritten as follows

$$\begin{aligned}&\sup_{\pi_i\in\Pi_i}\Big\{V_t^i+[r\hat{x}_i+a_i(\theta_i+\eta_{i_2}q_i)-\frac{\tau_i}{n-1}\sum_{m\neq i}a_m(\theta_m+\eta_{m_2}q_m^*)+\delta f(\pi_{ip},\pi_{mp}^*)(t)\\&+\sum_{j=1}^{n}(\mu_j-r)f(\pi_{ij},\pi_{mj}^*)]V_{\hat{x}_i}^i+\sum_{j=1}^{n}\mu_js_jV_{s_j}^i+\frac{1}{2}[\sum_{j=1}^{n}f^2(\pi_{ij},\pi_{mj}^*)\sigma_j^2s_j^{2\beta}+\sigma_{i_0}^2q_i^2\\&+\frac{\tau_i^2}{(n-1)^2}\sum_{m\neq i}\sigma_{m_0}^2q_m^{*2}]V_{\hat{x}_i\hat{x}_i}^i+\sum_{j=1}^{n}f(\pi_{ij},\pi_{mj}^*)\sigma_j^2s_j^{2\beta+1}V_{\hat{x}_is_j}^i+\frac{1}{2}\sum_{j=1}^{n}\sigma_j^2s_j^{2\beta+2}V_{s_js_j}^i\\&+h^p[V^i(t,\bar{s},\hat{x}_i-\zeta f(\pi_{ip},\pi_{mp}^*),1)-V^i(t,\bar{s},\hat{x}_i,0)]\Big\}=0.\end{aligned}\tag{3.17}$$

Similarly, we guess that $V^i(t,\hat{x}_i,\bar{s},0)$ is of the following form

$$V^i(t,\hat{x}_i,\bar{s},0)=V^i(t,\hat{x}_i,s_1,s_2,\cdots,s_n,0)=-\frac{1}{\gamma_i}e^{-\gamma_i[e^{r(T-t)}\hat{x}_i+\sum\limits_{j=1}^{n}G_{ij}(t,s_j)-F_{i0}(t)]}\tag{3.18}$$

with the boundary condition $F_{i0}(T)=0$. Inserting (3.5) and (3.18) into HJB equation (3.17), we derive

$$\begin{aligned}&\sup_{\pi_i\in\Pi_i}\Big\{-\sum_{j=1}^{n}\frac{\partial G_{ij}}{\partial t}+F_1'(t)-[a_i(\theta_i+\eta_{i_2}q_i)-\frac{\tau_i}{n-1}\sum_{m\neq i}a_m(\theta_m+\eta_{m_2}q_m^*)\\&\qquad+\sum_{j=1}^{n}(\mu_j-r)f(\pi_{ij},\pi_{mj}^*)+\delta f(\pi_{ip},\pi_{mp}^*)]e^{r(T-t)}-\sum_{j=1}^{n}\mu_js_j\frac{\partial G_{ij}}{\partial s_j}\\&\qquad+\frac{\gamma_i}{2}[\sum_{j=1}^{n}f^2(\pi_{ij},\pi_{mj}^*)\sigma_j^2s_j^{2\beta}+\sigma_{i_0}^2q_i^2+\frac{\tau_i^2}{(n-1)^2}\sum_{m\neq i}\sigma_{m_0}^2q_m^{*2}]e^{2r(T-t)}\\&\qquad+\gamma_ie^{r(T-t)}\sum_{j=1}^{n}f(\pi_{ij},\pi_{mj}^*)\sigma_j^2s_j^{2\beta+1}\frac{\partial G_{ij}}{\partial s_j}+\frac{\gamma_i}{2}\sum_{j=1}^{n}\sigma_j^2s_j^{2\beta+2}(\frac{\partial G_{ij}}{\partial s_j})^2\\&\qquad-\frac{1}{2}\sum_{j=1}^{n}\sigma_j^2s_j^{2\beta+2}\frac{\partial^2G_{ij}}{\partial s_j^2}+\frac{h^p}{\gamma_i}e^{\gamma_i[\zeta e^{r(T-t)}f(\pi_{ip},\pi_{mp}^*)+F_{i1}(t)-F_{i0}(t)]}-\frac{h^p}{\gamma_i}\Big\}=0.\end{aligned}\tag{3.19}$$

