

ROBUST OPTIMAL INVESTMENT AND REINSURANCE STRATEGIES WITH DELAY AND DEFAULT RISK UNDER DEPENDENT RISK MODEL

MU Rui, MA Shi-xia, ZHANG Xin-ru

(*School of Science, Hebei University of Technology, Tianjin 300401, China*)

Abstract: In this paper, we consider the robust optimal investment-reinsurance strategies with delay and default risk under risk dependent model. We assume that the ambiguity-averse insurer's wealth process have two dependent classes of insurance business and the surplus can invest in a risk-free asset, a defaultable bond and a risky asset whose price process satisfies Heston model. Using dynamic programming principle, we establish the robust Hamilton-Jacobi-Bellman(HJB) equations for the post-default case and the pre-default case, respectively. Furthermore, we obtain the robust optimal investment and reinsurance strategies and the corresponding value functions by maximizing the expected exponential utility of the terminal wealth. Finally, we provide numerical examples to illustrate the effects of some model parameters on the robust optimal strategies.

Keywords: robust optimal strategies; heston model; stochastic differential delay equation; dependent risk; default risk

2010 MR Subject Classification: 62P05; 91B30; 91B70

Document code: A **Article ID:** 0255-7797(2022)04-0300-15

1 Introduction

Reinsurance and investment are two important issues for insurers. Reinsurance is an effective way to reduce risk, while investment is the most common way to increase wealth. In recent years, based on different decision-making objectives, more attention have been paid to the research on the optimal reinsurance-investment problem. For example, Liang et al.[1] considered the objective function of minimizing the ruin probability; Guan and Liang[2] studied the optimal problems aiming to maximize the expected utility of terminal wealth; Wang et al.[3] investigated optimal reinsurance and investment problems under mean variance criterion, and so on.

Besides the different decision-making objectives mentioned above, many scholars also studied the optimal investment-reinsurance strategy in the following aspects. Firstly, the model uncertainties do exist widely in finance and insurance, some scholars have investigated the impact of ambiguity on optimal reinsurance and investment problems. For example, Chen and Yang[4] considered the robust optimal reinsurance-investment strategy

* **Received date:** 2021-04-29

Accepted date: 2021-09-09

Foundation item: Supported by the Natural Science Foundation of China(12071107).

Biography: Mu Rui(1997-), female, born at Hengshui, Hebei, postgraduate, major in stochastic process and its application.

selection problem for an ambiguity-averse insurer (AAI). Wang et al.[5] investigated a class of robust non-zero-sum reinsurance-investment stochastic differential games between two competing insurers. More often, Zhang and Chen[6] and Sun et al.[7] also took the ambiguity into account. Secondly, the default of one company usually has strong influence on other companies, high yield bonds with default risk have become attractive to investors. Many scholars have studied the reinsurance-investment problem with default risk, such as Zhang and Chen[6], Yang[8], and so on. Thirdly, due to the insurer's current wealth existing capital inflow/outflow, it is necessary to take delay into account. Refer to Shen and Zeng[9], the optimal investment-reinsurance strategy is obtained under delay risk. Moreover, based on the different risk asset models, A and Li[10] and Zhang and Chen[11] studied an optimal excess-of-loss reinsurance and investment strategies with delay. In addition, some insurance businesses are usually correlated in practice. For example, a traffic accident may cause property loss or medical claims or death claims, these insurance businesses will be correlated. Therefore, it is necessary to take dependent risk into account. Bi et al.[12] studied the optimal problems with two dependent classes of insurance business under the criterion of mean-variance. Furthermore, with the multiple dependent risk, Yuen et al.[13] and Sun et al.[7] derived the optimal strategies and value function under the different risk models.

Inspired by the above references, we consider the robust optimal investment and reinsurance strategies with delay and default risk under risk dependent model. Comparing with the existing literature, the main contributions of this paper include: (i) We consider delay, default, dependent risk and ambiguity aversion simultaneously in an optimal reinsurance-investment problem; (ii) We obtain the robust optimal strategies and corresponding value function in the post-default and pre-default case under risk dependent model.

The remainder of this paper is organized as follows. In section 2, we describe the model and the necessary assumptions. In Section 3, we state the robust optimal control problem for an AAI. Section 4 derives the robust optimal investment-reinsurance strategies under the post-default case and the pre-default case respectively. In section 5, we present some numerical illustrations to analyze our theoretical results.

2 Model Assumption

We consider a filtered complete probability space $(\Omega, \mathcal{F}, \{F_t\}_{t \in [0, T]}, P)$, where T represents the terminal time and is a positive finite constant and F_t stands for the information of the market available up to time t . Assume that all processes introduced below are well-defined and adapted processes in this space.

2.1 Surplus Process

Assume that the insurer's surplus process is described by the following Cramer-Lundberg model

$$dX(t) = c dt - d\left(\sum_{i=1}^{N_1(t)+N(t)} X_i\right) - d\left(\sum_{i=1}^{N_2(t)+N(t)} Y_i\right), \quad (2.1)$$

where X_i is the i th claim size from the first class; $\{X_i, i \geq 1\}$ are assumed to be independent and identically distributed positive random variables with common distribution function $F_X(\cdot)$, finite first moment $E(X_i) = \mu_1 > 0$, and second moment $E(X_i^2) = \nu_1$; Similarly, Y_i is the i th claim size from the second class, $E(Y_i) = \mu_2 > 0$, and $E(Y_i^2) = \nu_2$; $\{N_1(t), t \geq 0\}$, $\{N_2(t), t \geq 0\}$ and $\{N(t), t \geq 0\}$ are three independent Poisson processes with positive intensity parameters λ_1 , λ_2 and λ , respectively. $\{X_i, i \geq 1\}$, $\{Y_i, i \geq 1\}$, $\{N_1(t), t \geq 0\}$, $\{N_2(t), t \geq 0\}$ and $\{N(t), t \geq 0\}$ are mutually independent. Further, the insurer's premium rate is calculated according to the expected value principle, i.e., $c = (1 + \eta)[(\lambda_1 + \lambda)\mu_1 + (\lambda_2 + \lambda)\mu_2]$, where $\eta > 0$ is the insurer's safety loading. From Grandell[14], we know that the compound Poisson processes can be approximated by the following diffusion process :

$$d\left(\sum_{i=1}^{N_1(t)+N(t)} X_i\right) = a_1 dt - \sigma_1 dB_1(t), \quad d\left(\sum_{i=1}^{N_2(t)+N(t)} Y_i\right) = a_2 dt - \sigma_2 dB_2(t), \quad (2.2)$$

where $a_1 = \mu_1(\lambda_1 + \lambda)$, $\sigma_1^2 = \nu_1(\lambda_1 + \lambda)$, $a_2 = \mu_2(\lambda_2 + \lambda)$, $\sigma_2^2 = \nu_2(\lambda_2 + \lambda)$. Here $B_1(t)$ and $B_2(t)$ are standard Brownian motions with correlation coefficient $\rho_0 \in (-1, 1)$. To be protected from potential large claims, the insurer is allowed to purchase proportional reinsurance to disperse risk, let $q_1(t), q_2(t) \in [0, 1]$ be the reinsurance retention levels for the 1th and 2th line of business at time t , respectively. Then by Liang and Yuen[15], the dynamics of $X^q(t)$ can be described by

$$\begin{aligned} dX^q(t) &= (c - c_1) dt - q_1(t)d\left(\sum_{i=1}^{N_1(t)+N(t)} X_i\right) - q_2(t)d\left(\sum_{i=1}^{N_2(t)+N(t)} Y_i\right) \\ &= [c - c_1 - a_1q_1(t) - a_2q_2(t)] dt + \sqrt{\sigma_1^2q_1^2(t) + \sigma_2^2q_2^2(t) + 2q_1(t)q_2(t)\lambda\mu_1\mu_2} dW_0(t), \end{aligned} \quad (2.3)$$

where $c_1 = (1 + \theta)[\mu_1(\lambda_1 + \lambda)(1 - q_1(t)) + \mu_2(\lambda_2 + \lambda)(1 - q_2(t))]$ is the reinsurance premium rate, $W_0(t)$ is a standard Brownian motion.

