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GRADIENT ESTIMATE FOR A NONLINEAR PARABOLIC EQUATION UNDER GEOMETRIC FLOW

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Abstract: In this paper, through the Li-Yau gradient estimate and Jun Sun' s research on the gradient estimate of heat equation under general geometric flow, we will derive local gradient estimates for positive solutions of a nonlinear parabolic equation on Riemannian manifold under general geometric flow. These results can be regarded as a generlization of Wang's results. At the same time, we give a corresponding Harnack inequality.

Keywords: gradient estimate; geometric flow; a nonlinear parabolic equation; Harnack inequality

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1 Introduction

Starting with the classical work of Li and Yau [1], Li and Yau proved the celebrated Li-Yau gradient bound for positive solutions of the heat equation and obtained the classical Harnack inequlity.

Let $(\mathbf{M}^n, g_{ij})$ be an n-dimensional complete Riemannian manifold. For positive solutions of the heat equation

$$\frac{\partial u}{\partial t} = \Delta u, \tag{1.1}$$

we suppose Ric$\geq -K$, where $K \geq 0$ and Ric is the Ricci curvature of $\mathbf{M}$. Then any positive solution of (1.1) satisfies

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} \leq \frac{n\alpha^2 K}{2(\alpha-1)} + \frac{n\alpha^2}{2t}, \quad \forall \alpha > 1.$$

In the special case where Ric $\geq$0, one has the optimal Li-Yau bound

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} \leq \frac{n\alpha^2}{2t}, \quad \forall \alpha > 1.$$

The gradient estimate is an important tool in the study of elliptic and parabolic type equations. Generalization of Li-Yau gradient bounds have been studied by many researchers.

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Hamilton [2] discovered a matrix Li-Yau type bound for the heat equation. A certain matrix Li-Yau bound under weaker conditions was subsequently established by Cao-Ni [3] on Kähler manifolds. In 2009, S. Liu [4] proved a gradient estimate for positive solutions of the heat equation along the Ricci flow. Sun [5] generalized Liu's results to a general geometric flow.

In recent years, there are more and more researches on the gradient estimates for positive solutions to some nonlinear parabolic equations.

For exemple, in 2009, Lu, Ni, Vázquez and Villani studied the porous medium equation (PME for short)

$$\partial_t u = \Delta u^m \tag{1.2}$$

on manifolds [6]. They got a local Aronson-Bénilan estimate for PME. Huang, Huang and Li in [7] generalized the results of Lu, Ni, Vázquez and Villani [6] on the PME and obtained Li-Yau type, Hamilton type and Li-Xu type gradient estimates. Wang, Xie and Zhou [8] had uniformly promoted these results to the Ricci flow.

In this paper, we focus on the gradient estimates of positive solutions to an extremely important nonlinear parabolic equation. This eqution was originated from Ma in [9], who considered a local gradient estimate of positive solutions for the following parabolic equation

$$\Delta u + au\log u + bu = 0 \quad \text{in} \quad M^n,$$

where a, b $\in \mathbb{R}$ are constants for complete noncompact manifolds with a fixed metric and curvature locally bounded below.

In [10], Yang generalized Ma's result [9] and derived a local gradient estimates for positive solutions to the equation

$$\frac{\partial u}{\partial t} = \Delta u + au\log u + bu,$$

where a, b $\in \mathbb{R}$ are constants for complete noncompact manifolds with a fixed metric and curvature locally bounded below.

Replacing u by $e^{b/a}u$, the equation becomes

$$u_t = \Delta u + au\log u. \tag{1.3}$$

In 2020, Wen Wang and P. Zhang in [11] investigated gradient estimates for positive solutions to (1.2) along Ricci flow. His results can be regarded as generalizations of the results of Li-Yau, J. Y. Li, Hamilton and Li-Xu to a more general nonlinear parabolic equation along the Ricci flow.

In this paper, we will follow closely [11] and derive local gradient estimates for positive solutions of (1.3) on Riemannian manifolds under general geometric flow. This can be regarded as a generlization of Wang' results. The general geometric flow equation is given as follows:

$$\frac{\partial g_{ij}}{\partial t} = 2h_{ij}, \tag{1.4}$$

where h_{ij} is a second-order symmetric tensor. The special case is the Ricci flow

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij},$$

which was introduced by Hamilton [12] in 1982.

To state our main rsult, we introduce three C^1 functions $\alpha(t)$, $\varphi(t)$ and $\gamma(t) : (0, +\infty) \to (0, +\infty)$. Suppose that three C^1 functions satisfy the following conditions:

(C1) $\alpha(t) > 1$.

(C2) $\alpha(t)$ and $\varphi(t)$ satisfy the following system

$$\begin{cases} \frac{2\varphi}{n} - 2\alpha K_1 \geqslant \left(\frac{2\varphi}{n} - \alpha'\right)\frac{1}{\alpha}, \\ \frac{2\varphi}{n} - \alpha' > 0, \\ \frac{\varphi^2}{n} + \alpha\varphi' \geqslant 0. \end{cases}$$

(C3) $\gamma(t)$ satisfies $\frac{\gamma'}{\gamma} - \left(\frac{2\varphi}{n} - \alpha'\right)\frac{1}{\alpha} \leqslant 0$.

(C4) $\gamma(t)$ is non-decreasing, and $\alpha(t)$ is also non-decreasing and bounded uniformly. Here $\alpha' = \frac{d\alpha}{dt}$, $\varphi' = \frac{d\varphi}{dt}$ and $\gamma' = \frac{d\gamma}{dt}$.

Detailed calculation of some specific functions $\alpha(t)$, $\varphi(t)$ and $\gamma(t)$ can be found in [11]. We state our results as follows.