The first order maximizing condition for the optimal investment strategy is

$$q_i^*(t)=\frac{a_i\eta_{i_2}}{\gamma_i\sigma_{i_0}^2e^{r(T-t)}},\tag{3.20}$$

$$\pi_{ij}^*(t)=\frac{\mu_j-r}{\gamma_i\sigma_j^2s_j^{2\beta}e^{r(T-t)}}-\frac{s_j}{e^{r(T-t)}}\frac{\partial G_{ij}}{\partial s_j}+\frac{\tau_i}{n-1}\sum_{m\neq i}\pi_{mj}^*(t),\tag{3.21}$$

$$\pi_{ip}^*(t)=\frac{ln\frac{\delta}{\zeta h^p}}{\zeta\gamma_ie^{r(T-t)}}-\frac{F_{i1}(t)-F_{i0}(t)}{\zeta e^{r(T-t)}}+\frac{\tau_i}{n-1}\sum_{m\neq i}\pi_{mp}^*(t).\tag{3.22}$$

Inputting Eqs.(3.20), (3.21) and (3.22) into Eq.(3.19), we get

$$\begin{aligned}&F_{i0}'(t)-\frac{\delta}{\zeta}F_{i0}(t)+\frac{\delta}{\zeta}F_{i1}(t)-[a_i\theta_i-\frac{\tau_i}{n-1}\sum_{m\neq i}a_m\theta_m]e^{r(T-t)}\\&\qquad+\frac{\delta}{\zeta\gamma_i}(1-ln\frac{\delta}{\zeta h^p})-\frac{h^p}{\gamma_i}-(1+\frac{\tau_i}{n-1})^2\frac{a_i^2\eta_{i_2}^2}{2\gamma_i\sigma_{i_0}^2}=0,\qquad F_{i0}(T)=0.\end{aligned}\tag{3.23}$$

By solving Eq.(3.23), we obtain

$$F_{i0}(t) = \frac{\delta}{\zeta} e^{\frac{\delta}{\zeta}t} \int_t^T F_{i1}(u) e^{-\frac{\delta}{\zeta}u} du - f_{i1}[e^{-\frac{\delta}{\zeta}(T-t)} - e^{-r(T-t)}] - \frac{\zeta}{\delta} f_{i2}[e^{-\frac{\delta}{\zeta}(T-t)} - 1], \quad (3.24)$$

here $f_{i1} = \frac{1}{r-\frac{\delta}{\zeta}}[a_i\theta_i - \frac{\tau_i}{n-1}\sum\limits_{m\neq i} a_m\theta_m]$ and $f_{i2} = \frac{\delta}{\zeta\gamma_i}(1 - ln\frac{\delta}{\zeta h^p}) - \frac{h^p}{\gamma_i} - (1+\frac{\tau_i}{n-1})^2 \frac{a_i^2\eta_{i_2}^2}{2\gamma_i\sigma_{i_0}^2}$.

Lemma 3.2 Eqs.(3.22) has a unique root $\pi_{ip}^*(t)$ which is given by

$$\pi_{ip}^*(t) = \frac{\varphi_i(t)}{1+\frac{\tau_i}{n-1}} e^{-r(T-t)} + \frac{\frac{\tau_i}{n-1}}{1+\frac{\tau_i}{n-1}} \frac{1}{1-\psi_n} \sum_{i=1}^n \frac{\varphi_i(t)}{1+\frac{\tau_i}{n-1}} e^{-r(T-t)}, \ \ i = 1, 2, \cdots, n,$$

here,

$$\varphi_i(t) = \frac{ln\frac{\delta}{\zeta h^p}}{\zeta\gamma_i} - \frac{F_{i1}(t) - F_{i0}(t)}{\zeta}. \quad (3.25)$$

Proof Similar to Lemma 3.1, thus we omit it.