2.2 Financial Market

The insurer is assumed to invest in a risk-free asset whose price process is governed by $dB(t) = rB(t)dt$, $B(0) = b_0$ and a risky asset whose price process follows Heston model:

$$dS(t) = S(t) \left[(r + \alpha L(t))dt + \sqrt{L(t)}dW_1(t) \right], \quad S(0) = s_0; \quad (2.4)$$

$$dL(t) = k(m - L(t))dt + \sigma\sqrt{L(t)}d\bar{W}_2(t), \quad L(0) = l_0, \quad (2.5)$$

where positive constant r is the risk-free interest rate, α, k, m , and σ are all positive constants, and $W_1(t)$ and $\bar{W}_2(t)$ are two standard Brownian motions with $E[W_1(t)\bar{W}_2(t)] = \rho t, \rho \in [-1, 1]$. By standard Gaussian linear regression, $\bar{W}_2(t)$ can be rewritten as

$$d\bar{W}_2(t) = \rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t), \quad (2.6)$$

where $W_2(t)$ is another standard Brownian motion. We assume that $W_0(t), W_1(t)$ and $W_2(t)$ are mutually independent. Moreover, we require $2km \geq \sigma^2$ to ensure that $L(t)$ is almost surely nonnegative. Similar to Bi et al.[12], we assume that there exists a defaultable bond with a maturity date T_1 and that the default time is τ . The dynamics of the defaultable bond price under P is given by

$$dp(t, T_1) = p(t-, T_1) [rdt + (1 - Z(t))\delta(1 - \Delta)dt - (1 - Z(t-))\varsigma dM^P(t)], \tag{2.7}$$

where $\delta = h^Q\varsigma$ is the risk neutral credit spread, $1/\Delta \geq 1$ is the default risk premium, $\varsigma \in [0, 1]$ is the loss rate of the bond when a default occurs, $M^P(t) = Z(t) - \int_0^1 (1 - Z(s))h^p ds$ is a martingale and $Z(t) = \mathbf{1}_{\{\tau \leq t\}}$ is a default process.

2.3 Wealth Process with Delay

Let $Y^\pi(t), \bar{Y}^\pi(t)$ and $G^\pi(t)$ be the delayed wealth and average and pointwise performance of the wealth in the past horizon $[t - h, t]$ respectively, i.e.

$$Y^\pi(t) = \int_{-h}^0 e^{\beta s} X^\pi(t + s) ds, \bar{Y}^\pi(t) = \frac{Y^\pi(t)}{\int_{-h}^0 e^{\beta s} ds}, G^\pi(t) = X^\pi(t - h), \tag{2.8}$$

for $\forall t \in [0, T]$, where $\beta \geq 0$ is an average parameter and $h > 0$ is the delay parameter. Denote by the function $f(t, X^\pi(t) - \bar{Y}^\pi(t), X^\pi(t) - G^\pi(t))$ the capital inflow/outflow amount, which is proportional to the past performance of the insurer's wealth, i.e.

$$\begin{aligned} f(t, X^\pi(t) - \bar{Y}^\pi(t), X^\pi(t) - G^\pi(t)) &= k_1 (X^\pi(t) - \bar{Y}^\pi(t)) + k_2 (X^\pi(t) - G^\pi(t)) \\ &= (k_1 + k_2) X^\pi(t) - k_1 \bar{Y}^\pi(t) - k_2 G^\pi(t), \end{aligned} \tag{2.9}$$

where k_1 and k_2 are two nonnegative constants, $X^\pi(t) - G^\pi(t)$ represents the absolute performance of wealth, $X^\pi(t) - \bar{Y}^\pi(t)$ stands for the average performance of the wealth. Let $\pi(t) := \{q_1(t), q_2(t), \pi_1(t), \pi_2(t)\}_{t \in [0, T]}$ be the reinsurance-investment strategy. Following Shen et al.[9] and A et al.[10], when we consider the capital inflow/outflow function, the stochastic differential delay equation (SDDE) can be given as follows:

$$\begin{aligned} dX^\pi(t) &= dX^q(t) + \pi_1(t) \frac{dS(t)}{S(t)} + \pi_2(t) \frac{dp(t, T_1)}{p(t-, T_1)} + (X^\pi(t) - \pi_1(t) - \pi_2(t)) \frac{dB(t)}{B(t)} \\ &\quad - f(t, X^\pi(t) - \bar{Y}^\pi(t), X^\pi(t) - G^\pi(t)) dt \\ &= [AX^\pi(t) + c - c_1 - a_1q_1(t) - a_2q_2(t) + \alpha l\pi_1(t) + \delta(1 - z)(1 - \Delta)\pi_2(t) \\ &\quad + \bar{k}_1 Y^\pi(t) + k_2 G^\pi(t)] dt + \pi_1(t) \sqrt{l} dW_1(t) - \pi_2(t)(1 - Z(t-))\varsigma dM^P(t) \\ &\quad + \sqrt{\sigma_1^2 q_1^2(t) + \sigma_2^2 q_2^2(t) + 2\lambda\mu_1\mu_2 q_1(t)q_2(t)} dW_0(t), \end{aligned} \tag{2.10}$$

where

$$A = r - k_1 - k_2, \bar{k}_1 = \frac{k_1}{\int_{-h}^0 e^{\beta s} ds}. \tag{2.11}$$

Furthermore, the initial value of the average performance wealth is $Y^\pi(0) = \frac{x_0(1 - e^{-\beta h})}{\beta}$.

Definition 1 (admissible strategy) For any fixed $t \in [0, T]$, the strategy π is said to be admissible if it is F_t -progressively measurable and satisfies

- (i) $q_i(t) \in [0, 1], i = 1, 2;$
- (ii) $E \left(\int_0^T \|\pi(s)\|^2 ds \right) < \infty$, where $\|\pi(t)\|^2 = q_1^2(t) + q_2^2(t) + \pi_1^2(t) + \pi_2^2(t);$
- (iii) $\forall (t, x, y, l) \in [0, T] \times R \times R \times R^+$, the SDDE (2.10) has a pathwise unique solution $\{X^\pi(t), t \in [0, T]\}$ with $E_{t,x,y,l}^{Q^*} U[(X^\pi(T), Y^\pi(T))] < \infty$, where Q^* is the chosen model to describe the worst case, $E_{t,x,y,l}[\cdot]$ is the condition expectation given $X^\pi(t) = x, Y^\pi(t) = y, L(t) = l$.