Main Theorem 1 Let $(M^n, g(t))_{t\in[0,T]}$ be a smooth one parameter family of complete Riemannian manifolds evolving by (1.4) for t in some time interval [0, T]. Let M be complete under the initial metric $g(0)$. Suppose that there exist three functions $\alpha(t)$, $\varphi(t)$ and $\gamma(t)$ which satisfy the above conditions $(C1), (C2)(C3)$ and $(C4)$. Given $x_0 \in M$ and $R > 0$, let u be a positive solution to the equation (1.3) $\partial_t u = \Delta u + au\log u$ in the cube $B_{2R,T} := \{(x,t) \mid d(x, x_0, t) \leqslant 2R, 0 \leqslant t \leqslant T\}$, where a is a constant. Suppose that there exist constants $K_1, K_2, K_3, K_4 \geq 0$ such that

$$\mathrm{Ric} \geq -K_1 g, \quad -K_2 g \leq h \leq K_3 g, \quad |\nabla h| \leq K_4 \tag{1.5}$$

on $Q_{2R,T}$. Then for $(x,t) \in Q_{R,T}$, we have if $\frac{\gamma\alpha^4}{\alpha-1} \leqslant C_1$ for some constant C_1, then

$$\begin{aligned} \frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha a\log u \leqslant & C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K_1}}{R} + a + K_2\right) + \frac{n^2 C}{R^2\gamma} \\ & + \frac{3^{\frac{4}{3}} n\alpha^2 K_4^{\frac{2}{3}}}{4^{\frac{2}{3}}(\alpha-1)^{\frac{1}{3}}} + \frac{\sqrt{3}(K_1+K_2)n\alpha^2}{\alpha-1} \\ & + \alpha n^{\frac{1}{2}}(K_2+K_3) + a^{\frac{1}{2}} n^{\frac{1}{2}} \varphi^{\frac{1}{2}} + \alpha\varphi, \end{aligned}$$

where $C = C(n, C_1)$ is a constant. If $\frac{\gamma}{\alpha-1} \leqslant C_2$ for some constant C_2, then

$$\begin{aligned} \frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha a\log u \leqslant & C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K_1}}{R} + a + K_2\right) + \frac{n^2 C\alpha^4}{R^2\gamma} \\ & + \frac{3^{\frac{4}{3}} n\alpha^2 K_4^{\frac{2}{3}}}{4^{\frac{2}{3}}(\alpha-1)^{\frac{1}{3}}} + \frac{\sqrt{3}(K_1+K_2)n\alpha^2}{\alpha-1} \\ & + \alpha n^{\frac{1}{2}}(K_2+K_3) + a^{\frac{1}{2}} n^{\frac{1}{2}} \varphi^{\frac{1}{2}} + \alpha\varphi, \end{aligned}$$

where $C = C(n, C_2)$ is a constant.

Corollary 1.1 Let $(M^n, g(t))_{t\in[0,T]}$ be a smooth oneparameter family of complete Riemannian manifolds evolving by (1.4) for t in some time interval [0, T] . Let M be complete under the initial metric $g(0)$. Suppose that there exist constants $K_1, K_2, K_3, K_4 \geq 0$ such that $\mathrm{Ric} \geq -K_1 g, -K_2 g \leq h \leq K_3 g, |\nabla h| \leq K_4$, Given $x_0 \in M$ and $R > 0$, let u be a positive solution to the equation (1.3) in the cube $B_{2R,T} := \{(x,t) \mid d(x, x_0, t) \leqslant 2R, 0 \leqslant t \leqslant T\}$. Then the following special estimates are valid.

1. Li-Yau type

$$\alpha(t) = \text{constant}, \quad \varphi(t) = \frac{\alpha n}{t} + \frac{nK_1\alpha^2}{\alpha - 1}, \gamma(t) = t^\theta \quad \text{with} \quad 0 < \theta \leqslant 2.$$

$$\begin{aligned}\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha a \log u \leqslant & C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K_1}}{R} + \frac{\alpha^2}{\alpha-1}\frac{1}{R^2} + a + K_2\right) \\ & + \frac{3^{\frac{4}{3}} n\alpha^2 K_4^{\frac{2}{3}}}{4^{\frac{2}{3}}(\alpha-1)^{\frac{1}{3}}} + \frac{\sqrt{3}(K_1+K_2)n\alpha^2}{\alpha-1} \\ & + \alpha n^{\frac{1}{2}}(K_2+K_3) + a^{\frac{1}{2}} n^{\frac{1}{2}} \varphi^{\frac{1}{2}} + \alpha\varphi.\end{aligned}$$

2. Hamilton type

$$\alpha(t) = e^{2Kt}, \quad \varphi(t) = \frac{n}{t}e^{4K_1 t}, \quad \gamma(t) = te^{2K_1 t}.$$

$$\begin{aligned}\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha a \log u \leqslant & C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K_1}}{R} + a + K_2\right) \\ & + \frac{C\alpha^4}{R^2 t e^{2K_1 t}} + \frac{3^{\frac{4}{3}} n\alpha^2 K_4^{\frac{2}{3}}}{4^{\frac{2}{3}}(\alpha-1)^{\frac{1}{3}}} + \frac{\sqrt{3}(K_1+K_2)n\alpha^2}{\alpha-1} \\ & + \alpha n^{\frac{1}{2}}(K_2+K_3) + a^{\frac{1}{2}} n^{\frac{1}{2}} \varphi^{\frac{1}{2}} + \alpha\varphi.\end{aligned}$$

3. Li-Xu type

$$\alpha(t) = 1 + \frac{\sinh(K_1 t)\cosh(K_1 t) - K_1 t}{\sinh^2(K_1 t)}, \quad \varphi(t) = 2nK_1[1 + \coth(K_1 t)], \quad \gamma(t) = \tanh(K_1 t).$$

$$\begin{aligned}\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha a \log u \leqslant & C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K_1}}{R} + a + K_2\right) \\ & + \frac{C}{R^2\tanh(K_1 t)} + \frac{3^{\frac{4}{3}} n\alpha^2 K_4^{\frac{2}{3}}}{4^{\frac{2}{3}}(\alpha-1)^{\frac{1}{3}}} + \frac{\sqrt{3}(K_1+K_2)n\alpha^2}{\alpha-1} \\ & + \alpha n^{\frac{1}{2}}(K_2+K_3) + a^{\frac{1}{2}} n^{\frac{1}{2}} \varphi^{\frac{1}{2}} + \alpha\varphi.\end{aligned}$$

4. Linear Li-Xu type

$$\alpha(t) = 1 + 2K_1 t, \varphi(t) = \frac{n}{t} + nK_1(1 + 2K_1 t + \mu K_1 t), \gamma(t) = K_1 t \quad \text{with} \quad \mu \geqslant \frac{1}{4}.$$

$$\frac{|\nabla u|^2}{u^2}-\alpha\frac{u_t}{u}+\alpha a\log u \leqslant C\alpha^2\left(\frac{1}{R^2}+\frac{\sqrt{K_1}}{R}+a+K_2\right)$$
$$+\frac{C\alpha^4}{R^2K_1t}+\frac{3^{\frac{4}{3}}n\alpha^2K_4^{\frac{2}{3}}}{4^{\frac{2}{3}}(\alpha-1)^{\frac{1}{3}}}+\frac{\sqrt{3}(K_1+K_2)n\alpha^2}{\alpha-1}$$
$$+\alpha n^{\frac{1}{2}}\left(K_2+K_3\right)+a^{\frac{1}{2}}n^{\frac{1}{2}}\varphi^{\frac{1}{2}}+\alpha\varphi.$$

The local estimates above imply global estimates.