Theorem 3.1 For the wealth process (2.3) with n insurers and Problem 1, the ith $(i = 1, 2, \cdots, n)$ insurer's equilibrium reinsurance and investment strategy $\pi_i^*(t)$ is given by

$$\begin{aligned} q_i^*(t) =& \frac{a_i\eta_{i_2}}{\gamma_i\sigma_{i_0}^2} e^{-r(T-t)}, \\ \pi_{ij}^*(t) =& \frac{\phi_{ij}(t)}{1+\frac{\tau_i}{n-1}} e^{-r(T-t)} + \frac{\frac{\tau_i}{n-1}}{1+\frac{\tau_i}{n-1}} \frac{1}{1-\psi_n} \sum_{i=1}^n \frac{\phi_{ij}(t)}{1+\frac{\tau_i}{n-1}} e^{-r(T-t)}, \ \ j = 1, 2, \cdots, n, \\ \pi_{ip}^*(t) =& \frac{\varphi_i(t)}{1+\frac{\tau_i}{n-1}} e^{-r(T-t)} + \frac{\frac{\tau_i}{n-1}}{1+\frac{\tau_i}{n-1}} \frac{1}{1-\psi_n} \sum_{i=1}^n \frac{\varphi_i(t)}{1+\frac{\tau_i}{n-1}} e^{-r(T-t)}, \ \ t \in [0, \tau \wedge T], \end{aligned}$$

where ψ_n and $\phi_{ij}(t)$ are shown in Eq. (3.16), $\varphi_i(t)$ is given by Eq. (3.25).

Besides, the equilibrium value functions are

$$\begin{aligned} V^i(t, \hat{x}_i, s_1, s_2, \cdots, s_n, 1) =& -\frac{1}{\gamma_i} e^{-\gamma_i[e^{r(T-t)}\hat{x}_i + \sum\limits_{j=1}^n G_{ij}(t,s_j) - F_{i1}(t)]}, \\ V^i(t, \hat{x}_i, s_1, s_2, \cdots, s_n, 0) =& -\frac{1}{\gamma_i} e^{-\gamma_i[e^{r(T-t)}\hat{x}_i + \sum\limits_{j=1}^n G_{ij}(t,s_j) - F_{i0}(t)]}, \end{aligned}$$

where $G_{ij}(t, s_j)$ are given by (3.12), $F_{i1}(t)$ and $F_{i0}(t)$ are given by (3.15) and (3.24).

The following theorem verifies that the solution to HJB equation (3.3) given in Theorem 3.1 is indeed the solution to Problem 1.

Theorem 3.2 (Verification theorem) If $W^i(t, \hat{x}_i, \bar{s}, z)$ be the solution of HJB equation (3.3) with the boundary condition $V^i(T, \hat{x}_i, \bar{s}, z) = U_i(\hat{x}_i)$, and for $\forall j = 1, 2, \cdots, n$, the parameters satisfy one of the following conditions

$$(a)\ r > (1 - \frac{1}{\sqrt{6}})\mu_j;$$

$$(b)\ r < (1 - \frac{1}{\sqrt{6}})\mu_j \text{ and } T < \frac{1}{\beta\sqrt{6(\mu_j - r)^2 - r^2}} \arctan(-\frac{\sqrt{6(\mu_j - r)^2 - r^2}}{r}),$$

then the optimal value function is $V^i(t, \hat{x}_i, \bar{s}, z) = W^i(t, \hat{x}_i, \bar{s}, z)$, and the optimal strategy is $\pi_i^*(t)$, which is given in Theorem 3.1.

The proof of the verification theorem can be adapted from Theorem 2 of Gu et.al [7]. We hence omit it here.