3 Robust Optimal Control Problem for an AAI

We assume that the insurer is concerned with $X^\pi(T)$ and $Y^\pi(T)$ in the time interval $[T - h, T]$. Therefore, the insurer has the following exponential utility function:

$$U(X^\pi(T), Y^\pi(T)) = -\frac{1}{n} e^{-n(X^\pi(T) + \varepsilon Y^\pi(T))}. \quad (3.1)$$

For simplicity, we call the combination $X^\pi(T) + \varepsilon Y^\pi(T)$ the terminal wealth. Traditionally, the insurer is assumed to be an ambiguity-neutral investor (ANI) with objective function as

$$\sup_{\pi \in \Pi} E[U(X^\pi(T), Y^\pi(T)) | X^\pi(t) = x, Y^\pi(t) = y, L(t) = l], \quad (3.2)$$

where Π is the set of all admissible strategies. Actually, the insurer would like to consider the alternative probability measure rather than reference probability measure P because of uncertainty on the parameters. The alternative models are defined by $Q := \{Q | Q \sim P\}$. Next, define a process $\{\theta(t) = (\theta_0(t), \theta_1(t), \theta_2(t)) | t \in [0, T]\}$ satisfying that (i) $\theta(t)$ is F_t -measurable, $\forall t \in [0, T]$; (ii) $E \left[\exp \left\{ (1/2) \int_0^T \|\theta(t)\|^2 dt \right\} \right] < \infty$, where $\|\theta(t)\|^2 = \sum_{i=0}^2 \theta_i^2(t)$. We denote Θ for the space of all such processes θ . For $\forall \theta \in \Theta$, we define a real-valued process $\{\Lambda^\theta(t) | t \in [0, T]\}$ on (Ω, F, P) by

$$\Lambda^\theta(t) = \left\{ - \int_0^t \theta(u) dW(u) - \frac{1}{2} \int_0^t \|\theta(u)\|^2 du \right\}, \quad (3.3)$$

where $W(t) = (W_0(t), W_1(t), W_2(t))'$. By its differentiation rule,

$$d\Lambda^\theta(t) = \Lambda^\theta(t)[- \theta(t) dW(t)]. \quad (3.4)$$

Thus $\Lambda^\theta(t)$ is a P -martingale. Hence, $E[\Lambda^\theta(t)] = 1$. For $\forall \theta \in \Theta$, a new real-world probability measure Q absolutely continuous to P on F_T is defined by $\frac{dQ}{dP} \Big|_{F_T} := \Lambda^\theta(t)$. So far, we have constructed a family of real-world probability measures Q parameterized by $\theta \in \Theta$. Applying Girsanov's theorem, we can see that

$$dW_i^Q(t) = dW_i(t) + \theta_i(t) dt, \quad i = 0, 1, 2. \quad (3.5)$$

Thus, the stochastic process (2.5) under Q is

$$\begin{aligned} dL(t) = & \left[r(m - L(t)) - \sigma \rho \theta_1(t) \sqrt{L(t)} - \sigma \theta_2(t) \sqrt{L(t)} \sqrt{1 - \rho^2} \right] dt \\ & + \sigma \rho \sqrt{L(t)} dW_1^Q(t) + \sigma \sqrt{L(t)} \sqrt{1 - \rho^2} dW_2^Q(t). \end{aligned} \quad (3.6)$$

And the wealth process (2.10) under Q is rewritten as

$$\begin{aligned} dX^\pi(t) = & [AX^\pi(t) + (\eta - \theta)(a_1 + a_2) + \theta[a_1q_1(t) + a_2q_2(t)] + \alpha l\pi_1(t) + \delta(1 - z)(1 - \Delta)\pi_2(t) \\ & + \bar{k}_1Y^\pi(t) + k_2G^\pi(t) - \theta_0(t)\sqrt{\sigma_1^2q_1^2(t) + \sigma_2^2q_2^2(t) + 2\lambda\mu_1\mu_2q_1(t)q_2(t)} \\ & - \theta_1(t)\pi_1(t)\sqrt{l}] dt + \sqrt{\sigma_1^2q_1^2(t) + \sigma_2^2q_2^2(t) + 2\lambda\mu_1\mu_2q_1(t)q_2(t)}dW_0^Q(t) \\ & + \pi_1(t)\sqrt{l}dW_1^Q(t) - \pi_2(t)(1 - Z(t-))\varsigma dM^P(t). \end{aligned} \tag{3.7}$$

Inspired by Zhang and Chen[6], we formulate the following robust control problem to modify problem (3.2), i.e.

$$V(t, x, y, l, z) = \sup_{\pi \in \Pi} \inf_{\theta \in \Theta} E^Q \left[U(X^\pi(T), Y^\pi(T)) + \int_t^T \Psi(s, X^\pi(s), Y^\pi(s), \theta(s)) ds \right], \tag{3.8}$$

where

$$\Psi(t) = \frac{\theta_0^2(t)}{2\psi_0(t)} + \frac{\theta_1^2(t)}{2\psi_1(t)} + \frac{\theta_2^2(t)}{2\psi_2(t)} + h^P(1 - Z(t))\frac{\theta_3(t) \ln \theta_3(t) - \theta_3(t) + 1}{\psi_3(t)}, \tag{3.9}$$

$\psi_i(t) = -\frac{\beta_i}{nV(t,x,y,l,z)}$, $i = 0, 1, \dots, 3$, and $\beta_i > 0, i = 0, 1, \dots, 3$ are the ambiguity-aversion coefficients representing the ambiguity-averse level of the insurer to the model uncertainty.

In order to solve the optimization problem (3.8) and derive the optimal strategy $\pi^*(t)$. By dynamic programming principle, the HJB equation for (3.8) can be derived as

$$\sup_{\pi \in \Pi} \inf_{\theta \in \Theta} \left\{ L^{\pi, \theta} V + \frac{\theta_0^2}{2\psi_0} + \frac{\theta_1^2}{2\psi_1} + \frac{\theta_2^2}{2\psi_2} + h^P(1 - z)\frac{\theta_3 \ln \theta_3 - \theta_3 + 1}{\psi_3} \right\} = 0, \tag{3.10}$$

with the boundary condition $V(t, x, y, l, z) = U(X^\pi(T), Y^\pi(T))$, where $L^{\pi, \theta}$ is the variational operator and is defined by

$$\begin{aligned} L^{\pi, \theta} V = & V_t + V_x [Ax + (\eta - \theta)(a_1 + a_2) + \theta[a_1q_1 + a_2q_2] + \alpha l\pi_1 + \delta(1 - z)(1 - \Delta)\pi_2 + \bar{k}_1y \\ & + k_2g - \theta_0\sqrt{\sigma_1^2q_1^2 + \sigma_2^2q_2^2 + 2\lambda\mu_1\mu_2q_1q_2} - \theta_1\pi_1\sqrt{l}] + V_y (x - \beta y - e^{-\beta h}g) \\ & + V_l [k(m - l) - \sigma\rho\theta_1\sqrt{l} - \sigma\theta_2\sqrt{l}\sqrt{1 - \rho^2}] + \frac{1}{2}V_{xx} [\pi_1^2l + \sigma_1^2q_1^2 + \sigma_2^2q_2^2 + 2\lambda\mu_1\mu_2q_1q_2] \\ & + \frac{1}{2}V_{ll}\sigma^2l + V_{xl}\sigma\rho l\pi_1 + \theta_3h^P(1 - z) [V(t, x - \pi_2\varsigma, y, l, 1) - V(t, x, y, l, 0)], \end{aligned} \tag{3.11}$$

here $V_t, V_x, V_y, V_l, V_{xx}, V_{ll}$ and V_{xl} represent the partial derivatives of value function with respect to the corresponding variables.

4 Robust Optimal Investment and Reinsurance Strategies

In the following, we set out to derive the explicit solutions to the HJB equation (3.10) for the post-default case and the pre-default case, respectively.

