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STRONG ATOMIC DECOMPOSITIONS OF TWO-PARAMETER B-VALUED WEAK ORLICZ STRONG MARTINGALE SPACES

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Abstract: In this paper, we study the two-parameter B-valued strong martingale on weak Orlicz space, with emphasis on the strong atomic decomposition theorem of two-parameter B-valued strong martingale space $w\widetilde{H}_{\Phi}^{\sigma}$, by using atomic decomposition theorem, the sufficient conditions for boundedness of sublinear operator $\|Tf\|_{wL_{\Phi}} \leq C\|f\|_{w\widetilde{H}_{\Phi}^{\sigma}}$ is given. The results as above generalize the conclusion of weak L_p martingale space.

Keywords: atomic decompositions; weak Orlicz spaces; strong martingale; two-parameter B-valued martingale

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1 Introduction

With the continuous improvement of martingale theory, the applications of harmonic analysis in weak function space and martingale space have become more and more extensive. Herz [1] gave the atomic decomposition method from the modulation analysis into the study of the theory. The atomic decomposition method has been widely used in the martingale theory study, especially the two-parameter atomic decomposition became particularly important to prove inequality theory. Weisz [2] established real-valued single-parameter martingale, the atomic decomposition theory of two-parameter martingale space and the strong atomic decomposition theory of two-parameter strong martingale space in the literature. In the meantime, Weisz used atomic decomposition methods to prove important inequality and duality theory in martingale space. The literature [3-4] studied the situation of single-parameter B-valued martingale space and weak martingale space. They gave the atomic decomposition theories in B-valued martingale space and weak martingale space when the value with certain geometric properties. Chen [5] obtained the atomic decomposition in the two-parameter B-valued martingale space. Ye [6] defined the strong atom and established

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a strong atomic decomposition theorem in the two-parameter B-valued strong martingale space of $s\widetilde{\sum}^{p}_{\alpha}$. Liu [7] introduced a class of important function space–weak Orlicz space. And in the article [8] they studied some basic properties of weak Orlicz space and some applications in harmonic analysis.

In the paper, we will discuss the atomic decomposition for two-parameter B-valued weak Orlicz strong martingale space.

2 Preliminaries and Notations

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, (B, $\|\cdot\|$) be a Banach space with dimension not less than 2. $\{\mathcal{F}_n, n \in N^2\}$ is a non-decreasing partial order sub σ algebraic sequence of $\mathcal{F}$ in N^2 and which satisfies the condition F_4. For $\forall n = (n_1, n_2) \in N^2$, we define $\mathcal{F}_n^- = \sigma(\mathcal{F}_{n_1-1,n_2} \bigcup \mathcal{F}_{n_1,n_2-1})$, $\mathcal{F}_n^+ = \sigma(\mathcal{F}_{n_1+1,n_2} \bigcup \mathcal{F}_{n_1,n_2+1})$ and $\mathcal{F}^* = \sigma(\bigcup\limits_{n\in N} \mathcal{F}_\infty)$. We denote E, E_n, E_n^-, E_n^+ and E_n^* respectively represent the expectation operator conditional expectation operator, and with respect to $\mathcal{F}_n$, $\mathcal{F}_n^-$, $\mathcal{F}_n^+$ and $\mathcal{F}^*$.

For two sets $G, H \subset N^2$ consisting of incomparable pairs of numbers, we write $H \not\ll G$(or $H \not\gg G$) if $\forall m \in G$ there exists $v_t \in H$ such that $v_t \not\ll m$(or $v_t \not\gg m$). Denote by $\inf H$: the set $v_t \in H$: there does not exist $m \in H$ such that $v_t < m$.

We define $d_n f$ as follows:

$$d_n f = \begin{cases} 0, n_1 n_2 = 0 \\ f_{n_1,n_2} - f_{n_1-1,n_2} - f_{n_1,n_2-1} + f_{n_1-1,n_2-1}, n_1 n_2 \neq 0. \end{cases}$$

Let Φ be convex and non-decreasing in $[0, \infty)$ which satisfies $\Phi(0) = 0$ and $\lim\limits_{t\to\infty} \Phi(t) = \infty$, $\varphi(t)$ is a right continuous derivative function for Φ. There are two major parameters for the function Φ,

$$q_\Phi = \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)}, \quad p_\Phi = \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)}. \tag{2.1}$$

It is called that Φ satisfies the condition of Δ_2, if for any $a > 1$, there exists a constant $C_a > 0$ such that $\Phi(at) \leq C_a \Phi(t)$, $\forall t > 0$. This condition is equivalent to $p_\Phi < \infty$. Φ is strictly convex, if $q_\Phi > 1$. In particular, if $0 < q_\Phi \leq p_\Phi < \infty$, the function $\frac{\Phi(t)}{t^{q_\Phi}}$ is monotonically increasing on $(0, \infty)$, and $\frac{\Phi(t)}{t^{p_\Phi}}$ is monotonically decreasing on $(0, \infty)$.