Corollary 1.2 Let $(M^n, g(0))$ be a complete noncompact Riemannian manifold without boundary and $g(t)$ evolving by (1.4). Suppose that there exist constants $K_1, K_2, K_3, K_4 \geq 0$ such that $\mathrm{Ric} \geq -K_1g, -K_2g \leq h \leq K_3g, |\nabla h| \leq K_4$. Let $u(x,t)$ be a positive solution to the equation (1.3) . For $(x,t) \in M^n \times (0,T]$, then

$$\frac{|\nabla u|^2}{u^2}-\alpha\frac{u_t}{u}+\alpha a\log u \leqslant C\alpha^2(K_1+a)+\frac{3^{\frac{4}{3}}n\alpha^2K_4^{\frac{2}{3}}}{4^{\frac{2}{3}}(\alpha-1)^{\frac{1}{3}}}+\frac{\sqrt{3}(K_1+K_2)n\alpha^2}{\alpha-1}$$
$$+\alpha n^{\frac{1}{2}}\left(K_2+K_3\right)+a^{\frac{1}{2}}n^{\frac{1}{2}}\varphi^{\frac{1}{2}}+\alpha\varphi.$$

As a consequence of the gradient estimate, we can obtain the following Harnack inequlity.

Corollary 1.3 Let $(M^n, g(0))$ be a complete noncompact Riemannian manifold without boundary or a closed Riemannian manifold. Assume $g(t)$ evolves by (1.4). Suppose that there exist constants $K_1, K_2, K_3, K_4 \geq 0$ such that $\mathrm{Ric} \geq -K_1g, -K_2g \leq h \leq K_3g, |\nabla h| \leq K_4$. Let $u(x,t)$ be a positive solution to the equation (1.3) . Then for all $(x_1,t_1) \in M^n\times(0,T)$ and $(x_2,t_2) \in M^n \times (0,T)$ such that $t_1 < t_2$, we have

$$u\left(x_2,t_2\right) \leqslant u\left(x_1,t_1\right)\times\exp\left(\int_0^1\frac{|\gamma'(s)|^4}{2\left(t_2-t_1\right)^2}ds+\int_{t_1}^{t_2}\frac{\alpha^2(t)}{32}dt+\int_{t_1}^{t_2}[Q+|a\log N|]dt\right), \tag{1.6}$$

where $N = \max_{M^n\times[0,T]} u$, and

$$Q=\frac{1}{\alpha(t)}\left[C\alpha^2(K_1+a)+\frac{3^{\frac{4}{3}}n\alpha^2K_4^{\frac{2}{3}}}{4^{\frac{2}{3}}(\alpha-1)^{\frac{1}{3}}}+\frac{\sqrt{3}(K_1+K_2)n\alpha^2}{\alpha-1}+\alpha n^{\frac{1}{2}}\left(K_2+K_3\right)+a^{\frac{1}{2}}n^{\frac{1}{2}}\varphi^{\frac{1}{2}}+\alpha\varphi\right].$$

2 Proof of Main Theorems

To prove Theorem 1, the following lemma is needed. Let $f = \log u$. Then

$$\left(\Delta-\partial_t\right)f=-|\nabla f|^2-af. \tag{2.1}$$

Let $F = |\nabla f|^2 - \alpha f_t + \alpha a f - \alpha\varphi$, where $\alpha = \alpha(t)$ and $\varphi = \varphi(t)$. Then

$$\Delta f=f_t-af-|\nabla f|^2=-\frac{F}{\alpha}-\left(\frac{\alpha-1}{\alpha}\right)|\nabla f|^2-\varphi. \tag{2.2}$$

Lemma 2.1 ([5], Lemma 3) Suppose the metric evolves by (2.1) . Then, for any smooth function f, we have $\frac{\partial}{\partial t}|\nabla f|^2 = -2h(\nabla f, \nabla f)+2\langle \nabla f, \nabla f_t\rangle$, and $\frac{\partial}{\partial t}\Delta f = \Delta \frac{\partial}{\partial t} f - 2\langle h, \nabla^2 f\rangle - 2\langle \operatorname{div} h - \frac{1}{2}\nabla(\operatorname{tr}_g h), \nabla f\rangle$. Here, div h is the divergence of h .

Lemma 2.2 Suppose $(M, g(t))$ satisfies the hypotheses of Main Theorem 1. We have

$$\begin{aligned}(\Delta-\partial_t) F \geqslant & \left|f_{ij}+\frac{\varphi}{n}\delta_{ij}\right|^2+\left(\frac{2\varphi}{n}-\alpha'\right)\frac{1}{\alpha}F-\alpha^2 n\left(K_2+K_3\right)^2-3\sqrt{n}\alpha K_4|\nabla f| \\ & -2(K_1+K_2)|\nabla f|^2-2\nabla f\nabla F+2a(\alpha-1)|\nabla f|^2+a\alpha\Delta f.\end{aligned} \tag{2.3}$$

Proof We calculate directly by using the Lemma 2.1.

$$\begin{aligned}&\Delta F\\ =&\Delta|\nabla f|^2-\alpha\Delta\left(f_t\right)+\alpha a\Delta f\\ =&2\left|f_{ij}\right|^2+2f_j f_{iij}+2R_{ij}f_i f_j-\alpha(\Delta f)_t-2\alpha\left\langle h,\nabla^2 f\right\rangle-2\alpha\langle\operatorname{div} h-\frac{1}{2}\nabla\left(\operatorname{tr}_g h\right),\nabla f\rangle+\alpha a\Delta f\\ =&2\left|f_{ij}\right|^2+2f_j f_{iij}+2R_{ij}f_i f_j-\alpha(f_t-af-|\nabla f|^2)_t-2\alpha\left\langle h,\nabla^2 f\right\rangle\\ &-2\alpha\langle\operatorname{div} h-\frac{1}{2}\nabla\left(\operatorname{tr}_g h\right),\nabla f\rangle+\alpha a\Delta f,\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\partial_t F=&\left(|\nabla f|^2\right)_t-\alpha f_{tt}-\alpha' f_t+\alpha' af+\alpha af_t-\alpha\varphi'-\alpha'\varphi\\ =&2\nabla f\nabla\left(f_t\right)-2h_{ij}f_i f_j-\alpha f_{tt}-\alpha' f_t+\alpha' af+\alpha af_t-\alpha\varphi'-\alpha'\varphi.\end{aligned} \tag{2.5}$$