4.1 Post-Default Case

In the post-default case, i.e. $z = 1$, we have that $p(t, T_1) = 0, \tau \leq t \leq T$. Thus $\pi_2(t) = 0$ for $\tau \leq t \leq T$. Then the HJB equation (3.10) can be rewritten as the following form:

$$\begin{aligned} \sup_{\pi \in \Pi} \inf_{\theta \in \Theta} \left\{ V_t + V_x \left[Ax + (\eta - \theta)(a_1 + a_2) + \theta[a_1 q_1 + a_2 q_2] + \alpha l \pi_1 + \bar{k}_1 y + k_2 g - \theta_1 \pi_1 \sqrt{l} \right. \right. \\ \left. \left. - \theta_0 \sqrt{\sigma_1^2 q_1^2 + \sigma_2^2 q_2^2 + 2\lambda \mu_1 \mu_2 q_1 q_2} \right] + V_y (x - \beta y - e^{-\beta h} g) + V_l \left[k(m - l) - \sigma \rho \theta_1 \sqrt{l} \right. \right. \\ \left. \left. - \sigma \theta_2 \sqrt{l} \sqrt{1 - \rho^2} \right] + \frac{1}{2} V_{xx} \left[\pi_1^2 l + \sigma_1^2 q_1^2 + \sigma_2^2 q_2^2 + 2\lambda \mu_1 \mu_2 q_1 q_2 \right] + \frac{1}{2} V_{ll} \sigma^2 l + V_{xl} \sigma \rho l \pi_1 \right. \\ \left. - \frac{nV}{2\beta_0} \theta_0^2 - \frac{nV}{2\beta_1} \theta_1^2 - \frac{nV}{2\beta_2} \theta_2^2 \right\} = 0, \end{aligned} \quad (4.1)$$

with the boundary condition $V(T, x, y, l, 1) = -\frac{1}{n} e^{-n(x+\varepsilon y)}$. According to A and Li[10], we assume the parameters satisfy the following conditions:

$$k_2 = \varepsilon e^{-\beta h}, \quad \bar{k}_1 = (\beta + A + \varepsilon)\varepsilon. \quad (4.2)$$

According to (4.1) and the boundary condition of $V(T, x, y, l, 1)$, we conjecture that the optimal value function has the following form:

$$V(T, x, y, l, 1) = -\frac{1}{n} e^{-n(x+\varepsilon y)H(t)+M(t)l+N(t)}. \quad (4.3)$$

The boundary condition implies that $H(T) = 1, M(T) = 0, N(T) = 0$. We get partial derivative

$$\begin{aligned} V_t = [-n(x + \varepsilon y)H'(t) + M'(t)l + N'(t)]V, \quad V_x = -nH(t)V, \quad V_y = -n\varepsilon H(t)V \\ V_l = M(t)V, \quad V_{xx} = n^2 H^2(t)V, \quad V_{ll} = M^2(t)V, \quad V_{xl} = -nH(t)M(t)V. \end{aligned} \quad (4.4)$$

Inserting (4.4) into (4.1), fix π and maximize over θ , by the first-order conditions we get

$$\begin{aligned} \theta_0^*(t) &= \beta_0 H(t) \sqrt{\sigma_1^2 q_1^2 + \sigma_2^2 q_2^2 + 2\lambda \mu_1 \mu_2 q_1 q_2}, \\ \theta_1^*(t) &= \beta_1 H(t) \pi_1 \sqrt{l} - \frac{\beta_1}{n} M(t) \sigma \rho \sqrt{l}, \\ \theta_2^*(t) &= -\frac{\beta_2}{n} M(t) \sigma \sqrt{l} \sqrt{1 - \rho^2}. \end{aligned} \quad (4.5)$$

Inserting (4.4) and (4.5) into (4.1), by the first-order conditions we get

$$\pi_1^*(t) = \frac{1}{H(t)} \left[\frac{\alpha}{n + \beta_1} + \frac{\sigma \rho}{n} M(t) \right], \quad \pi_2^*(t) = 0. \quad (4.6)$$

Taking the above conditions into (4.1), we get

$$\begin{aligned} -n(x + \varepsilon y)H'(t) + M'(t)l + N'(t) - nH(t) \left[Ax + (\eta - \theta)(a_1 + a_2) + \bar{k}_1 y + k_2 g \right] - n\varepsilon H(t) \\ (x - \beta y - e^{-\beta h} g) + r(m - l)M(t) + \frac{1}{2} \left[1 + \frac{\beta_2}{n} \right] l(1 - \rho^2) \sigma^2 M^2(t) - \frac{n l \alpha^2}{2(n + \beta_1)} - \alpha l \sigma \rho M(t) \\ + \inf_{q_1, q_2} \{f(q_1, q_2, t)\} = 0, \end{aligned} \quad (4.7)$$

where $f(q_1, q_2, t) = -n\theta(a_1q_1 + a_2q_2)H(t) + \frac{n(n+\beta_0)}{2}(\sigma_1^2q_1^2 + \sigma_2^2q_2^2 + 2\lambda\mu_1\mu_2q_1q_2)H^2(t)$.

For any $t \in [0, T]$, we get the derivative of $f(q_1, q_2, t)$ with respect to (w.r.t.) q_1 and q_2

$$\begin{aligned} \frac{\partial f}{\partial q_1} &= -n\theta a_1 H(t) + n(n + \beta_0)[\sigma_1^2 q_1 + \lambda\mu_1\mu_2 q_2]H^2(t), \quad \frac{\partial^2 f}{\partial q_1^2} = n(n + \beta_0)\sigma_1^2 H^2(t), \\ \frac{\partial f}{\partial q_2} &= -n\theta a_2 H(t) + n(n + \beta_0)[\sigma_2^2 q_2 + \lambda\mu_1\mu_2 q_1]H^2(t), \quad \frac{\partial^2 f}{\partial q_2^2} = n(n + \beta_0)\sigma_2^2 H^2(t), \\ \frac{\partial^2 f}{\partial q_1 \partial q_2} &= \frac{\partial^2 f}{\partial q_2 \partial q_1} = n(n + \beta_0)\lambda\mu_1\mu_2 H^2(t). \end{aligned} \tag{4.8}$$

So we have following Hessian matrix

$$\begin{bmatrix} \frac{\partial^2 f}{\partial q_1^2} & \frac{\partial^2 f}{\partial q_1 \partial q_2} \\ \frac{\partial^2 f}{\partial q_2 \partial q_1} & \frac{\partial^2 f}{\partial q_2^2} \end{bmatrix} = n(n + \beta_0)H^2(t)B, \quad B = \begin{bmatrix} \sigma_1^2 & \lambda\mu_1\mu_2 \\ \lambda\mu_1\mu_2 & \sigma_2^2 \end{bmatrix}. \tag{4.9}$$

By the Lemma 1 of Yuen et al.[13], B is a positive definite matrix, which means that the Hessian matrix is also a positive definite matrix. Now substituting $q^*(t) = (q_1^*(t), q_2^*(t))$ into (4.7) and separating the variables, we derive the following equations:

$$(x + \varepsilon y) H'(t) + [Ax + \bar{k}_1 y + k_2 g + \varepsilon(x - \beta y - e^{-\beta h} g)] H(t) = 0, \tag{4.10}$$

$$M'(t) - (k + \alpha\sigma\rho)M(t) + \frac{n+\beta_2}{2n}\sigma^2(1 - \rho^2)M^2(t) - \frac{n\alpha^2}{2(n+\beta_1)} = 0, \tag{4.11}$$

$$N'(t) + kmM(t) - n(\eta - \theta)(a_1 + a_2)H(t) + f(q_1^*, q_2^*, t) = 0, \tag{4.12}$$

with the boundary conditions $H(T) = 1, M(T) = 0, N(T) = 0$. Solving (4.10) and (4.11), we get

$$H(t) = e^{(A+\varepsilon)(T-t)}, \quad 0 \leq t \leq T, \tag{4.13}$$

$$M(t) = \frac{b_3[1 - e^{b_1(T-t)}]}{2b_1 + (b_1 + b_2)[e^{b_1(T-t)} - 1]}, \tag{4.14}$$

where $b_1 = \sqrt{(k + \alpha\sigma\rho)^2 + \frac{n+\beta_2}{n+\beta_1}\sigma^2\alpha^2(1 - \rho^2)}$, $b_2 = k + \alpha\sigma\rho$, $b_3 = \frac{n\alpha^2}{n+\beta_1}$.