4 Numerical Analysis

In this section, we present a series of numerical simulations to analyze the effects of different model parameters on the equilibrium reinsurance-investment strategies. In the following examples, unless otherwise specified, we select the basic parameters as $t = 0$, $T = 10$, $r = 0.03$, $\beta = 1$, $\zeta = 0.4$, $h^p = 0.005$, $\delta = 0.02$, and other model parameters are given in the Table 1.

Table 1 Values of parameters in numerical simulations

First insurer's parameters		Second insurer's parameters		Third insurer's parameters	
γ_1	0.4	γ_2	0.5	γ_3	0.3
τ_1	0.6	τ_2	0.5	τ_3	0.7
a_1	1.5	a_2	1.4	a_3	1.6
η_{1_1}	0.8	η_{2_1}	0.7	η_{3_1}	0.6
η_{1_2}	2.0	η_{2_2}	1.8	η_{3_2}	2.2
σ_{1_0}	1.0	σ_{2_0}	1.2	σ_{3_0}	0.8
First risky asset's parameters		Second risky asset's parameters		Third risky asset's parameters	
μ_1	0.07	μ_2	0.09	μ_3	0.05
s_1	1.2	s_2	1.4	s_3	1.3
σ_1	0.35	σ_2	0.4	σ_3	0.3

4.1 Effects of Parameters on Equilibrium Reinsurance Strategy

Form the expression of $q_i^*(t)$, we derive $\frac{\partial q_i^*(t)}{a_i} > 0$, $\frac{\partial q_i^*(t)}{\eta_{i_2}} > 0$, $\frac{\partial q_i^*(t)}{\gamma_i} < 0$, $\frac{\partial q_i^*(t)}{\sigma_{i_0}} < 0$ and $\frac{\partial q_i^*(t)}{r} < 0$, which mean that as the expectation of claim a_i or the safety loading of the reinsurer η_{i_2} increases, the cost of reinsurance becomes higher, and thus the ith insurer will take more risks by buying less reinsurance or acquiring more new businesses. When the ith insurer's risk aversion coefficient γ_i is higher, or the volatility σ_{i_0} is larger, the ith insurer prefers to transfer more risks into reinsurance, so as to purchase more reinsurance or acquire less new business. If the interest rate r is larger, the insurer will invest more funds in risk-free asset. As a result, the insurer will also buy more reinsurance or acquire less new business.

4.2 Effects of Parameters on Equilibrium Investment Strategy

In this subsection, without loss of generality, we assume that the financial market con-

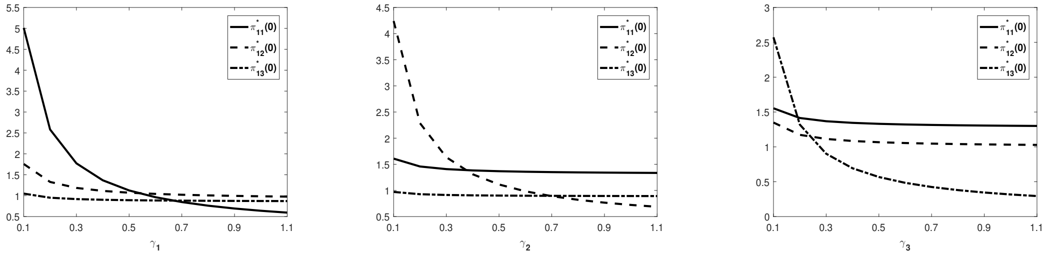

Figure 1: Effects of $\gamma_1, \gamma_2, \gamma_3$ on $\pi^*_{11}(0), \pi^*_{12}(0), \pi^*_{13}(0)$