We follow that from (2.4) and (2.5)

$$\begin{aligned}(\Delta-\partial_t) F=&2\left|f_{ij}\right|^2+2f_j f_{iij}+2R_{ij}f_i f_j-\alpha(f_t-af-|\nabla f|^2)_t-2\alpha\left\langle h,\nabla^2 f\right\rangle\\ &-2\alpha\left\langle\operatorname{div} h-\frac{1}{2}\nabla\left(\operatorname{tr}_g h\right),\nabla f\right\rangle+\alpha a\Delta f+2h_{ij}f_i f_j\\ &-2\nabla f\nabla\left(f_t\right)+\alpha f_{tt}+\alpha' f_t-\alpha' af-\alpha af_t+\alpha\varphi'+\alpha'\varphi\\ =&2\left|f_{ij}\right|^2+2f_j f_{iij}+2R_{ij}f_i f_j-\alpha(|\nabla f|^2)_t-2\alpha\left\langle h,\nabla^2 f\right\rangle\\ &-2\alpha\left\langle\operatorname{div} h-\frac{1}{2}\nabla\left(\operatorname{tr}_g h\right),\nabla f\right\rangle+\alpha a\Delta f+2h_{ij}f_i f_j\\ &-2\nabla f\nabla\left(f_t\right)+\alpha' f_t-\alpha' af+\alpha\varphi'+\alpha'\varphi\\ =&2\left|f_{ij}\right|^2+-2\alpha\left\langle h,\nabla^2 f\right\rangle-2\alpha\left\langle\operatorname{div} h-\frac{1}{2}\nabla\left(\operatorname{tr}_g h\right),\nabla f\right\rangle+\alpha a\Delta f\\ &+2R_{ij}f_i f_j+2h_{ij}f_i f_j-2\alpha h_{ij}f_i f_j+2\nabla f\nabla(\Delta f)+2\alpha\nabla f\nabla\left(f_t\right)\\ &-2\nabla f\nabla\left(f_t\right)+\alpha' f_t-\alpha' af+\alpha\varphi'+\alpha'\varphi\\ =&2\left|f_{ij}\right|^2+-2\alpha\left\langle h,\nabla^2 f\right\rangle-2\alpha\left\langle\operatorname{div} h-\frac{1}{2}\nabla\left(\operatorname{tr}_g h\right),\nabla f\right\rangle+\alpha a\Delta f\\ &+2R_{ij}f_i f_j+2h_{ij}f_i f_j-2\alpha h_{ij}f_i f_j-2\nabla f\nabla F+2a(\alpha-1)|\nabla f|^2\\ &\alpha' f_t-\alpha' af+\alpha\varphi'+\alpha'\varphi.\end{aligned} \tag{2.6}$$

Our assumption implies that

$$|h|^2 \leq (K_2+K_3)^2\,|g|^2 = n\,(K_2+K_3)^2\,.$$

Applying those bounds and Young's inequality yields

$$\left|\alpha\left\langle h,\nabla^2 f\right\rangle\right| \leq \frac{1}{2}\left|\nabla^2 f\right|^2 + \frac{1}{2}\alpha^2|h|^2 \leq \frac{1}{2}\left|\nabla^2 f\right|^2 + \frac{1}{2}\alpha^2 n\,(K_2+K_3)^2\,.$$

On the other hand,

$$\left|\operatorname{div} h - \frac{1}{2}\nabla\left(\operatorname{tr}_g h\right)\right| = \left|g^{ij}\nabla_i h_{jl} - \frac{1}{2}g^{ij}\nabla_l h_{ij}\right| \leq \frac{3}{2}|g||\nabla h| \leq \frac{3}{2}\sqrt{n}K_4.$$

Therefore, we arrive at

$$\begin{aligned}(\Delta-\partial_t)\,F \geqslant{}& |f_{ij}|^2 - \alpha^2 n\,(K_2+K_3)^2 - 3\sqrt{n}\alpha K_4\nabla f + \alpha a\Delta f\\ &+ 2R_{ij}f_if_j + 2h_{ij}f_if_j - 2\alpha h_{ij}f_if_j - 2\nabla f\nabla F + 2a(\alpha-1)|\nabla f|^2\\ &\alpha' f_t - \alpha' af + \alpha\varphi' + \alpha'\varphi.\end{aligned} \tag{2.7}$$

Further, by utilizing the unit matrix $(\delta_{ij})_{n\times n}$ and (2.7), we obtain

$$\begin{aligned}(\Delta-\partial_t)\,F ={}& \left|f_{ij}+\frac{\varphi}{n}\delta_{ij}\right|^2 + \left(\frac{2\varphi}{n}-2\alpha K_2\right)|\nabla f|^2 - \left(\frac{2\varphi}{n}-\alpha'\right)f_t - \left(\frac{2\varphi}{n}-\alpha'\right)af\\ &- \alpha^2 n\,(K_2+K_3)^2 - 3\sqrt{n}\alpha K_4|\nabla f| + 2R_{ij}f_if_j + 2h_{ij}f_if_j - 2\nabla f\nabla F\\ &+ 2a(\alpha-1)|\nabla f|^2 + a\alpha\Delta f + \frac{\varphi^2}{n}\\ ={}& \left|f_{ij}+\frac{\varphi}{n}\delta_{ij}\right|^2 + \left(\frac{2\varphi}{n}-\alpha'\right)|\nabla f|^2 - \left(\frac{2\varphi}{n}-\alpha'\right)f_t - \left(\frac{2\varphi}{n}-\alpha'\right)af\\ &- \alpha^2 n\,(K_2+K_3)^2 - 3\sqrt{n}\alpha K_4|\nabla f| + 2R_{ij}f_if_j + 2h_{ij}f_if_j - 2\nabla f\nabla F\\ &+ 2a(\alpha-1)|\nabla f|^2 + a\alpha\Delta f\\ \geqslant{}& \left|f_{ij}+\frac{\varphi}{n}\delta_{ij}\right|^2 + \left(\frac{2\varphi}{n}-\alpha'\right)\frac{1}{\alpha}F - \alpha^2 n\,(K_2+K_3)^2 - 3\sqrt{n}\alpha K_4|\nabla f|\\ &- 2(K_1+K_2)|\nabla f|^2 - 2\nabla f\nabla F + 2a(\alpha-1)|\nabla f|^2 + a\alpha\Delta f.\end{aligned}$$

This finishes the proof of the lemma.