As well known, the space wL_Φ is the collection of all measurable functions f for which

$$\|f\|_{wL_\Phi} = \inf\{c > 0 : \sup_{t>0} \Phi(\frac{t}{c}) P(\|f\| > t) \leq 1\}.$$

Definition 2.1 The B-valued martingale $f = (f_n, n \in N^2)$ is a strong martingale, when $E_{n-1}^* d_n f = 0$, $\forall n \in N^2$.

This article uses $\|f\|_p$, f^*, $S^{(p)}(f)$, $\sigma^{(p)}(f)$ to indicate its L_p-modulus, maximum function, p-mean square function and conditional p-mean square function related to $\mathcal{F}_{n-1}$:

$$\|f\|_p = \sup_{n\in N^2} \|f_n\|_p, \quad f_n^* = \sup_{i\leq n} \|f_i\|, \quad f^* = \sup_{n\in N^2} \|f_n\|;$$

$$S_n^{(p)}(f) = (\sum_{0\le i\le n} \|d_i f\|^p)^{\frac{1}{p}}, \qquad S^{(p)}(f) = \sup_{n\in N^2} S_n^{(p)}(f), 1 < p < \infty;$$

$$\sigma_n^{(p)}(f) = (\sum_{0\le i\le n} E_{i-1}\|d_i f\|^p)^{\frac{1}{p}}, \qquad \sigma^{(p)}(f) = \sup_{n\in N^2} \sigma_n^{(p)}(f), 1 < p < \infty.$$

Moreover, for the two-parameter B-valued strong martingale $f = (f_n, n \in N^2)$, we introduce a conditional p-mean square function about $\mathcal{F}_n^-$:

$$\widetilde{\sigma}^{(p)}(f) := (\sum_{n\in N^2} E_n^- \parallel d_n f \parallel^p)^{\frac{1}{p}}.$$

We define the weak Orlicz spaces as follows:

$$\begin{aligned}
wL_\Phi &= \{f = (f_n, n \in N^2) : \|f\|_{wL_\Phi} = \sup_{n\in N^2} \|f_n\|_{wL_\Phi} < \infty\}, \\
wH_\Phi^* &= \{f = (f_n, n \in N^2) : \|f\|_{wH_\Phi} = \|f^*\|_{wL_\Phi} < \infty\}, \\
wH_\Phi^S &= \{f = (f_n, n \in N^2) : \|f\|_{wH_\Phi^S} = \|S^{(p)}(f)\|_{wL_\Phi} < \infty\}, \\
wH_\Phi^\sigma &= \{f = (f_n, n \in N^2) : \|f\|_{wH_\Phi^s} = \|\sigma^{(p)}(f)\|_{wL_\Phi} < \infty\}, \\
w\widetilde{H}_\Phi^\sigma &= \{f = (f_n, n \in N^2) : \|f\|_{wH_\Phi^s} = \|\widetilde{\sigma}^{(p)}(f)\|_{wL_\Phi} < \infty\}.
\end{aligned}$$

Definition 2.2 A strong martingale $g \in L_p$ is said to be a strong (α, p) atom $(0 < \alpha \le p, 1 < p \le \infty)$, if there is a stopping time $v \in T(\mathcal{F}_n^+)$ such that:

(1) if $\{v \not\ll n\}$, $g_n := E_n g = 0$;

(2) $\|g^*\|_p \le P(v \ne \infty)^{\frac{1}{p}-\frac{1}{\alpha}}$.

Lemma 2.1([6]) Let B be a Banach space, $1 < p \le 2$, then the following statements are equivalent :

(1) B is p smooth;

(2) For any two-parameter B-valued strong martingale $f = (f_n, n \in N^2)$, the f_n converges in probability when $\sum\limits_{n\in N^2} E\|d_n f\|^p < \infty$.