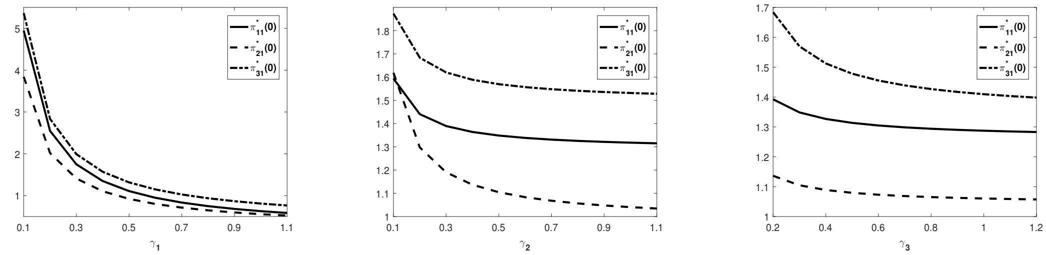

Figure 2: Effects of $\gamma_1, \gamma_2, \gamma_3$ on $\pi^*_{11}(0), \pi^*_{21}(0), \pi^*_{31}(0)$

sists of one risk-free bond, one defaultable bond and three risky assets, and that there are three insurers investing in the financial market.

Fig. 1 and Fig. 2 show the influences of γ_1, γ_2 and γ_3 on $\pi^*_{11}(0)$, $\pi^*_{12}(0)$, $\pi^*_{13}(0)$, $\pi^*_{21}(0)$ and $\pi^*_{31}(0)$. From Fig. 1 and Fig. 2 we can see that $\pi^*_{11}(0)$, $\pi^*_{12}(0)$, $\pi^*_{13}(0)$, $\pi^*_{21}(0)$ and $\pi^*_{31}(0)$ decrease with regard to γ_1, γ_2 and γ_3, respectively. It is observed that with the increase of $\gamma_i (1 \leq i \leq 3)$ the insurers tend to present stronger risk-aversion. Hence, a larger $\gamma_i (1 \leq i \leq 3)$ would cause less investment in risk assets. Because we consider the mutual influence among insurers, the increase in one insurer's risk-aversion coefficient would cause that the other two insurers also decrease their investment. However, from Fig. 1, we notice that each insurer is more sensitive to his own risk-aversion coefficient.

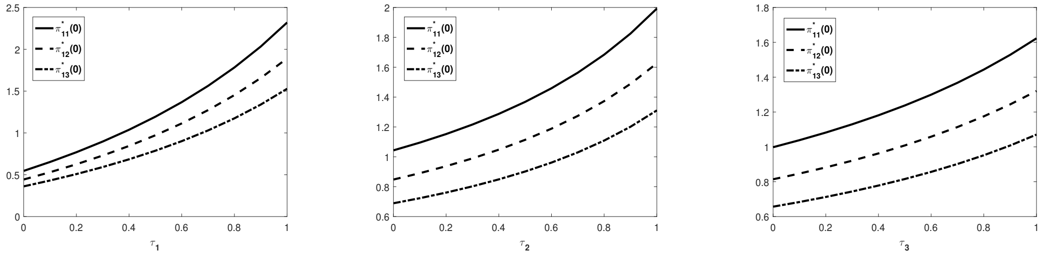

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Figure 3: Effects of τ_1, τ_2, τ_3 on $\pi^*_{11}(0), \pi^*_{12}(0), \pi^*_{13}(0)$

Fig. 3 displays the effects of τ_1, τ_2 and τ_3 on $\pi^*_{11}(0)$, $\pi^*_{12}(0)$ and $\pi^*_{13}(0)$, respectively. We can observe from Fig. 3 (a) that, with the increase of τ_1, the first insurer increases his investments in three risky assets. Here, a larger τ_1 means that the first insurer would put more weight on the relative average performance, and that the competition becomes more