Proof of the Main Theorem 1 Let $G=\gamma(t)F$ and $\gamma(t)>0$ be non-decreasing. Then

$$\begin{aligned}(\Delta-\partial_t)\,G ={}& \gamma\,(\Delta-\partial_t)\,F - \gamma' F\\ \geqslant{}& \gamma\left|f_{ij}+\frac{\varphi}{n}g_{ij}\right|^2 + \left(\frac{2\varphi}{n}-\alpha'\right)\frac{1}{\alpha}G - \gamma\alpha^2 n\,(K_2+K_3)^2 - 3\gamma\sqrt{n}\alpha K_4|\nabla f|\\ &- 2\gamma(K_1+K_2)|\nabla f|^2 - 2\nabla f\nabla G + 2a\gamma(\alpha-1)|\nabla f|^2 + a\gamma\alpha\Delta f - \gamma' F\\ ={}& \gamma\left|f_{ij}+\frac{\varphi}{n}g_{ij}\right|^2 + \left[\left(\frac{2\varphi}{n}-\alpha'\right)\frac{1}{\alpha} - \frac{\gamma'}{\gamma}\right]G - \gamma\alpha^2 n\,(K_2+K_3)^2 - 3\gamma\sqrt{n}\alpha K_4|\nabla f|\\ &- 2\gamma(K_1+K_2)|\nabla f|^2 - 2\nabla f\nabla G + 2a\gamma(\alpha-1)|\nabla f|^2 + a\gamma\alpha\Delta f.\end{aligned} \tag{2.8}$$

By our assumption of the bounds of h and the evolution of the metric, we know that $g(t)$ is uniformly equivalent to the initial metric $g(0)$, that is,

$$e^{-2K_2T}g(0) \le g(t) \le e^{2K_3T}g(0).$$

Thus we know that $(M, g(t))$ is also complete for $t \in [0,T]$. Now let $\psi(r)$ be a C^2 function on $[0,+\infty)$ such that

$$\psi(r) = \begin{cases} 1 & \text{if } r \in [0,1], \\ 0 & \text{if } r \in [2,+\infty), \end{cases}$$

$$0 \le \psi(r) \le 1, \quad \psi'(r) \le 0, \quad \psi''(r) \ge -C, \quad \frac{|\psi'(r)|^2}{\psi(r)} \le C,$$

where C is an absolute constant. Define

$$\phi(x,t) = \phi\left(d\left(x,x_0,t\right)\right) = \psi\left(\frac{d\left(x,x_0,t\right)}{R}\right) = \psi\left(\frac{\rho(x,t)}{R}\right),$$

where $\rho(x,t) = d\left(x,x_0,t\right)$. For the purpose of applying the maximum principle, the argument of [Calabi 1958] allows us to assume that the function $\phi(x,t)$, with support in $Q_{2R,T}$, is C^2 at the maximum point.

For any $0 < T_1 \le T$, let (x_1,t_1) be the point in Q_{2R,T_1}, at which ϕG achieves its maximum value. We can assume that this value is positive, because in the other case the proof is trivial. As $G(x,0)=0$, we know that $t_1 > 0$. Then at the point (x_1,t_1), we have

$$\nabla(\phi G) = F\nabla\phi + \phi\nabla G = 0, \quad \Delta(\phi G) \le 0, \quad \frac{\partial}{\partial t}(\phi G) \ge 0. \tag{2.9}$$

Therefore,

$$0 \ge \left(\Delta - \partial_t\right)(\phi G) = (\Delta\phi)G - \phi_t G + \phi\left(\Delta - \partial_t\right)G + 2\nabla\phi\cdot\nabla G. \tag{2.10}$$

Using the Laplacian comparison theorem, we have

$$\Delta\phi = \psi'\frac{\Delta\rho}{R} + \psi''\frac{|\nabla\rho|^2}{R^2} \ge -\frac{C}{R^2} - \frac{C}{R}\sqrt{K_1}. \tag{2.11}$$

Furthermore, we have

$$\frac{|\nabla\phi|^2}{\phi} = \frac{\left(\psi'\right)^2}{\psi}\frac{|\nabla\rho|^2}{R^2} \le \frac{C}{R^2}. \tag{2.12}$$

By our assumption, $G\left(x_1,t_1\right) > 0$. By the evolution formula of the geodesic length under geometric flow [Hamilton 1995a], we calculate at the point (x_1,t_1)

$$\begin{aligned} -\phi_t G &= -\psi'\left(\frac{\rho}{R}\right)\frac{1}{R}\frac{d\rho}{dt}G = -\psi'\left(\frac{\rho}{R}\right)\frac{1}{R}\int_{\gamma_{t_1}} h(S,S)dsG \\ &\ge \psi'\left(\frac{\rho}{R}\right)\frac{1}{R}K_2\rho G \ge -\sqrt{C}K_2 G, \end{aligned} \tag{2.13}$$

where γ_{t_1} is the geodesic connecting x and x_0 under the metric $g(t_1)$, S is the unite tangent vector to γ_{t_1} and ds is the element of arc length.

All the following computations are at the point (x_1, t_1). We have

$$\begin{aligned}\left|f_{ij}+\frac{\varphi}{n}\delta_{ij}\right|^2 &\geqslant \frac{1}{n}\left(\operatorname{tr}\left|f_{ij}+\frac{\varphi}{n}\delta_{ij}\right|\right)^2=\frac{1}{n}(\Delta f+\varphi)^2=\frac{1}{n}\left[-\frac{1}{\alpha}F-\frac{1}{\alpha}(\alpha-1)|\nabla f|^2\right]^2\\ &=\frac{1}{\alpha^2 n}\left[\frac{G}{\gamma}+(\alpha-1)|\nabla f|^2\right]^2,\end{aligned}\tag{2.14}$$

and

$$\Delta f=f_t-|\nabla f|^2-af=-\frac{F}{\alpha}-\frac{\alpha-1}{\alpha}|\nabla f|^2-\varphi<0.\tag{2.15}$$