Lemma 2.2([6]) Let B be a Banach space, $1 < p \le 2$, $0 < \alpha \le p$, then the following statements are equivalent :

(1) B is p smooth;

(2) There is a constant C such that for any two-parameter B-valued strong martingale f: $\|S^{(p)}(f)\|_\alpha \le C\|\widetilde{\sigma}^{(p)}(f)\|_\alpha$;

(3) There is a constant C such that for any two-parameter B-valued strong martingale f: $\|f^*\|_\alpha \le C\|\widetilde{\sigma}^{(p)}(f)\|_\alpha$.

3 Atomic Decomposition

Theorem 3.1 Let B be a Banach space, $1 < p \le 2$, Φ satisfies the condition of Δ_2. If B is p smooth, $0 < \alpha \le p$, then for any two-parameter B-valued strong martingale $f = (f_n, n \in N^2) \in w\widetilde{H}_\Phi^\sigma$, there is a strong (α, p) atom $(g^{(k)}, k \in Z)$ and a column of non-negative real number $\mu = \mu_k, k \in Z \in l_\alpha$, so that for all $n \in N^2$:

$$f_n = \sum_{k=-\infty}^{\infty} \mu_k E_n g^{(k)}, a.e., \tag{3.1}$$

$$(\sum_{k\in Z} \mu_k^\alpha)^{\frac{1}{\alpha}} \leq C\|f\|_{w\widetilde{H}_\Phi^\sigma}, \tag{3.2}$$

where C is only related to p and α.

Proof Let $f \in w\widetilde{H}_\Phi^\sigma$, for every $k \in Z$, we denote $F_k = \{\widetilde{\sigma}^{(p)}(f) > 2^k\}$ and stopping time $v_k = \inf\{n \in N^2 : E_n^+\chi_{(F_k)} > \frac{1}{2}\}$. Obviously, v_k is non-decreasing stopping time, and $v_k \to \infty$ when $k \to \infty$.

Let $f_n^{(v_k)} = \sum\limits_{m\leq n} \chi(v_k \ngtr m) d_m f$, so $(f_n^{(v_k)}, n \in N^2)$ is stopping martingale and

$$f_n^{(v_{k+1})} - f_n^{(v_k)} = \sum_{m\leq n} \chi(v_k \ll m \ngtr v_{k+1}) d_m f. \tag{3.3}$$

For $\forall n \in N^2$, we have

$$f_n = \sum_{k\in Z} \{f_n^{(v_{k+1})} - f_n^{(v_k)}\}. \tag{3.4}$$

The detail of (3.4) proving process can be seen in Reference [6].

Suppose $\mu_k = (\frac{p}{p-1})^2 (2C)^{\frac{1}{p}} 2^{k+1} P(v_k \neq \infty)^{\frac{1}{\alpha}}$, $g_n^{(k)} = (\frac{1}{\mu_k})(f_n^{(v_{k+1})} - f_n^{(v_k)})$, where C is the smooth coefficient of p. It is easy to verify for all $k \in Z$, $g^{(k)} = (g_n^{(k)}, n \in N^2)$ which is a strong martingale. When $v_k \ngtr n$, $f_n^{(v_{k+1})} = f_n^{(v_k)}$ then $g_n^{(k)} = 0$. Since B is p smooth, there is $C > 0$ such that

$$\begin{aligned} E\|f_n^{(v_{k+1})} - f_n^{(v_k)}\|^p &\leq CE(\sum_{m\leq n} \|d_n f\|^p \chi(v_k \ll m \ngtr v_{k+1})) \\ &\leq C \times 2 \times 2^{p(k+1)} P(v_k \neq \infty). \end{aligned} \tag{3.5}$$

Since the $v_k \ll m$ is $\mathcal{F}_m^-$ measurable, obtained by (3.5) and Lemma 2.1:

$$\begin{aligned} E(g^{(k)^*})^p &\leq (\frac{p}{(p-1)})^{2p} \sup_{n\in N^2} \|g^{(k)}\|_p^p \\ &\leq \sup_{n\in N^2} (\frac{p}{(p-1)})^{2p} \frac{C \times 2 \times 2^{p(k+1)}}{\mu^p} P(v_k \neq \infty) \\ &= P(v_k \neq \infty)^{1-\frac{p}{\alpha}}, \end{aligned}$$

thus

$$\|g^{(k)^*}\|_p \leq P(v_k \neq \infty)^{\frac{1}{p}-\frac{1}{\alpha}} < \infty.$$

Because B has RN property, there exists $g^{(k)} \in L_p$ such that $E_n g^{(k)} = g_n^{(k)}$. From the definition of strong atom we know that $g^{(k)}$ is a strong (α, p) atom, which proved (3.1).