The Hessian matrix (4.9) is a positive definite matrix, thus $f(q_1, q_2, t)$ is a convex function w.r.t. q_1 and q_2 . Therefore, using the first-order optimization conditions, we obtain

$$\hat{q}_1(t) = \frac{\theta D_1}{(n+\beta_0)e^{(A+\varepsilon)(T-t)} D_3}, \quad \hat{q}_2(t) = \frac{\theta D_2}{(n+\beta_0)e^{(A+\varepsilon)(T-t)} D_3}, \tag{4.15}$$

where $D_1 = a_1\sigma_1^2 - a_2\lambda\mu_1\mu_2$, $D_2 = a_2\sigma_2^2 - a_1\lambda\mu_1\mu_2$, $D_3 = \sigma_1^2\sigma_2^2 - \lambda^2\mu_1^2\mu_2^2$. Note that $D_1, D_2, D_3 > 0$, thus $\hat{q}_1(t) > 0, \hat{q}_2(t) > 0$.

Furthermore, let

$$t_1 = \begin{cases} T, & \theta D_1 \leq (n + \beta_0) D_3, \\ T - \frac{1}{A+\varepsilon} \ln \frac{\theta D_1}{(n+\beta_0)D_3}, & (n + \beta_0) D_3 < \theta D_1 < (n + \beta_0) D_3 e^{(A+\varepsilon)T}, \\ 0, & \theta D_1 \geq (n + \beta_0) D_3 e^{(A+\varepsilon)T}, \end{cases} \tag{4.16}$$

$$t_2 = \begin{cases} T, & \theta D_2 \leq (n + \beta_0) D_3, \\ T - \frac{1}{A+\varepsilon} \ln \frac{\theta D_2}{(n+\beta_0)D_3}, & (n + \beta_0) D_3 < \theta D_2 < (n + \beta_0) D_3 e^{(A+\varepsilon)T}, \\ 0, & \theta D_2 \geq (n + \beta_0) D_3 e^{(A+\varepsilon)T}, \end{cases} \tag{4.17}$$

We now discuss the optimal reinsurance strategy and $N(t)$ in different cases.

Case 1 $D_1 \leq D_2$, then $t_1 \geq t_2$.

(1) When $0 \leq t < t_2$, the optimal reinsurance proportion is $q^*(t) = (\hat{q}_1(t), \hat{q}_2(t))$, where $\hat{q}_1(t)$ and $\hat{q}_2(t)$ are given by (4.15). So we can obtain

$$N_1(t) = \int_t^T [-n(\eta - \theta)(a_1 + a_2)H(s) + kmM(s) + f(\hat{q}_1(s), \hat{q}_2(s))] ds + e_1, \quad (4.18)$$

(2) When $t \geq t_2$, we have $\hat{q}_2(t) \geq 1$, so $q_2^*(t) = 1$. Substituting $q_2^*(t) = 1$ into $\inf_{q_1, q_2} \{f(q_1, q_2, t)\}$ and by the first-order conditions we get

$$\bar{q}_1(t) = \frac{\theta a_1 - \lambda \mu_1 \mu_2 (n + \beta_0) e^{(A+\varepsilon)(T-t)}}{\sigma_1^2 + (n + \beta_0) e^{(A+\varepsilon)(T-t)}}. \quad (4.19)$$

Let

$$t'_1 = T - \frac{1}{(A+\varepsilon)} \ln \frac{\theta a_1}{(n + \beta_0) (\sigma_1^2 + \lambda \mu_1 \mu_2)}, \quad (4.20)$$

then for $t_2 \leq t < t'_1$, the optimal reinsurance strategy is $q^*(t) = (\bar{q}_1(t), 1)$. So,

$$N_2(t) = \int_t^T [-n(\eta - \theta)(a_1 + a_2)H(s) + kmM(s) + f(\bar{q}_1(s), 1)] ds + e_2, \quad (4.21)$$

(3) For $t'_1 \leq t \leq T$, it is easy to see that $q^*(t) = (1, 1)$. so we can obtain

$$N_3(t) = \int_t^T [-n(\eta - \theta)(a_1 + a_2)H(s) + kmM(s) + f(1, 1)] ds. \quad (4.22)$$

Case 2 $D_1 > D_2$, then $t_1 < t_2$.

(1) When $0 \leq t < t_1$, the optimal reinsurance proportion is $q^*(t) = (\hat{q}_1(t), \hat{q}_2(t))$, where $\hat{q}_1(t)$ and $\hat{q}_2(t)$ are given by (4.15). So we can obtain

$$N_4(t) = \int_t^T [-n(\eta - \theta)(a_1 + a_2)H(s) + kmM(s) + f(\hat{q}_1(s), \hat{q}_2(s))] ds + e_3, \quad (4.23)$$

(2) When $t \geq t_1$, we have $\hat{q}_1(t) \geq 1$, so $q_1^*(t) = 1$. Substituting $q_1^*(t) = 1$ into $\inf_{q_1, q_2} \{f(q_1, q_2, t)\}$ and by the first-order conditions we get

$$\bar{q}_2(t) = \frac{\theta a_2 - \lambda \mu_1 \mu_2 (n + \beta_0) e^{(A+\varepsilon)(T-t)}}{\sigma_2^2 + (n + \beta_0) e^{(A+\varepsilon)(T-t)}}. \quad (4.24)$$

Let

$$t'_2 = T - \frac{1}{(A+\varepsilon)} \ln \frac{\theta a_2}{(n + \beta_0) (\sigma_2^2 + \lambda \mu_1 \mu_2)}, \quad (4.25)$$

then for $t_1 \leq t < t'_2$, the optimal reinsurance strategy is $q^*(t) = (1, \bar{q}_2(t))$. So,

$$N_5(t) = \int_t^T [-n(\eta - \theta)(a_1 + a_2)H(s) + kmM(s) + f(1, \bar{q}_2(s))] ds + e_4, \quad (4.26)$$

(3) For $t'_2 \leq t \leq T$, it is easy to see that $q^*(t) = (1, 1)$, so we can obtain

$$N_3(t) = \int_t^T [-n(\eta - \theta)(a_1 + a_2)H(s) + kmM(s) + f(1, 1)] ds. \quad (4.27)$$

where $e_1 = \int_{t_2}^T [f(\bar{q}_1(s), 1) - f(\hat{q}_1(s), \hat{q}_2(s))] ds + e_2$, $e_2 = \int_{t'_1}^T [f(1, 1) - f(\bar{q}_1(s), 1)] ds$, $e_3 = \int_{t_1}^T [f(1, \bar{q}_2(s)) - f(\hat{q}_1(s), \hat{q}_2(s))] ds + e_4$ and $e_4 = \int_{t'_2}^T [f(1, 1) - f(1, \bar{q}_2(s))] ds$.

Summarizing the above results, we can obtain the following theorem.

