Vol. 42 ( 2022 )
No. 2

数 学 杂 志
J. of Math. (PRC)

# RESEARCH ANNOUNCEMENTS ON “STABILITY OF RAREFACTION WAVES OF THE COMPRESSIBLE NAVIER-STOKES-POISSON SYSTEM WITH LARGE INITIAL PERTURBATION”

ZHANG Lan[1], ZHAO Hui-jiang[2], ZHAO Qing-song[2]
(*1. Center for Mathematical Sciences and Department of Mathematics, Wuhan University of Technology, Wuhan 430070, China*)
(*2. School of Mathematics and Statistics, Wuhan University, Hubei Key Laboratory of Computational Science, Wuhan 430072, China*)

## 1 Introduction and Main Results

In this paper, we study the dynamics of compressible viscous charged particles consisting of two-species particles (e.g., ions and electrons) under the influence of the self-consistent electrostatic potential force in semiconductor device or plasma physics (cf. [1, 6]) which is modeled by the compressible Navier-Stokes-Poisson system (called NSP system in the sequel for simplicity). We focus on the one-dimensional case which can be formulated in the Eulerian coordinates as follows:

$$\begin{aligned}
\partial_t n_i+\partial_x(n_iu_i) &= 0,\\
m_in_i\left(\partial_t u_i+u_i\partial_x u_i\right)+T_i\partial_x n_i-n_i\partial_x\phi &= \mu_i\partial_x^2u_i,\\
\partial_t n_e+\partial_x(n_eu_e) &= 0,\\
m_en_e\left(\partial_t u_e+u_e\partial_x u_e\right)+T_e\partial_x n_e+n_e\partial_x\phi &= \mu_e\partial_x^2u_e.\\
\partial_x^2\phi &= n_i-n_e.
\end{aligned}\tag{1.1}$$

Here, the unknown functions $n_\alpha=n_\alpha(t,x)>0$ and $u_\alpha=u_\alpha(t,x)$ stand for the density and velocity of ions ($\alpha=i$) and electrons ($\alpha=e$), respectively. The positive constants $m_\alpha$, $T_\alpha$, and $\mu_\alpha$ represent the mass, the absolute temperature, and the viscosity coefficient of $\alpha$ charged particles, respectively. $\phi=\phi(t,x)$ is the potential of the self-consistent electrostatic field.

Since we are concerned with the Cauchy problem, the two-fluid model (1.1) is equipped with the following initial data

$$(n_\alpha(0,x),u_\alpha(0,x))=(n_{\alpha0}(x),u_{\alpha0}(x)),\quad \alpha=i,e,\tag{1.2}$$

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**Foundation item:** Supported by the National Natural Science Foundation of China (12101472; 11731008).

**Biography:** Zhao Huijiang(1967–), male, born at Hanchuan, Hubei, professor, major in partial differential equations. E-mail:hhjjzhao@whu.edu.cn.

where the initial data $(n_{\alpha 0}(x), u_{\alpha 0}(x))$ and the electric potential $\phi(t,x)$ are assumed to satisfy the following far-field conditions

$$\begin{aligned}\lim_{x\to\pm\infty}(n_{\alpha 0}(x), u_{\alpha 0}(x)) &= (n_{\pm}, u_{\pm}),\\ \lim_{x\to\pm\infty}\phi(t,x) &= \phi_{\pm}, \quad t\geq 0.\end{aligned} \tag{1.3}$$

Here $n_{\pm}>0, u_{\pm}$, and $\phi_{\pm}$ are constants satisfying $(n_+, u_+, \phi_+)\neq(n_-, u_-, \phi_-)$.

A closely related but somewhat simplified model for the case of compressible viscous single ions flow under the Boltzmann relation $n_e=e^{-\phi}$, after a suitable normalization, takes the form of

$$\begin{aligned}\partial_t n+\partial_x(nu) &= 0,\\ n\left(\partial_t u+u\partial_x u\right)+A\partial_x n-n\partial_x\phi &= \partial_x^2 u,\\ \partial_x^2\phi &= n-e^{-\phi}\end{aligned} \tag{1.4}$$

with prescribed initial data

$$(n(0,x), u(0,x))=(n_0(x), u_0(x)) \tag{1.5}$$

and the following far-field conditions

$$\begin{aligned}\lim_{x\to\pm\infty}(n_0(x), u_0(x)) &= (n_{\pm}, u_{\pm}),\\ \lim_{x\to\pm\infty}\phi(t,x) &= \phi_{\pm}, \quad t\geq 0,\end{aligned} \tag{1.6}$$

where $n=n(t,x)>0$ and $u=u(t,x)$ are the density and velocity of ions, $\phi=\phi(t,x)$ is the potential of the self-consistent electric field, and the positive constant $A>0$ is the absolute temperature of ions. For more physical background of the system (1.4) and its formally mathematical derivation, those interested are referred to [1, 5] and the references cited therein.