For any $k \in Z$, we have

$$
\begin{aligned}
(\sum_{k\in Z}\mu_k^{\alpha})^{\frac{1}{\alpha}} &\le c\Big(\sum_{k\in Z}2^{k\alpha}P(\nu_k\ne\infty)\Big)^{\frac{1}{\alpha}}\\
&\le c\Big(\sum_{k\imath Z}2^{k\alpha}P(\sup_{n\in N^2}E_n(\chi_{(F_k)})>\frac{1}{2})\Big)^{\frac{1}{\alpha}}\\
&\le c\Big(\sum_{k\imath Z}2^{k\alpha}E(\sup_{n\in N^2}E_n(\chi_{(F_k)})^p)\Big)^{\frac{1}{\alpha}}\\
&\le c\Big(\sum_{k\imath Z}2^{k\alpha}P(F_k)\Big)^{\frac{1}{\alpha}}\\
&\le c\Big(\sum_{k\in Z}2^{k\alpha}P\big(\widetilde{\sigma}^{(p)}(f)>2^k\big)\Big)^{\frac{1}{\alpha}}\\
&\le c\Big(\sum_{k\in Z}\int_{2^{k-1}}^{2^k}y^{\alpha-1}P(\widetilde{\sigma}^{(p)}(f)>y)dy\Big)^{\frac{1}{\alpha}}\\
&\le c\Big(\int_0^{\infty}y^{\alpha-1}P\big(\widetilde{\sigma}^{(p)}(f)>y\big)dy\Big)^{\frac{1}{\alpha}}\\
&\le c\Big(1+\int_1^{\infty}y^{\alpha-1}P\big(\widetilde{\sigma}^{(p)}(f)>y\big)dy\Big)^{\frac{1}{\alpha}}. \qquad (3.6)
\end{aligned}
$$

Since the function $\frac{\Phi(t)}{t_{q_\Phi}}$ is monotonically increasing in $(0,\infty)$, and $\sup\limits_{t>0}\Phi(t)P\big(\widetilde{\sigma}^{(p)}(f)>t\big)\le\|f\|_{w\widetilde{H}_\Phi^\sigma}=1$, then

$$
\begin{aligned}
(\sum_{k\in Z}\mu_k^{\alpha})^{\frac{1}{\alpha}} &\le c\Big(1+\int_1^{\infty}\frac{y^{q_\Phi}y^{\alpha-q_\Phi-1}}{\Phi(y)}\Phi(y)P\big(\widetilde{\sigma}^{(p)}(f)>y\big)dy\Big)^{\frac{1}{\alpha}}\\
&\le c\Big(1+\frac{1}{\Phi(1)}\int_1^{\infty}y^{\alpha-q_\Phi-1}dy\Big)^{\frac{1}{\alpha}}\\
&= c\Big(1+\frac{1}{q_\Phi-\alpha}\Phi(1)\Big)^{\frac{1}{\alpha}}. \qquad (3.7)
\end{aligned}
$$

Denote by C the last item, then

$$
(\sum_{k\in Z}\mu_k^{\alpha})^{\frac{1}{\alpha}}\le C\|f\|_{w\widetilde{H}_\Phi^\sigma},
$$

where C is a constant independent of f. Thus Theorem 3.1 is proved.

Theorem 3.2 Let B be a Banach space, $1<p\le\min\{q_\Phi,2\}$, Φ satisfies the condition of Δ_2, the statements are equivalent:

(1) B is p-uniform smooth;

(2) If $0 < \alpha \le 1$, then for any two-parameter B-valued strong martingale $f = (f_n, n \in N^2) \in w\widetilde{H}^\sigma_\Phi$, there is a strong (α, p) atom $(g^{(k)}, k \in Z)$ and a non-negative real number sequence $\mu = (\mu_k, k \in Z) \in l_\alpha$, for all $n \in N^2$, $f_n = \sum\limits_{k=-\infty} \mu_k E_n g^{(k)}$, a.e..