Substituting the above inequality (2.15) into (2.8), we obtain

$$\begin{aligned}(\Delta-\partial_t)\,G \geqslant&\gamma\left|f_{ij}+\frac{\varphi}{n}\delta_{ij}\right|^2+\left[\left(\frac{2\varphi}{n}-\alpha'\right)\frac{1}{\alpha}-\frac{\gamma'}{\gamma}\right]G\\ &-\gamma\alpha^2 n\,(K_2+K_3)^2-3\gamma\sqrt{n}\alpha K_4|\nabla f|-2\gamma(K_1+K_2)|\nabla f|^2+2a\gamma(\alpha-1)|\nabla f|^2\\ &-2\nabla f\nabla G-aG-a\gamma(\alpha-1)|\nabla f|^2-a\gamma\varphi\\ =&\gamma\left|f_{ij}+\frac{\varphi}{n}\delta_{ij}\right|^2+\left[\left(\frac{2\varphi}{n}-\alpha'\right)\frac{1}{\alpha}-\frac{\gamma'}{\gamma}\right]G\\ &-\gamma\alpha^2 n\,(K_2+K_3)^2-3\gamma\sqrt{n}\alpha K_4|\nabla f|-2\gamma(K_1+K_2)|\nabla f|^2+a\gamma(\alpha-1)|\nabla f|^2\\ &-2\nabla f\nabla G-aG-a\gamma\varphi.\end{aligned}$$

Using (2.10),(2.13) and (2.14), we infer

$$\begin{aligned}0\geqslant&(\Delta-\partial_t)\,(\phi G)\\ =&G\left(\Delta\phi-2\frac{|\nabla\phi|^2}{\phi}\right)+\phi\,(\Delta-\partial_t)\,G-G\phi_t\\ \geqslant&G\left(\Delta\phi-2\frac{|\nabla\phi|^2}{\phi}\right)+\frac{\phi\gamma}{\alpha^2 n}\left[\frac{G}{\gamma}+(\alpha-1)|\nabla f|^2\right]^2+\left[\left(\frac{2\varphi}{n}-\alpha'\right)\frac{1}{\alpha}-\frac{\gamma'}{\gamma}\right]\phi G\\ &-\gamma\phi\alpha^2 n\,(K_2+K_3)^2-3\gamma\phi\sqrt{n}\alpha K_4|\nabla f|-2\gamma\phi(K_1+K_2)|\nabla f|^2-2\phi\nabla f\nabla G\\ &-a\phi G+a\gamma\phi(\alpha-1)|\nabla f|^2-a\gamma\phi\varphi-G\sqrt{C}K_2.\end{aligned}$$

Multiplying with ϕ, we have

$$\begin{aligned}0\geqslant&\phi G\left[\Delta\phi-2\frac{|\nabla\phi|^2}{\phi}+\phi\left(\frac{2\varphi}{n}-\alpha'\right)\frac{1}{\alpha}-\phi\frac{\gamma}{\gamma}\right]+\frac{\phi^2\gamma}{\alpha^2 n}\left[\frac{G}{\gamma}+(\alpha-1)|\nabla f|^2\right]^2\\ &-\gamma\phi^2\alpha^2 n\,(K_2+K_3)^2-3\gamma\phi^2\sqrt{n}\alpha K_4|\nabla f|-2\gamma\phi^2(K_1+K_2)|\nabla f|^2-2\phi^2\nabla f\nabla G\\ &-a\phi^2 G+a\gamma\phi^2(\alpha-1)|\nabla f|^2-a\gamma\phi^2\varphi-G\phi\sqrt{C}K_2\\ \geqslant&\phi G\left[\Delta\phi-2\frac{|\nabla\phi|^2}{\phi}+\phi\left(\frac{2\varphi}{n}-\alpha'\right)\frac{1}{\alpha}-\phi\frac{\gamma'}{\gamma}\right]+\frac{\phi^2 G^2}{\alpha^2 n\gamma}+\frac{\phi^2(\alpha-1)^2\gamma}{n\alpha^2}|\nabla f|^4\\ &+\frac{2\phi^2(\alpha-1)}{n\alpha^2}G|\nabla f|^2-\gamma\phi^2\alpha^2 n\,(K_2+K_3)^2-3\gamma\phi^2\sqrt{n}\alpha K_4|\nabla f|-2\gamma\phi^2(K_1+K_2)|\nabla f|^2\\ &-2\phi^2\nabla f\nabla G-a\phi^2 G+a\gamma\phi^2(\alpha-1)|\nabla f|^2-a\gamma\phi^2\varphi-G\phi\sqrt{C}K_2.\end{aligned}$$

Young's inequality yields

$$\frac{2\phi^2(\alpha-1)}{n\alpha^2}G|\nabla f|^2+2\phi G\nabla\phi\nabla f\geqslant-\frac{n\alpha^2}{2(\alpha-1)}\frac{|\nabla\phi|^2}{\phi}\phi G, \tag{2.16}$$

$$\frac{\gamma\phi^2(\alpha-1)^2}{3n\alpha^2}|\nabla f|^4+a\gamma\phi^2(\alpha-1)|\nabla f|^2\geqslant-\frac{3\gamma\phi^2a^2n\alpha^2}{4}, \tag{2.17}$$

$$\frac{\gamma\phi^2(\alpha-1)^2}{3n\alpha^2}|\nabla f|^4-3\gamma\phi^2\sqrt{n}\alpha K_4|\nabla f|\geqslant-\frac{3^{\frac{8}{3}}\gamma\phi^2n\alpha^2K_4^{\frac{4}{3}}}{4^{\frac{4}{3}}(\alpha-1)^{\frac{2}{3}}}, \tag{2.18}$$

$$\frac{\gamma\phi^2(\alpha-1)^2}{3n\alpha^2}|\nabla f|^4-2\gamma\phi^2(K_1+K_2)|\nabla f|^2\geqslant-\frac{3\gamma\phi^2(K_1+K_2)^2n\alpha^2}{(\alpha-1)^2}. \tag{2.19}$$