It is well-known that the time-asymptotical behaviors of the global solutions of the Cauchy problem (1.1), (1.2), (1.3) of the two-fluid model and the Cauchy problem (1.4), (1.5), (1.6) of the one-fluid model are uniquely determined by the structure of the unique global entropy solution of the resulting Riemann problem of the corresponding quasineutral Euler system. Since $(n_+, u_+, \phi_+)\neq(n_-, u_-, \phi_-)$, the unique global entropy solution of the resulting Riemann problem of the quasineutral Euler system consists of rarefaction waves, shock waves, and/or their linear superposition, consequently one can expect that the large time behaviors of the global solutions of the Cauchy problem (1.1), (1.2), (1.3) of the two-fluid model and the Cauchy problem (1.4), (1.5), (1.6) of the one-fluid model also exhibit rich nonlinear wave phenomena.

The main purpose of this paper is concentrated on the case when the unique global entropy solution of the resulting Riemann problem of the corresponding quasineutral Euler system consists of rarefaction waves only. Our results contain two parts: The first part focuses on the Cauchy problem (1.4), (1.5), (1.6) of the one-fluid model; the second part is devoted to the Cauchy problem (1.1), (1.2), (1.3) of the two-fluid model.

### 1.1 Main Result for the One-Fluid Model

As shown in [2], the large time behaviors of the global solution $(n(t,x),u(t,x),\phi(t,x))$ of the Cauchy problem (1.4), (1.5), (1.6) can be determined by the structure of the unique global entropy solution $(N^r(x/t),U^r(x/t),\Phi^r(x/t))$ of the resulting Riemann problem of the following quasineutral Euler system

$$\begin{aligned} \partial_t n+\partial_x(nu) &= 0,\\ n\left(\partial_t u+u\partial_x u\right)+A\partial_x n-n\partial_x\phi &= 0,\\ \phi &= -\ln n \end{aligned} \tag{1.7}$$

with Riemann initial data

$$(n(0,x),u(0,x))=\left(n_0^R(x),u_0^R(x)\right)=\begin{cases}(n_-,u_-), & x>0,\\ (n_+,u_+), & x>0.\end{cases} \tag{1.8}$$

It is easy to check that the quasineutral Euler system (1.7) has two distinct eigenvalues

$$\lambda_1=u-\sqrt{A+1},\quad \lambda_2=u+\sqrt{A+1},$$

which are genuinely nonlinear, thus we can construct two rarefaction wave curves $R_i(n_-,u_-)(i=1,2)$ as follows:

$$\begin{aligned} & R_1(n_-,u_-)\\ \equiv\ & \left\{(n,u)\in\mathbb{R}_+\times\mathbb{R}\ \middle|\ u+\sqrt{A+1}\ln n=u_-+\sqrt{A+1}\ln n_-,n<n_-,u>u_-\right\},\\ & R_2(n_-,u_-)\\ \equiv\ & \left\{(n,u)\in\mathbb{R}_+\times\mathbb{R}\ \middle|\ u-\sqrt{A+1}\ln n=u_--\sqrt{A+1}\ln n_-,n>n_-,u>u_-\right\}. \end{aligned} \tag{1.9}$$

If $(n_+,u_+)\in R_2(n_-,u_-)$ and $\phi_\pm$ satisfy the compatibility condition

$$n_\pm=e^{-\phi_\pm}, \tag{1.10}$$

then one can easily deduce that the unique global entropy solution $(N^r(x/t),U^r(x/t),\Phi^r(x/t))$ of the Riemann problem (1.7), (1.8) is the rarefaction wave $\left(N_1^{R_2}(x/t),U_1^{R_2}(x/t),\Phi_1^{R_2}(x/t)\right)$ of the second family whose precise definition can be given by

$$\begin{aligned} U_1^{R_2}(x/t) &= w^R(x/t)-\sqrt{A+1},\\ N_1^{R_2}(x/t) &= n_-\exp\left(\frac{U_1^{R_2}(x/t)-u_-}{\sqrt{A+1}}\right),\\ \Phi_1^{R_2}(x/t) &= -\ln N_1^{R_2}(x/t), \end{aligned} \tag{1.11}$$

where $w^R(x/t)$ is the unique global entropy solution of the Riemann problem of the Burgers equation

$$\begin{cases} w_t + ww_x = 0, \\ w(0,x) = w_0^R(x) = \begin{cases} w_-, & x < 0, \\ w_+, & x > 0 \end{cases} \end{cases} \tag{1.12}$$

with $w_\pm := u_\pm + \sqrt{A+1}$.