Proof (1)⇒(2) from Theorem 3.1.

(2)⇒(1). Without loss of generality, we assume $\|f\|_{w\widetilde{H}^\sigma_\Phi} = 1$. As the function $\frac{\Phi(t)}{t_{q_\Phi}}$ is monotonically increasing in $(0, \infty)$, and $\sup\limits_{t>0} \Phi(t)P\big(\widetilde{\sigma}^{(p)}(f) > t\big) \le \|f\|_{w\widetilde{H}^\sigma_\Phi} = 1$. We get from the assumption that

$$\begin{aligned}
\|\widetilde{\sigma}^{(p)}(f)\|_p^p &\le p\int_0^\infty t^{p-1}P(\|\widetilde{\sigma}^{(p)}f\| > t)dt \\
&\le c\bigg(1 + \int_1^\infty \frac{y^{q_\Phi}y^{p-q_\Phi-1}}{\Phi(y)}\Phi(y)P\big(\widetilde{\sigma}^{(p)}(f) > y\big)dy\bigg) \\
&\le c\bigg(1 + \frac{1}{\Phi(1)}\int_1^\infty y^{p-q_\Phi-1}dy\bigg)\|f\|_{w\widetilde{H}^\sigma_\Phi} \\
&= c\bigg(1 + \frac{1}{q_\Phi - p}\Phi(1)\bigg)\|f\|_{w\widetilde{H}^\sigma_\Phi} < \infty. \qquad (3.8)
\end{aligned}$$

If g is a strong (α, p) atom, we can get

$$\|g^*\|_\alpha^\alpha = E(g^{*\alpha}\chi_{\tau\ne\infty}) \le E(g^{*p})^{\frac{\alpha}{p}}P(\tau \ne \infty)^{1-\frac{\alpha}{p}} \le 1. \qquad (3.9)$$

From the hypothesis, there is a strong (α, p) atom $(g^{(k)}, k \in Z)$ and a non-negative real number sequence $\mu = (\mu_k, k \in Z) \in l_\alpha$, for all $n \in N^2$, $f_n = \sum\limits_{k=-\infty}^{\infty} \mu_k E_n g^{(k)}$, a.e..

Therefore, from $0 < \alpha \le 1$, such that

$$\begin{aligned}
E\|f_m - f_n\|^\alpha &= \sum_{k=-\infty}^{\infty} \mu_k^\alpha E\|g_m^{(k)} - g_n^{(k)}\|^\alpha \\
&\le C\sum_{|k|>k_0} \mu_k^\alpha + \sum_{|k|>k_0} \mu_k^\alpha (E\|g_m^{(k)} - g_n^{(k)}\|)^\alpha. \qquad (3.10)
\end{aligned}$$

Set $m \to \infty, n \to \infty$ and $k_0 \to \infty$, therefore $(f_n, n \in N^2)$ is L_α Cauchy convergence. Thus f_n converges according in probability. From Lemma 2.1, B is p-smooth.

Theorem 3.2 is proved.

4 Boundedness of Sublinear Operators

Suppose $T : X \to Y$ is the mapping, where X is the weak Orlicz strong martingale space on $(\Omega, \mathcal{F}_n, P)$, and Y is the space of measurable function on $(\Omega, \mathcal{F}, P)$. T is sub-linear, if

$$|T(f + g)| \le |Tf| + |Tg|, \quad |T(\alpha f)| = |\alpha||Tf|, a.e.. \qquad (4.1)$$

If the $Tf = 0$ is on the set $f = 0$. T is called bounded, if

$$\|T\| = \sup_{\|f\|_X \le 1} \|Tf\|_Y < \infty. \qquad (4.2)$$

Theorem 4.1 Let $T : L_p \to L_p$ be a bounded sublinear operator, B be isomorphic to p-smooth Banach space which satisfies for any (α, p) atom a,

$$P(|Ta| > 0) \le C_0 P(\nu \neq \infty), \tag{4.3}$$

where ν is the stopping time associated with a, and C_0 is a constant independent of a. If $0 < p < q_\Phi \le p_\Phi < \infty$ and $p \le 2$, then there is $C > 0$ such that

$$\|Tf\|_{wL_\Phi} \le C\|f\|_{w\widetilde{H}^\sigma_\Phi}, \quad \forall f = (f_n) \in w\widetilde{H}^\sigma_\Phi(B). \tag{4.4}$$