Theorem 4.1 In the post-default case, for problem (3.8) with the exponential utility function (3.1), the robust optimal investment strategies are given by

$$\begin{aligned} \pi_1^*(t) &= \frac{1}{H(t)} \left[\frac{\alpha}{n+\beta_1} + \frac{\sigma\rho}{n} M(t) \right], \\ \pi_2^*(t) &= 0. \end{aligned}$$

The robust optimal reinsurance strategy and the optimal value function are given as follows:

(1) When $D_1 \leq D_2$, the robust optimal reinsurance strategies are

$$(q_1^*(t), q_2^*(t)) = \begin{cases} (\hat{q}_1(t), \hat{q}_2(t)), & 0 \leq t < t_2, \\ (\bar{q}_1(t), 1), & t_2 \leq t < t'_1, \\ (1, 1), & t'_1 \leq t < T, \end{cases}$$

with t_2 and t'_1 given in equations (4.17) and (4.20). The value function is given by

$$V(t, x, y, l, 1) = \begin{cases} -\frac{1}{n} \exp^{-n(x+\varepsilon y)H(t)+lM(t)+N_1(t)}, & 0 \leq t < t_2, \\ -\frac{1}{n} \exp^{-n(x+\varepsilon y)H(t)+lM(t)+N_2(t)}, & t_2 \leq t < t'_1, \\ -\frac{1}{n} \exp^{-n(x+\varepsilon y)H(t)+lM(t)+N_3(t)}, & t'_1 \leq t < T, \end{cases}$$

(2) When $D_1 > D_2$, the robust optimal reinsurance strategies are

$$(q_1^*(t), q_2^*(t)) = \begin{cases} (\hat{q}_1(t), \hat{q}_2(t)), & 0 \leq t < t_1, \\ (1, \bar{q}_2(t)), & t_1 \leq t < t'_2, \\ (1, 1), & t'_2 \leq t < T, \end{cases}$$

with t_1 and t'_2 given in equations (4.16) and (4.25). The value function is given by

$$V(t, x, y, l, 1) = \begin{cases} -\frac{1}{n} \exp^{-n(x+\varepsilon y)H(t)+lM(t)+N_4(t)}, & 0 \leq t < t_1, \\ -\frac{1}{n} \exp^{-n(x+\varepsilon y)H(t)+lM(t)+N_5(t)}, & t_1 \leq t < t'_2, \\ -\frac{1}{n} \exp^{-n(x+\varepsilon y)H(t)+lM(t)+N_3(t)}, & t'_2 \leq t < T, \end{cases}$$

with $H(T)$, $M(t)$ and $N_i(t)$, $i = 1, 2, \dots, 5$ given in equations (4.13), (4.14), (4.18), (4.21) – (4.23) and (4.26), respectively.

4.2 Pre-Default Case

In the pre-default case, i.e. $z = 0$. Then we can rewrite the HJB equation (3.10) as

$$\begin{aligned} \sup_{\pi \in \Pi} \inf_{\theta \in \Theta} \left\{ V_t + V_x \left[Ax + (\eta - \theta)(a_1 + a_2) + \theta[a_1 q_1 + a_2 q_2] + \alpha l \pi_1 + \bar{k}_1 y + k_2 g - \theta_1 \pi_1 \sqrt{l} \right. \right. \\ \left. \left. - \theta_0 \sqrt{\sigma_1^2 q_1^2 + \sigma_2^2 q_2^2 + 2\lambda \mu_1 \mu_2 q_1 q_2} \right] + V_y (x - \beta y - e^{-\beta h} g) + V_l \left[k(m - l) - \sigma \rho \theta_1 \sqrt{l} \right. \right. \\ \left. \left. - \sigma \theta_2 \sqrt{l} \sqrt{1 - \rho^2} \right] + \frac{1}{2} V_{xx} \left[\pi_1^2 l + \sigma_1^2 q_1^2 + \sigma_2^2 q_2^2 + 2\lambda \mu_1 \mu_2 q_1 q_2 \right] + \frac{1}{2} V_{ll} \sigma^2 l + V_{xl} \sigma \rho l \pi_1 \right. \\ \left. + \theta_3 h^P [V(t, x - \pi_2 \varsigma, y, l, 1) - V(t, x, y, l, 0)] - \frac{nV}{2\beta_0} \theta_0^2 - \frac{nV}{2\beta_1} \theta_1^2 - \frac{nV}{2\beta_2} \theta_2^2 \right. \\ \left. - \frac{nV}{\beta_3} h^P (\theta_3 \ln \theta_3 - \theta_3 + 1) \right\} = 0. \end{aligned} \quad (4.28)$$

We conjecture the value function

$$V(T, x, y, l, 0) = -\frac{1}{n} e^{-n(x+\varepsilon y)\bar{H}(t)+\bar{M}(t)l+\bar{N}(t)}. \quad (4.29)$$

The boundary condition $V(T, x, y, l, 0) = -\frac{1}{n} e^{-n(x+\varepsilon y)\bar{H}(t)+\bar{M}(t)l+\bar{N}(t)}$ implies that $\bar{H}(T) = 1, \bar{M}(T) = 0, \bar{N}(T) = 0$. Similarly to the former subsection, we obtain

$$\begin{aligned} \theta_0^*(t) &= \beta_0 \bar{H}(t) \sqrt{\sigma_1^2 q_1^2 + \sigma_2^2 q_2^2 + 2\lambda \mu_1 \mu_2 q_1 q_2}, \\ \theta_1^*(t) &= \beta_1 \bar{H}(t) \pi_1 \sqrt{l} - \frac{\beta_1}{n} \bar{M}(t) \sigma \rho \sqrt{l}, \\ \theta_2^*(t) &= -\frac{\beta_2}{n} \bar{M}(t) \sigma \sqrt{l} \sqrt{1 - \rho^2}, \\ \theta_3^*(t) &= e^{\frac{\beta_3}{n} [e^{n\varsigma H(t)\pi_2 - n(x+\varepsilon y)(H(t)-\bar{H}(t))+ (M(t)-\bar{M}(t))l + (N(t)-\bar{N}(t))} - 1]}], \end{aligned} \quad (4.30)$$

and the optimal investment strategies

$$\begin{aligned} \pi_1^*(t) &= \frac{1}{\bar{H}(t)} \left[\frac{\alpha}{n+\beta_1} + \frac{\sigma \rho}{n} \bar{M}(t) \right], \\ \pi_2^*(t) &= \frac{1}{n\varsigma \bar{H}(t)} \left[\ln \frac{\delta \bar{H}(t)}{\theta_3^* h^P \varsigma \bar{H}(t)} + n(x + \varepsilon y)(H(t) - \bar{H}(t)) - (M(t) - \bar{M}(t))l - (N(t) - \bar{N}(t)) \right]. \end{aligned} \quad (4.31)$$

Combining $\theta_3^*(t)$ with $\pi_2^*(t)$, we further have the following nonlinear equation for $\theta_3^*(t)$:

$$\frac{nh^P}{\beta_3} \theta_3^*(t) \ln \theta_3^*(t) + h^P \theta_3^*(t) - \frac{\delta \bar{H}(t)}{\varsigma \bar{H}(t)} = 0. \quad (4.32)$$

Lemma 1 Nonlinear equation $\frac{nh^P}{\beta_3} \theta_3^*(t) \ln \theta_3^*(t) + h^P \theta_3^*(t) - \frac{\delta \bar{H}(t)}{\varsigma \bar{H}(t)} = 0$ exists a unique positive root $\theta_3^*(t)$.

Proof The proof of this lemma is similar to lemma 1 in zhang and chen[6], so we omit it here.