Since $(n_+, u_+) \in R_2(n_-, u_-)$, we have $w_- := u_- + \sqrt{A+1} < w_+ = u_+ + \sqrt{A+1}$ and consequently the unique global entropy solution $w^R(x/t)$ of the Riemann problem (1.12) is the rarefaction wave solution connecting $w_-$ and $w_+$, which can be defined as follows:

$$w^R(x/t) = \begin{cases} w_-, & \frac{x}{t} \le w_-, \\ \frac{x}{t}, & w_- \le \frac{x}{t} \le w_+, \\ w_+, & \frac{x}{t} \ge w_+. \end{cases} \tag{1.13}$$

Under the above preparations, our first result on the Cauchy problem (1.4), (1.5), (1.6) for the one-fluid model can be stated as in the following theorem.

**Theorem 1.1** Let $\delta := |n_+ - n_-| + |u_+ - u_-|$ be the strength of the rarefaction wave solution $\big(N_1^{R_2}(x/t),\ U_1^{R_2}(x/t), \Phi_1^{R_2}(x/t)\big)$ of the second family of the Riemann problem (1.7), (1.8) and assume that $(n_+, u_+) \in R_2(n_-, u_-)$, $\phi_\pm = -\ln n_\pm$. If we assume further that the initial data $(n_0(x), u_0(x))$ satisfy

$$\begin{aligned} \big(n_0(x) - n_0^R(x), u_0(x) - u_0^R(x)\big) &\in L^2(\mathbb{R}), \\ (\partial_x n_0(x), \partial_x u_0(x)) &\in L^2(\mathbb{R}), \\ \inf_{x\in\mathbb{R}} \big\{n_0(x)\big\} &> 0, \end{aligned} \tag{1.14}$$

then there exists a sufficiently small positive constant $\delta_0$ such that if $0 < \delta \le \delta_0$, the Cauchy problem (1.4), (1.5), (1.6) for the one-fluid model admits a unique global strong solution $(n(t,x), u(t,x), \phi(t,x))$, which satisfies

$$C_1^{-1} \le n(t,x) \le C_1, \quad |\phi(t,x)| \le C_2 \tag{1.15}$$

for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ and

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \big|\big(n(t,x) - N_1^{R_2}(x/t), u(t,x) - U_1^{R_2}(x/t), \phi(t,x) + \ln N_1^{R_2}(x/t)\big)\big| = 0. \tag{1.16}$$

Here $\delta_0$, $C_1 \ge 1$, and $C_2$ are some positive constants depending only on $\inf\limits_{x\in\mathbb{R}}\{n_0(x)\}$, $\|(\partial_x n_0, \partial_x u_0)\|_{L^2(\mathbb{R})}$, and $\big\|\big(n_0 - n_0^R, u_0 - u_0^R\big)\big\|_{L^2(\mathbb{R})}$.

### 1.2 Main Result for the Two-Fluid Model

For the two-fluid case, the construction of the rarefaction wave is slightly different. More precisely, the large-time behaviors of its global solutions $(n_\alpha(t,x), n_\alpha(t,x), \phi(t,x))(\alpha =$

$i, e)$ can be completely described by the structure of the unique entropy solution $(N^r(x/t), U^r(x/t), \Phi^r(x/t))$ of the following Riemann problem of the quasineutral Euler system

$$\begin{aligned} \partial_t n + \partial_x (nu) &= 0, \\ n(\partial_t u + u\partial_x u) + \frac{T_i + T_e}{m_i + m_e}\partial_x n &= 0, \\ \phi &= \frac{T_i m_e - T_e m_i}{m_i + m_e}\ln n \end{aligned} \tag{1.17}$$

with Riemann initial data

$$(n(0,x), u(0,x)) = \left(n_0^R(x), u_0^R(x)\right) = \begin{cases} (n_-, u_-), & x < 0, \\ (n_+, u_+), & x > 0. \end{cases} \tag{1.18}$$

If $\phi_\pm$ satisfies the compatibility condition

$$\phi_\pm = \frac{T_i m_e - T_e m_i}{m_i + m_e}\ln n_\pm, \tag{1.19}$$

then one can easily deduce that the unique global entropy solution $(N^r(x/t), U^r(x/t), \Phi^r(x/t))$ of the Riemann problem (1.17), (1.18) is the rarefaction wave $\left(N_2^{R_2}(x/t), U_2^{R_2}(x/t), \Phi_2^{R_2}(x/t)\right)$ of the second family whose precise definition can be given by

$$\begin{aligned} U_2^{R_2}(x/t) &= w^R(x/t) - \sqrt{\frac{T_i + T_e}{m_i + m_e}}, \\ N_2^{R_2}(x/t) &= n_- \exp\left(\frac{U_1^{R_2}(x/t) - u_-}{\sqrt{\frac{T_i + T_e}{m_i + m_e}}}\right), \\ \Phi_2^{R_2}(x/t) &= \frac{T_i m_e - T_e m_i}{m_i + m_e}\ln N_2^{R_2}(x/t), \end{aligned} \tag{1.20}$$

where $w^R(x/t)$ is given by (1.13).