Therefore, with the help of the inequalities (2.16)–(2.19) we arrive at

$$\begin{aligned}
0\geqslant&\phi G\left[\Delta\phi-2\frac{|\nabla\phi|^2}{\phi}+\phi\left(\frac{2\varphi}{n}-\alpha'\right)\frac{1}{\alpha}-\phi\frac{\gamma'}{\gamma}-a\phi-\sqrt{C}K_2\right]\\
&+\frac{\phi^2G^2}{\alpha^2n\gamma}-\frac{n\alpha^2}{2(\alpha-1)}\frac{|\nabla\phi|^2}{\phi}\phi G-\frac{3\gamma\phi^2a^2n\alpha^2}{4}-\frac{3^{\frac{8}{3}}\gamma\phi^2n\alpha^2K_4^{\frac{4}{3}}}{4^{\frac{4}{3}}(\alpha-1)^{\frac{2}{3}}}-\frac{3\gamma\phi^2(K_1+K_2)^2n\alpha^2}{(\alpha-1)^2}\\
&-\gamma\phi^2\alpha^2n\left(K_2+K_3\right)^2-a\gamma\phi^2\varphi\\
\geqslant&\phi G\left[\Delta\phi-2\frac{|\nabla\phi|^2}{\phi}+\phi\left(\frac{2\varphi}{n}-\alpha'\right)\frac{1}{\alpha}-\phi\frac{\gamma'}{\gamma}-\frac{n\alpha^2}{2(\alpha-1)}\frac{|\nabla\phi|^2}{\phi}-a\phi-\sqrt{C}K_2\right]\\
&+\frac{\phi^2G^2}{\alpha^2n\gamma}-\frac{3\gamma\phi^2a^2n\alpha^2}{4}-\frac{3^{\frac{8}{3}}\gamma\phi^2n\alpha^2K_4^{\frac{4}{3}}}{4^{\frac{4}{3}}(\alpha-1)^{\frac{2}{3}}}-\frac{3\gamma\phi^2(K_1+K_2)^2n\alpha^2}{(\alpha-1)^2}\\
&-\gamma\phi^2\alpha^2n\left(K_2+K_3\right)^2-a\gamma\phi^2\varphi\\
\geqslant&\left[-\frac{C}{R^2}(1+\sqrt{K_1}R)-\frac{2C}{R^2}+\phi\left(\frac{2\varphi}{n}-\alpha'\right)\frac{1}{\alpha}-\phi\frac{\gamma'}{\gamma}-\frac{n\alpha^2}{2(\alpha-1)}\frac{C}{R^2}-\sqrt{C}K_2\right]\phi G\\
&+\frac{\phi^2G^2}{\alpha^2n\gamma}-\frac{3\gamma\phi^2a^2n\alpha^2}{4}-\frac{3^{\frac{8}{3}}\gamma\phi^2n\alpha^2K_4^{\frac{4}{3}}}{4^{\frac{4}{3}}(\alpha-1)^{\frac{2}{3}}}-\frac{3\gamma\phi^2(K_1+K_2)^2n\alpha^2}{(\alpha-1)^2}\\
&-\gamma\phi^2\alpha^2n\left(K_2+K_3\right)^2-a\gamma\phi^2\varphi.
\end{aligned}$$

For the inequality $Ax^2-2Bx\leqslant C$, one has $x\leqslant\frac{2B}{A}+\left(\frac{C}{A}\right)^{\frac{1}{2}}$, where $A,B,C>0$

$$\begin{aligned}
\phi G\left(x,T_1\right)\leqslant&(\phi G)\left(x_1,t_1\right)\\
\leqslant&\Bigg\{n\gamma\alpha^2\left[\frac{C}{R^2}(1+\sqrt{K_1}R)+\frac{n\alpha^2}{2(\alpha-1)}\frac{C}{R^2}+a\phi+\sqrt{C}K_2\right]\\
&+n\gamma\alpha^2\phi\left[\frac{\gamma'}{\gamma}-\left(\frac{2\varphi}{n}-\frac{\alpha'}{\alpha}\right)\frac{1}{\alpha}\right]+\frac{\sqrt{3}\gamma\phi an\alpha^2}{2}+\frac{3^{\frac{4}{3}}\gamma\phi n\alpha^2K_4^{\frac{2}{3}}}{4^{\frac{2}{3}}(\alpha-1)^{\frac{1}{3}}}\\
&+\frac{\sqrt{3}\gamma\phi(K_1+K_2)n\alpha^2}{\alpha-1}+\gamma\phi\alpha n^{\frac{1}{2}}\left(K_2+K_3\right)+a^{\frac{1}{2}}n^{\frac{1}{2}}\gamma\phi\varphi^{\frac{1}{2}}\Bigg\}\left(x_1,t_1\right).
\end{aligned}$$

If γ is nondecreasing which satisfies the system

$$\begin{cases} \dfrac{\gamma'}{\gamma}-\left(\dfrac{2\varphi}{n}-\alpha'\right)\dfrac{1}{\alpha}\leqslant 0 \\ \qquad\qquad\qquad \dfrac{\gamma\alpha^4}{\alpha-1}\leqslant C_1 \end{cases}$$

Recalling that $\alpha(t)$ and $\gamma(t)$ are non-decreasing and $t_1<T_1$ we have

$$\begin{aligned}
\phi G\left(x,T_1\right)\leqslant&(\phi G)\left(x_1,t_1\right)\\
\leqslant& n\gamma\left(T_1\right)\alpha^2\left(T_1\right)\left[\frac{C}{R^2}(1+\sqrt{K_1}R)+a\phi+\sqrt{C}K_2\right]+\frac{n^2C}{R^2}+\frac{\sqrt{3}\gamma\left(T_1\right)\phi an\alpha^2\left(T_1\right)}{2}\\
&+\frac{3^{\frac{4}{3}}\gamma\left(T_1\right)\phi n\alpha^2\left(T_1\right)K_4^{\frac{2}{3}}}{4^{\frac{2}{3}}(\alpha-1)^{\frac{1}{3}}}+\frac{\sqrt{3}\gamma\left(T_1\right)\phi(K_1+K_2)n\alpha^2\left(T_1\right)}{\alpha-1}\\
&+\gamma\left(T_1\right)\phi\alpha\left(T_1\right)n^{\frac{1}{2}}\left(K_2+K_3\right)+a^{\frac{1}{2}}n^{\frac{1}{2}}\gamma\left(T_1\right)\phi\varphi^{\frac{1}{2}}\left(T_1\right).
\end{aligned}$$

Hence, we have for $\phi\equiv 1$ on $B_{R,T}$

$$\begin{aligned}
\sup_{B_R}F\left(x,T_1\right)\leqslant& n\alpha^2\left(T_1\right)\left[\frac{C}{R^2}(1+\sqrt{K_1}R)+a+\sqrt{C}K_2\right]+\frac{n^2C}{R^2\gamma\left(T_1\right)}\\
&+\frac{\sqrt{3}an\alpha^2\left(T_1\right)}{2}+\frac{3^{\frac{4}{3}}n\alpha^2\left(T_1\right)K_4^{\frac{2}{3}}}{4^{\frac{2}{3}}(\alpha-1)^{\frac{1}{3}}}+\frac{\sqrt{3}(K_1+K_2)n\alpha^2\left(T_1\right)}{\alpha-1}\\
&+\alpha\left(T_1\right)n^{\frac{1}{2}}\left(K_2+K_3\right)+a^{\frac{1}{2}}n^{\frac{1}{2}}\varphi^{\frac{1}{2}}\left(T_1\right).
\end{aligned}$$