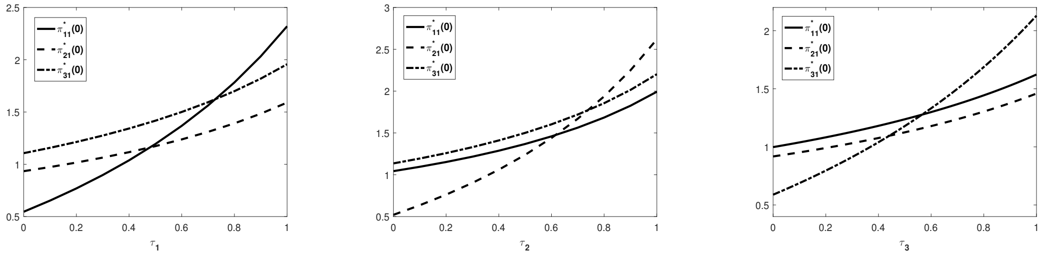

Figure 4: Effects of τ_1, τ_2, τ_3 on $\pi^*_{11}(0), \pi^*_{21}(0), \pi^*_{31}(0)$

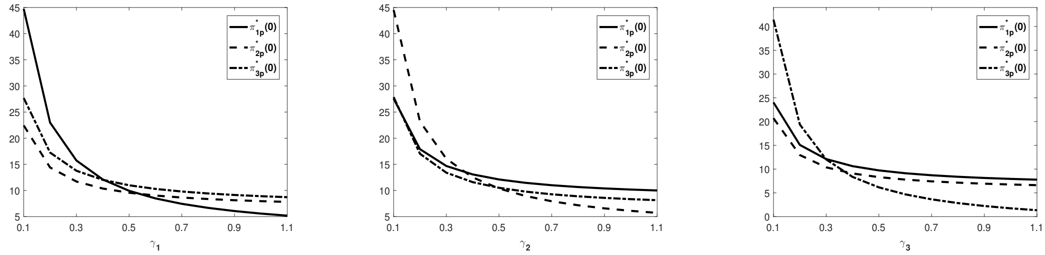

Figure 5: Effects of $\gamma_1, \gamma_2, \gamma_3$ on $\pi^*_{1p}(0), \pi^*_{2p}(0), \pi^*_{3p}(0)$

intense. To beat competitors, the first insurer would invest more of his surplus not only in the first risky asset, but also in the second and third risky asset. The similar phenomenon also appears in Fig. 3 (b) and (c). As τ_2 or τ_3 gradually increases, the first insurer would invest more of its wealth in risky assets in order to gain an advantage over the competition.

Fig. 4 further captures how $\pi^*_{11}(0)$, $\pi^*_{21}(0)$ and $\pi^*_{31}(0)$ vary with respect to the competition parameters τ_1, τ_2 and τ_3. Similar to the result for Fig. 3, the growth of $\tau_i (1 \le i \le 3)$ indicates that the ith insurer cares more about his competitor's performance at terminal time, and hence he tends to increase exposure on risky assets, i.e., investing more amount of surplus in the risky assets. It can also be seen that each insurer's relative concern parameter has a more significant influence on their own investment.

Fig. 5 illustrates the effects of γ_1, γ_2 and γ_3 on $\pi^*_{1p}(0)$, $\pi^*_{2p}(0)$, $\pi^*_{3p}(0)$. Here $\gamma_i (1 \le i \le 3)$ is the ith insurer's risk-aversion coefficient. Similar to that in Fig. 1, $\pi^*_{1p}(0)$, $\pi^*_{2p}(0)$, $\pi^*_{3p}(0)$ decrease with regard to γ_1, γ_2 and γ_3.

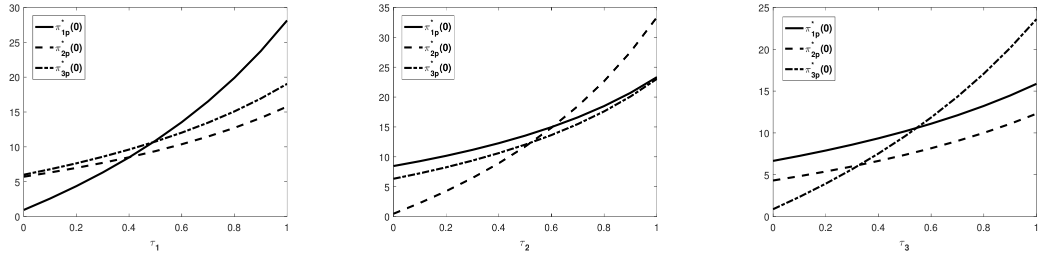

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Figure 6: Effects of τ_1, τ_2, τ_3 on $\pi^*_{1p}(0), \pi^*_{2p}(0), \pi^*_{3p}(0)$