Proof Let $f \in w\widetilde{H}^\sigma_\Phi(B)$ and $\|f\|_{w\widetilde{H}^\sigma_\Phi(B)} = \|\widetilde{\sigma}^{(p)}(f)\|_{wL_\Phi} = 1$. From Theorem 3.1, there exists atomic decomposition of f such that satisfies (3.1) and (3.2). For any $y > 0$, let $j \in Z$ such that $2^j \le y < 2^{j+1}$, we define

$$g_n = \sum_{k \le j-1} \mu_k a_n^k, \qquad h_n = \sum_{k \ge j} \mu_k a_n^k, \qquad n \ge 0. \tag{4.5}$$

Considering $h = (h_n, n \in N^2)$, we denote that each a_k satisfies $\Phi(2^k)P(\nu_k \neq \infty) \le c\Phi(2^k)\big(P\big(\widetilde{\sigma}^{(p)}(f) > 2^k\big)\big) \le c\sup\limits_{t>0}\Phi(t)\big(P\big(\widetilde{\sigma}^{(p)}(f) > t\big)\big) = c\|f\|_{w\widetilde{H}^\sigma_\Phi(B)} = c$.

Condition (4.3) shows

$$\begin{aligned}
\Phi(2^j)P(|Th| > 2^j) &\le \Phi(2^j)P(|Th| > 0) \le \Phi(2^j)P(\sum_{k\ge j}\mu_k|Ta^k| > 0) \\
&\le \sum_{k\ge j}\Phi(2^j)P(|Ta^k| > 0) \le C_0\sum_{k\ge j}\Phi(2^j)P(\nu_k \neq \infty) \\
&= C_0\sum_{k\ge j}\frac{\Phi(2^j)}{\Phi(2^k)}\Phi(2^k)P(\nu_k \neq \infty) \le C_0\sum_{k\ge j}\frac{\Phi(2^k 2^{-(k-j)})}{\Phi(2^k)} \\
&\le C_0\sum_{k\ge j}\frac{\Phi(2^k)2^{-(k-j)}}{\Phi(2^k)} \le C_0\sum_{k'=0}^{\infty}2^{-k'} = 2C_0.
\end{aligned}$$

The basic thing is $\Phi(at) \le a\Phi(t)$, $\forall t > 0$, $0 < a \le 1$, which derive from the convexity of Φ. For $g = (g_n, n \in N^2)$, we know $\|g^*\|_p \le C\|\widetilde{\sigma}^{(p)}(g)\|_p$ $(1 \le p \le 2)$ from Lemma 2.2. Since $T : L_p \to L_p$ is a bounded sublinear operator, then

$$\begin{aligned}
\|Tg\|_{L_p} &\le C\|g\|_{L_p} \le C\|g^*\|_{L_p} \le C'\|\widetilde{\sigma}^{(p)}(g)\|_{L_p} \\
&\le C'\|\sum_{k\le j-1}\mu_k\widetilde{\sigma}^{(p)}(a^k)\|_{L_p} \\
&\le C'\sum_{k\le j-1}\mu_k\|\widetilde{\sigma}^{(p)}(a^k)\|_{L_p} \\
&\le C'\sum_{k\le j-1}\mu_k(E\widetilde{\sigma}^{(p)}(a^k)^p\chi_{\{\nu_k\neq\infty\}})^{\frac{1}{p}} \\
&\le C''\sum_{k\le j-1}2^k P(\nu_k \neq \infty)^{\frac{1}{p}}
\end{aligned}$$

and

$$P(|Tg| > 2^j) \leq \frac{1}{2^{pj}}\|Tg\|_{L_p}^p \leq C'^p\left(\frac{1}{2^j}\sum_{k\leq j-1} 2^k P(\nu_k \neq \infty)^{\frac{1}{p}}\right)^p.$$