Because the robust optimal reinsurance strategies are same as the post-default case, substituting $q^*(t), \pi_1^*(t)$ and $\pi_2^*(t)$ into (4.28) and separating the variables, we obtain

$$(x + \varepsilon y) \bar{H}'(t) + [Ax + \bar{k}_1 y + k_2 g + \varepsilon (x - \beta y - e^{-\beta h} g)] \bar{H}(t) = 0, \quad (4.33)$$

$$\bar{M}'(t) - (k + \alpha \sigma \rho) \bar{M}(t) + \frac{n+\beta_2}{2n} \sigma^2 (1 - \rho^2) \bar{M}^2(t) - \frac{n\alpha^2}{2(n+\beta_1)} - \frac{\delta \bar{H}(t)}{\varsigma \bar{H}(t)} (\bar{M}(t) - M(t)) = 0, \quad (4.34)$$

$$\begin{aligned} &\bar{N}'(t) + km\bar{M}(t) - n(\eta - \theta)(a_1 + a_2)\bar{H}(t) + \frac{\delta\bar{H}(t)}{\varsigma H(t)} - \theta_3^* h^P - \frac{nh^P}{\beta_3}(\theta_3^* \ln \theta_3^* - \theta_3^* + 1) \\ &- \frac{\delta\bar{H}(t)}{\varsigma H(t)} \left[\ln \frac{\delta\bar{H}(t)}{\theta_3^* h^P \varsigma H(t)} + n(x + \varepsilon y)(H(t) - \bar{H}(t)) \right] - \frac{\delta\bar{H}(t)}{\varsigma H(t)}(\bar{N}(t) - N(t)) + f(q_1^*, q_2^*, t) = 0, \end{aligned} \tag{4.35}$$

with the boundary conditions $\bar{H}(T) = 1, \bar{M}(T) = 0, \bar{N}(T) = 0$. Similar to (4.13), we can get

$$\bar{H}(t) = H(t) = e^{(A+\varepsilon)(T-t)}, \quad 0 \leq t \leq T. \tag{4.36}$$

Because the solution of nonlinear first order riccati differential equation is unique, we obtain

$$\bar{M}(t) = M(t) = \frac{b_3 [1 - e^{b_1(T-t)}]}{2b_1 + (b_1 + b_2) [e^{b_1(T-t)} - 1]}. \tag{4.37}$$

Next, let $G_i(t) = \bar{N}_i(t) - N_i(t)$, then we have $G_i'(t) = \bar{N}_i'(t) - N_i'(t) = \frac{\delta}{\varsigma} G_i(t) - \frac{\delta}{\varsigma} + \theta_3^* h^P + \frac{nh^P}{\beta_3}(\theta_3^* \ln \theta_3^* - \theta_3^* + 1) + \frac{\delta}{\varsigma} \ln \frac{\delta}{\theta_3^* h^P \varsigma}$, with the boundary conditions $G_i(T) = \bar{N}_i(T) - N_i(T) = 0$, we obtain

$$G_i(t) = P e^{-\frac{\delta}{\varsigma}(T-t)} - P, \tag{4.38}$$

$$\bar{N}_i(t) = N_i(t) + G_i(t) = N_i(t) + P e^{-\frac{\delta}{\varsigma}(T-t)} - P, \tag{4.39}$$

where $P = -\frac{\delta}{\varsigma} + \theta_3^* h^P + \frac{nh^P}{\beta_3}(\theta_3^* \ln \theta_3^* - \theta_3^* + 1) + \frac{\delta}{\varsigma} \ln \frac{\delta}{\theta_3^* h^P \varsigma}$.

In the same way, summarizing the above results, we obtain the following theorem.

Theorem 4.2 In the pre-default case, the robust optimal investment strategies are given by

$$\begin{aligned} \pi_1^*(t) &= \frac{1}{H(t)} \left[\frac{\alpha}{n+\beta_1} + \frac{\sigma\rho}{n} \bar{M}(t) \right], \\ \pi_2^*(t) &= \frac{1}{n\varsigma H(t)} \left[\ln \frac{\delta}{\theta_3^* h^P \varsigma} + G_i(t) \right]. \end{aligned}$$

The robust optimal reinsurance strategies are same as post-default case and the optimal value function is given in different situations.

(1) When $D_1 \leq D_2$, the value function is given by

$$V(t, x, y, l, 0) = \begin{cases} -\frac{1}{n} \exp^{-n(x+\varepsilon y)\bar{H}(t)+l\bar{M}(t)+\bar{N}_1(t)}, & 0 \leq t < t_2, \\ -\frac{1}{n} \exp^{-n(x+\varepsilon y)\bar{H}(t)+l\bar{M}(t)+\bar{N}_2(t)}, & t_2 \leq t < t'_1, \\ -\frac{1}{n} \exp^{-n(x+\varepsilon y)\bar{H}(t)+l\bar{M}(t)+\bar{N}_3(t)}, & t'_1 \leq t < T, \end{cases}$$

(2) When $D_1 > D_2$, the value function is given by

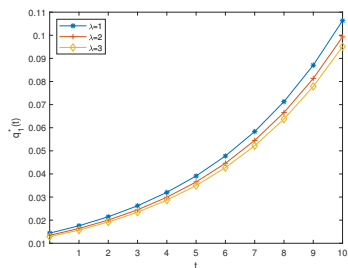
$$V(t, x, y, l, 0) = \begin{cases} -\frac{1}{n} \exp^{-n(x+\varepsilon y)\bar{H}(t)+l\bar{M}(t)+\bar{N}_4(t)}, & 0 \leq t < t_1, \\ -\frac{1}{n} \exp^{-n(x+\varepsilon y)\bar{H}(t)+l\bar{M}(t)+\bar{N}_5(t)}, & t_1 \leq t < t'_2, \\ -\frac{1}{n} \exp^{-n(x+\varepsilon y)\bar{H}(t)+l\bar{M}(t)+\bar{N}_3(t)}, & t'_2 \leq t < T, \end{cases}$$

where $H(t), \bar{H}(t), \bar{M}(t), G_i(t)$ and $\bar{N}_i(t)$ are given by equations (4.13) and (4.36)-(4.39), respectively.

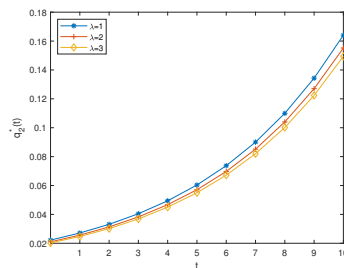
5. Numerical Analysis

In this section, we present some numerical examples to verify the theoretical results on the optimal strategy. Assuming that the claim sizes random variables X_i and Y_i are exponentially distributed with parameters α_1 and α_2 , respectively, then $\mu_1 = 1/\alpha_1$, $\mu_2 = 1/\alpha_2$, $\nu_1 = 1/\alpha_1^2$ and $\nu_2 = 1/\alpha_2^2$. The basic parameters are given by $\lambda = 1$, $\lambda_1 = 3$, $\lambda_2 = 4$, $\beta_0 = \beta_1 = \beta_2 = 1$, $\theta = 0.3$, $n = 1.5$, $A = 0.1$, $\varepsilon = 0.1$, $r = 0.04$, $\sigma = 1$, $\alpha = 2$, $\rho = 0.3$, $T = 10$, $t = 0$, $h^p = 0.003$ and $h^Q = 0.02$.

Based on the parameters we have set, it follows that $U_1 < U_2$ and $\theta U_2 < (n + \beta_0)U_3$ always hold, so we set $t_2 = T$ and the optimal reinsurance strategy $(q_1^*(t), q_2^*(t)) = (\hat{q}_1(t), \hat{q}_2(t))$. In Figure 1, the reinsurance proportion decreases with the increase of λ . It means that the higher degree of dependence of the two insurance businesses, the greater potential risks of the insurance will face. From Figure 2, we can see that the reinsurance proportion decreases with the ambiguity-averse level β_0 increasing.