To demonstrate the impacts of $\tau_i (1 \le i \le 3)$ on the equilibrium pre-default bond investment strategies $\pi^*_{1p}(0)$, $\pi^*_{2p}(0)$ and $\pi^*_{3p}(0)$, we refer to Fig. 6. From these subfigures, we find that $\pi^*_{1p}(0)$, $\pi^*_{2p}(0)$ and $\pi^*_{3p}(0)$ are increasing functions of $\tau_i (1 \le i \le 3)$ in all three

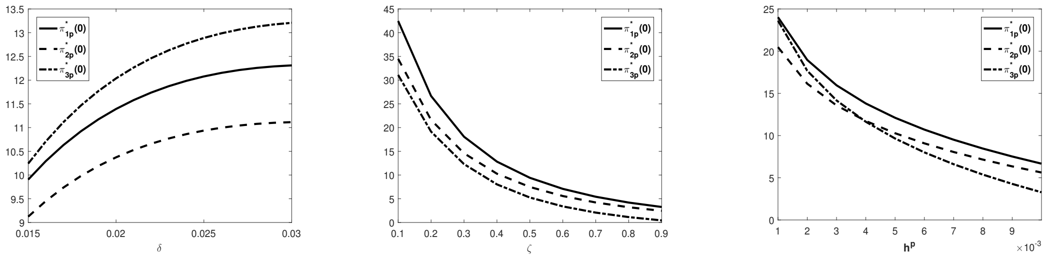

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Figure 7: Effects of δ, h^p, ζ on $\pi^*_{1p}(0), \pi^*_{2p}(0), \pi^*_{3p}(0)$

cases. In other words, if an insurer is more concerned with his relative terminal wealth, he would retain more risks, so more money would invest in the defaultable bond. In addition, from Fig. 6 (a), for fixed τ_2 and τ_3, a greater τ_1 produces a larger $\pi^*_{2p}(0)$ and $\pi^*_{3p}(0)$, which implies that the second and third insurer would invest more surplus in the corporate bond when his competitor is aggressive. Fig. 6 (b) and (c) reveal a similar phenomenon.

Fig. 7 depicts the effects of loss rate ζ, credit spread δ and default intensity h^p on the equilibrium investment strategies $\pi^*_{1p}(0)$, $\pi^*_{2p}(0)$ and $\pi^*_{3p}(0)$. As is well illustrated in Fig. 7 (a) and (b), $\pi^*_{1p}(0)$, $\pi^*_{2p}(0)$ and $\pi^*_{3p}(0)$ decrease w.r.t. ζ while increase w.r.t. δ. So the insurer will invest less in the defaultable bond when he has higher loss rate or lower credit spread. From Fig. 7 (c), we find that $\pi^*_{1p}(0)$, $\pi^*_{2p}(0)$ and $\pi^*_{3p}(0)$ have a positive relationship with h^p. Indeed, a larger h^p means a stronger default intensity, and therefore, the less money is invested in the defaultable bond.