If γ is nondecreasing which satisfies the system

$$\begin{cases} \frac{\gamma'}{\gamma}-\left(\frac{2\varphi}{n}-\alpha'\right)\frac{1}{\alpha}\leqslant 0 \\ \quad \frac{\gamma}{\alpha-1}\leqslant C_2 \end{cases}$$

Recalling that $\alpha(t)$ and $\gamma(t)$ are non-decreasing and $t_1<T_1$ we have

$$\begin{aligned}
\phi G\left(x,T_1\right)\leqslant&(\phi G)\left(x_1,t_1\right)\\
\leqslant& n\gamma\left(T_1\right)\alpha^2\left(T_1\right)\left[\frac{C}{R^2}(1+\sqrt{K_1}R)+\frac{Cn\alpha^4}{R^2}+a\phi+\sqrt{C}K_2\right]+\frac{\sqrt{3}\gamma\left(T_1\right)\phi an\alpha^2\left(T_1\right)}{2}\\
&+\frac{3^{\frac{4}{3}}\gamma\left(T_1\right)\phi n\alpha^2\left(T_1\right)K_4^{\frac{2}{3}}}{4^{\frac{2}{3}}(\alpha-1)^{\frac{1}{3}}}+\frac{\sqrt{3}\gamma\left(T_1\right)\phi(K_1+K_2)n\alpha^2\left(T_1\right)}{\alpha-1}\\
&+\gamma\left(T_1\right)\phi\alpha\left(T_1\right)n^{\frac{1}{2}}\left(K_2+K_3\right)+a^{\frac{1}{2}}n^{\frac{1}{2}}\gamma\left(T_1\right)\phi\varphi^{\frac{1}{2}}\left(T_1\right).
\end{aligned}$$

Hence, we have for $\phi\equiv 1$ on $B_{R,T}$

$$\begin{aligned}
F\left(x,T_1\right)\leqslant& n\alpha^2\left(T_1\right)\left[\frac{C}{R^2}(1+\sqrt{K_1}R)+a+\sqrt{C}K_2\right]+\frac{n^2C\alpha^4}{R^2\gamma\left(T_1\right)}\\
&+\frac{\sqrt{3}an\alpha^2\left(T_1\right)}{2}+\frac{3^{\frac{4}{3}}n\alpha^2\left(T_1\right)K_4^{\frac{2}{3}}}{4^{\frac{2}{3}}(\alpha-1)^{\frac{1}{3}}}+\frac{\sqrt{3}(K_1+K_2)n\alpha^2\left(T_1\right)}{\alpha-1}\\
&+\alpha\left(T_1\right)n^{\frac{1}{2}}\left(K_2+K_3\right)+a^{\frac{1}{2}}n^{\frac{1}{2}}\varphi^{\frac{1}{2}}\left(T_1\right).
\end{aligned}$$

Because T_1 is arbitrary in $0 < T_1 < T$, the conclusion is valid. This proof is complete.

Proof of the Corollary 1.1 The Main Theorem 1 implies this result obviously.

Proof of the Corollary 1.2 By the uniform equivalence of $g(t)$, we know that $(M, g(t))$ is complete noncompact for $t \in [0,T]$. Letting $R \to +\infty$ in the inquallities of Main Theorem 1 completes the proof.

Proof of the Corollary 1.3 The gradient estimates in Corollary 1.2 can be written as

$$\frac{|\nabla u(x,t)|^2}{u^2(x,t)} - \alpha \frac{u_t(x,t)}{u(x,t)} + a\alpha(t) \log u(x,t) \leq Q,$$

where

$$\begin{aligned} Q = & \frac{1}{\alpha(t)} \left[C\alpha^2(K_1 + a) + \frac{3^{\frac{4}{3}} n\alpha^2 K_4^{\frac{2}{3}}}{4^{\frac{2}{3}}(\alpha-1)^{\frac{1}{3}}} + \frac{\sqrt{3}(K_1+K_2)n\alpha^2}{\alpha - 1} + \alpha n^{\frac{1}{2}}(K_2 + K_3) \right. \\ & \left. + a^{\frac{1}{2}} n^{\frac{1}{2}} \varphi^{\frac{1}{2}} + \alpha\varphi \right]. \end{aligned}$$

Define $l(s) = \log u(\gamma(s), (1-s)t_2 + st_1)$. Obviously, we infer that $l(0) = \log u(x_2, t_2)$ and $l(1) = \log u(x_1, t_1)$. Direct calculation shows

$$\begin{aligned} \frac{\partial l(s)}{\partial s} &= (t_2 - t_1)\left(\frac{\nabla u}{u} \frac{\gamma'(s)}{t_2 - t_1} - \frac{u_t}{u} \right) \\ &\leqslant (t_2 - t_1)\left[\frac{\nabla u}{u} \frac{\gamma'(s)}{t_2 - t_1} - \frac{1}{\alpha(t)} \frac{|\nabla u|^2}{u^2} - a \log u(x,t) + Q \right] \\ &\leqslant \frac{\alpha(t)}{4} \frac{|\gamma'(s)|^2}{t_2 - t_1} + (t_2 - t_1)\left[Q + |a \log N| \right], \end{aligned}$$

where $N = \max_{M^n \times [0,T]} u$.

Integrating the above inequality over $\gamma(s)$, we obtain

$$\begin{aligned} \log \frac{u(x_1,t_1)}{u(x_2,t_2)} &= \int_0^1 \frac{\partial l(s)}{\partial s} ds \\ &\leqslant \int_0^1 \left[\frac{\alpha(t)}{4} \frac{|\gamma'(s)|^2}{t_2 - t_1} + (t_2 - t_1)\left[Q + |a \log N| \right] \right] ds \\ &\leqslant \int_0^1 \frac{|\gamma'(s)|^4}{2(t_2 - t_1)^2} ds + \int_{t_1}^{t_2} \frac{\alpha^2(t)}{32} dt + \int_{t_1}^{t_2} \left[Q + |a \log N| \right] dt, \end{aligned}$$

which implies the corollary.

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一种非线性抛物方程在一般几何流下的梯度估计

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摘要: 本文通过Li-Yau梯度估计的方法和Jun Sun对热方程在一般几何流下梯度估计的研究, 推导出一类重要的非线性抛物方程在一般几何流演化下的梯度估计, 并得到了哈拿克不等式等一些结论. 推广了Wang的结果.

关键词: 梯度估计; 几何流;一种非线性抛物方程; 哈拿克不等式

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