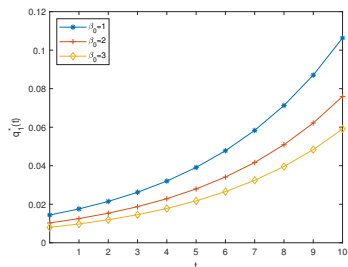


(a) The effect of λ on $q_1^*(t)$.

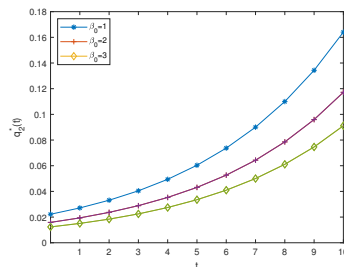


(b) The effect of λ on $q_2^*(t)$.

Figure 1 The effect of λ on $q^*(t)$.



(a) The effect of β_0 on $q_1^*(t)$.



(b) The effect of β_0 on $q_2^*(t)$.

Figure 2 The effect of β_0 on $q^*(t)$.

Next, we analyze the effect of some parameters on the $\pi_1^*(t)$ and $\pi_2^*(t)$. In Figure 3, we know that $\pi_1^*(t)$ decreases with the risk aversion coefficient n , which indicates that the insurer will pay less money to invest in stock market when n becomes larger. In Figure 4, with larger β_1 , the insurer has less confidence in the reference model, and hence the less risky asset the insurer wishes to invest. In fact, the parameter ρ is the correlation coefficient

between $W_1(t)$ and $\bar{W}_2(t)$. From Figure 5, with stronger positive correlation, the variance of the volatility becomes greater, which leads to that the AAI reduces the risk exposure. On the contrary, with stronger negative correlation, the AAI is willing to invest more wealth in the risky asset. In Figure 6, the wealth invested in the defaultable bond increases when the loss rate decreases. On the contrary, the insurer will invest more money in the defaultable bond with higher credit spread δ .

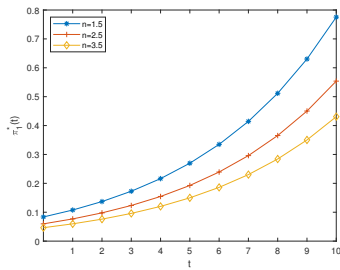


Figure 3 The effect of n on $\pi_1^*(t)$.

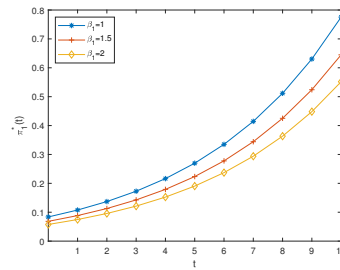


Figure 4 The effect of β_1 on $\pi_1^*(t)$.

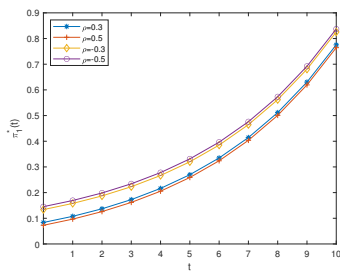


Figure 5 The effect of ρ on $\pi_1^*(t)$.

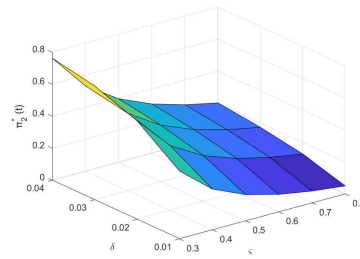


Figure 6 The effect of ζ and δ on $\pi_2^*(t)$.

References

- [1] Liang Xiaoqing, Liang Zhibin, Young V R. Optimal reinsurance under the mean-variance premium principle to minimize the probability of ruin[J]. Insurance Mathematics and Economics, 2020, 92(3): 128–146.
- [2] Guan Guohui, Liang Zongxia. Optimal reinsurance and investment strategies for insurer under interest rate and inflation risks[J]. Insurance Mathematics and Economics, 2014, 55(1): 105–115.
- [3] Wang Hao, Wang Rongming, Wei Jiaqin. Time-consistent investment-proportional reinsurance strategy with random coefficients for mean-variance insurers[J]. Insurance Mathematics and Economics, 2019, 85(1): 104–114.
- [4] Chen Zhiping, Yang Peng. Robust optimal reinsurance-investment strategy with price jumps and correlated claims[J]. Insurance Mathematics and Economics, 2020, 92(3): 27–46.
- [5] Wang Ning, Zhang Nan, Jin Zhuo, Qian Linyi. Reinsurance-investment game between two mean-variance insurers under model uncertainty[J]. Journal of Computational and Applied Mathematics, 2021, 382(1): 95–121.

- [6] Zhang Qiang, Chen Ping. Robust optimal proportional reinsurance and investment strategy for an insurer with defaultable risks and jumps[J]. Journal of Computational and Applied Mathematics, 2019, 356(1): 46–66.
- [7] Sun Jingyun, Yao Haixiang, Kang Zhilin. Robust optimal investment-reinsurance strategies for an insurer with multiple dependent risks[J]. Insurance Mathematics and Economics, 2019, 89(9): 157–170.
- [8] Yang Peng, Clementi F. Closed-loop equilibrium reinsurance-investment strategy with insider information and default risk[J]. Mathematical Problems in Engineering, 2021, 2021(2): 1–32.
- [9] Shen Yang, Zeng Yan. Optimal investment-reinsurance with delay for mean-variance insurers: A maximum principle approach[J]. Insurance Mathematics and Economics, 2014, 57(4): 1–12.
- [10] A Chunxiang, Li Zhongfei. Optimal investment and excess-of-loss reinsurance problem with delay for an insurer under Heston's SV model[J]. Insurance Mathematics and Economics, 2015, 61(1): 181–196.
- [11] Zhang Qiang, Chen Ping. Optimal reinsurance and investment strategy for an insurer in a model with delay and jumps[J]. Methodology and Computing in Applied Probability, 2020, 22(4): 777–801.
- [12] Bi Junna, Liang Zhibin, Xu Fangjun. Optimal mean-variance investment and reinsurance problems for the risk model with common shock dependence[J]. Insurance Mathematics and Economics, 2016, 70(6): 245–258.
- [13] Yuen K C, Liang Zhibin, Zhou Ming. Optimal proportional reinsurance with common shock dependence[J]. Insurance Mathematics and Economics, 2015, 64(4): 1–13.
- [14] Grandell J. Aspects of risk theory[M]. New York: Springer, 1991.
- [15] Liang Zhibin, Yuen K C. Optimal dynamic reinsurance with dependent risks: variance premium principle[J]. Scandinavian Actuarial Journal, 2016, 2016(1): 18–36.

相依风险模型下具有延迟和违约风险的鲁棒最优投资和再保险策略

慕蕊, 马世霞, 张欣茹

(河北工业大学理学院, 天津 300401)

摘要: 本文研究了在风险相依模型下具有延迟和违约风险的鲁棒最优投资再保险策略. 假设模糊厌恶型保险人的财富过程有两类相依的保险业务并且余额可以投资于无风险资产、可违约债券和价格过程遵循Heston模型的风险资产. 利用动态规划原则, 我们分别建立了违约后和违约前的鲁棒HJB方程. 另外, 通过最大化终端财富的期望指数效用, 我们得到了最优投资和再保险策略以及相应的值函数. 最后, 通过一些数值例子说明了某些模型参数对鲁棒最优策略的影响.

关键词: 鲁棒最优策略; Heston模型; 随机微分延迟方程; 相依风险; 违约风险

MR(2010)主题分类号: 62P05; 91B30; 91B70 中图分类号: O211.6; O29