Since the function $\frac{\Phi(t)}{t_{p_\Phi}}$ is monotonically decreasing, therefore $\frac{\Phi(2^j)}{\Phi(2^k)} \leq \frac{2^{jp_\Phi}}{2^{kp_\Phi}} = 2^{(j-k)p_\Phi}$, if $k \leq j$, such that

$$\begin{aligned}
\Phi(2^j)P(|Tg| > 2^j) &\leq C''^p\left(\sum_{k\leq j-1} 2^{k-j}\left(\frac{\Phi(2^j)}{\Phi(2^k)}\cdot\Phi(2^k)P(\nu_k \neq \infty)\right)^{\frac{1}{p}}\right)^p \\
&\leq C''^p\left(\sum_{k\leq j-1} 2^{k-j}\left(\frac{\Phi(2^j)}{\Phi(2^k)}\right)^{\frac{1}{p}}\right)^p \\
&\leq C''^p\left(\sum_{k\leq j-1} 2^{k-j}2^{(j-k)\frac{p_\Phi}{p}}\right)^p \\
&\leq C''^p\left(\sum_{k'\geq 1} 2^{-k'(1-\frac{p_\Phi}{p})}\right)^p \\
&\leq C'''^p 2^{p_\Phi - p},
\end{aligned}$$

which C''' is a constant independent of f, so

$$\begin{aligned}
\sup_{y>0}\Phi(2y)P(|Tf| > 2y) &\leq \sup_{y>0}\Phi(2y)\Big(P(|Tg| > y) + P(|Th| > y)\Big) \\
&\leq C\sup_{j\in Z}\Phi(2^j)\Big(P(|Tg| > 2^j) + P(|Th| > 2^j)\Big) \\
&\leq C(C'''^p 2^{p_\Phi - p} + 2C_0) := \tilde{C}.
\end{aligned}$$

We assume $\tilde{C} \geq 1$ with loss of generality, then

$$\sup_{y>0}\Phi(\frac{2y}{\tilde{C}})P(|Tf| > 2y) \leq \frac{1}{\tilde{C}}\sup_{y>0}\Phi(2y)(P(|Tf| > 2y) \leq 1.$$

Finally,

$$\|Tf\|_{wL_\Phi} \leq \tilde{C} = \tilde{C}\|f\|_{w\tilde{H}_\Phi^\sigma}.$$

Thus, we prove (4.1).

References

[1] Herz C. H_p space of martingale$(0 < p < 1)$[J]. Z. Wahrs. Verw Geb., 1974, 28: 189–205.

[2] Weisz F. Martingale Hardy spaces and their applications in Fourier analysis[M]. Berlin Heidelberg: Springer, 1994.

[3] Liu Peide, Hou Youliang. Atomic decompositions of Banach-space valued martingales[J]. Science in China Series A Mathematics, 1999, 42(1): 38–47.

[4] Hou Youliang, Ren Yanbo. Vector-valued weak martingale Hardy spaces and atomic decompositions[J]. Acta Mathematica Hungarica, 2007, 115(3): 235–246.

[5] Chen Riyan, Gan Shixin. Atomic decompositions for two-parameter vector-valued martingales and two-parameter vector-valued martingale spaces[J]. Acta Mathematica Hungarica, 2001, 93(1): 7–25.

[6] Ye Chen. Strong atom decomposition in two-index B-value strong martingale space[J]. Journal of Ningbo University (Science and Technology Edition), 2009, 22(02): 222–226.

[7] Liu Peide, Hou Youliang, Wang M F. Weak Orlicz space and its applications to the martingale theory[J]. Science China, 2010, 53(4): 905–916.

[8] Liu Peide, Wang Maofa, et al. Weak Orlicz spaces: Some basic properties and their applications to harmonic analysis[J]. Science China Mathematics, 2013(4): 789–802.

[9] Ren Yanbo. Weak atom decomposition of weak orlicz martingale space and application [J]. Annals of Mathematics, Series A (Chinese Edition), 2015, 36(2): 119–128.

[10] Liu Peide. A new progress in martingale space theory[J]. Journal of Mathematics, 2017, 37(03): 445–456.

两参数B值弱型Orlicz强鞅空间的强原子分解

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摘要: 本文研究了弱Orlicz空间上的两参数B值强鞅，重点研究了两参数B 值强鞅空间$w\widetilde{H}_\Phi^\sigma$ 的强原子分解定理, 利用原子分解定理，给出了次线性算子$\|Tf\|_{wL_\Phi} \le C\|f\|_{w\widetilde{H}_\Phi^\sigma}$有界性的充分条件，上述结果推广了弱$L_p$ 鞅空间的结论.

关键词: 原子分解; 弱Orlicz空间; 强鞅; 两参数B 值鞅

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