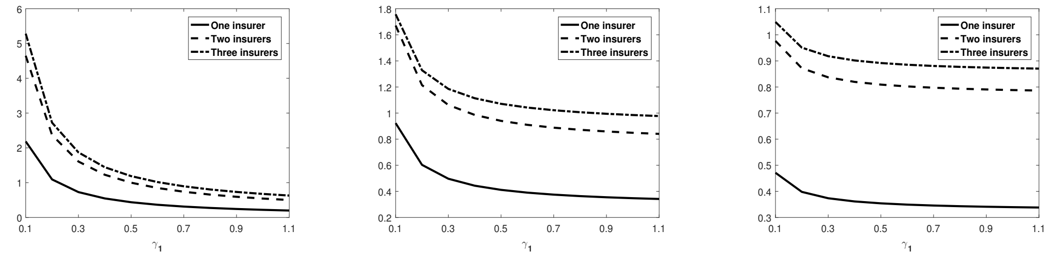

Figure 8: Effects of the number of insurers on $\pi^*_{11}(0), \pi^*_{12}(0), \pi^*_{13}(0)$

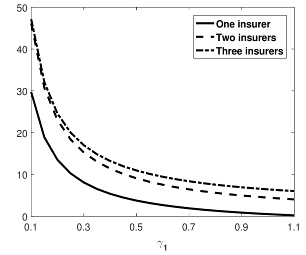

Figure 9: Effects of the number of insurers on $\pi^*_{1p}(0)$

4.3 Effects of the Number of Insurers on Equilibrium Investment Strategy

In this subsection, we take the first insurer as an example to illustrate the influences of the number of insurers on insurer's investment strategies. Fig. 8 and Fig. 9 give the sensitivity of $\pi^*_{11}(0)$, $\pi^*_{12}(0)$, $\pi^*_{13}(0)$ and $\pi^*_{1p}(0)$ to the number of insurers. Here we respectively assume that there are one insurer, two insurers and three insurers investing in the above financial market. We can see from Fig. 8 and Fig. 9 that the investment strategies $\pi^*_{11}(0)$, $\pi^*_{12}(0)$, $\pi^*_{13}(0)$ and $\pi^*_{1p}(0)$ increase with the rising number of insurers. This is because, with the increase of the number of insurers, the insurer will have access to more market information. That would boost his confidence and make him invest more of his surplus in risky assets and the defaultable bond.

5. Conclusion

In this paper, we investigate a non-zero-sum stochastic differential investment and reinsurance game involving a defaultable security for n competitive insurers. The insurer's claim process is assumed to follow a drifted Brownian motion. We allow each insurer to dynamically purchase proportional reinsurance and allocate his surplus to a risk-free asset, a defaultable bond and n risky assets. Moreover, we adopt the CEV model to describe each risky asset's price process. By adopting a stochastic control approach, we derive the HJB equations under both pre-default case and post-default case. Explicit expressions of equilibrium strategy that maximize the expected exponential utility of the terminal wealth relative to that of his competitors and corresponding equilibrium value functions are obtained. At last, numerical examples and sensitivity analyses are presented to illustrate the impacts of parameters on the equilibrium strategy. Results indicate that the risk-aversion coefficient of both the insurer and his competitors have an impact on the insurer's investment strategy. Furthermore, we find that competition will increase each insurer's investment in risky assets and defaultable bond, and the number of insurers will also influence each insurer's investment strategy.

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违约市场中的n个保险公司的最优投资和再保险问题

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摘要: 本文研究了n个保险公司之间的非零和随机微分投资再保险博弈问题. 每个保险公司可以购买比例再保险, 并将财富投资于一个由无风险资产, 可违约债券和n个风险资产组成的金融市场. 特别地, 风险资产的价格过程服从CEV模型, 可违约债券可在违约时收回一定比例的价值. 每个保险公司的目标是相对于竞争对手, 最大化终端财富的期望指数效用. 利用随机最优控制理论, 我们分别推导了均衡策略和均衡值函数的显式表达式. 数值例子分析了模型参数对均衡策略的影响. 此外, 我们还分析了保险公司数量对均衡投资策略的影响. 我们发现, 随着保险公司数量的增加, 每个保险公司将在风险资产和可违约债券上投入更多的资金.

关键词: 投资再保险; 非零和博弈; 可违约债券; 指数效应函数; CEV模型

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