It is straightforward to check that

$$\begin{aligned} & \left(N_i^{R_2}(x/t), U_i^{R_2}(x/t), N_e^{R_2}(x/t), U_e^{R_2}(x/t), \Phi^{R_2}(x/t)\right) \\ = & \left(N_2^{R_2}(x/t), U_2^{R_2}(x/t), N_2^{R_2}(x/t), U_2^{R_2}(x/t), \Phi_2^{R_2}(x/t)\right) \end{aligned} \tag{1.21}$$

is a rarefaction wave solution of the following Riemann problem

$$\begin{aligned} \partial_t n_i + \partial_x (n_i u_i) &= 0, \\ m_i n_i (\partial_t u_i + u_i \partial_x u_i) + T_i \partial_x n_i - n_i \partial_x \phi &= 0, \\ \partial_t n_e + \partial_x (n_e u_e) &= 0, \\ m_e n_e (\partial_t u_e + u_e \partial_x u_e) + T_e \partial_x n_e + n_e \partial_x \phi &= 0 \end{aligned} \tag{1.22}$$

with Riemann initial data

$$\begin{aligned}(n_i(0,x),u_i(0,x),n_e(0,x),u_e(0,x)) &= \left(n_{i0}^R(x),u_{i0}^R(x),n_{e0}^R(x),u_{e0}^R(x)\right)\\ &= \begin{cases}(n_-,u_-,n_-,u_-), & x>0,\\ (n_+,u_+,n_+,u_+), & x>0.\end{cases}\end{aligned} \tag{1.23}$$

For the Cauchy problem (1.1), (1.2), (1.3) of the two-fluid model, our main result on the nonlinear stability of the wave pattern $\left(N_i^{R_2}(x/t),U_i^{R_2}(x/t),N_e^{R_2}(x/t),U_e^{R_2}(x/t)\right)$ given by (1.21) can be stated as in the following theorem.

**Theorem 1.2** Let $\delta := |n_+ - n_-| + |u_+ - u_-|$ and assume that the following conditions hold:

(i) define $\bar{R}_2(n_-,n_+)$ by (1.9) with $A=\frac{T_i+T_e}{m_i+m_e}-1$ and assume

$$(n_+,u_+)\in \overline{R}_2(n_-,u_-),\quad \phi_\pm=\frac{T_im_e-T_em_i}{m_i+m_e}\ln n_\pm;$$

(ii) the initial data $(n_{i0}(x),u_{i0}(x),n_{e0}(x),u_{e0}(x),\phi_0(x))$ satisfy

$$\begin{aligned}\left(n_{\alpha0}(x)-n_0^R(x),u_{\alpha0}(x)-u_0^R(x)\right) &\in L^2(\mathbb{R}),\quad \alpha=i,e,\\ (\partial_xn_{\alpha0}(x),\partial_xu_{\alpha0}(x)) &\in L^2(\mathbb{R}),\quad \alpha=i,e,\\ \frac{\hat{n}_{i0}(\xi)-\hat{n}_{e0}(\xi)}{|\xi|} &\in L^2(\mathbb{R}),\\ \inf_{x\in\mathbb{R}}\left\{n_{\alpha0}(x)\right\} &> 0,\quad \alpha=i,e,\end{aligned} \tag{1.24}$$

where $\hat{n}_{i0}(\xi)$ and $\hat{n}_{e0}(\xi)$ denote the Fourier transform of $n_{i0}(x)$ and $n_{e0}(x)$, respectively.

Then there exists a sufficiently small positive constant $\delta_0$, which depends only on $\|(\partial_xn_{\alpha0},\partial_xu_{\alpha0},\partial_x\phi_0)\|_{L^2(\mathbb{R})}$, $\inf_{x\in\mathbb{R}}\{n_{\alpha0}(x)\}$, and $\left\|\left(n_{\alpha0}-n_0^R,u_{\alpha0}-u_0^R\right)\right\|_{L^2(\mathbb{R})}$, such that if $0<\delta\le\delta_0$, the Cauchy problem (1.1), (1.2), (1.3) of the two-fluid model admits a unique global strong solution $(n_\alpha(t,x),u_\alpha(t,x),\phi(t,x))$ which satisfies

(i) There exists a positive constant $C_3\ge1$ depending only on $\|(\partial_xn_{\alpha0},\partial_xu_{\alpha0},\partial_x\phi_0)\|_{L^2(\mathbb{R})}$, $\inf\limits_{x\in\mathbb{R}}\{n_{\alpha0}(x)\}$, and $\left\|\left(n_{\alpha0}-n_0^R,u_{\alpha0}-u_0^R\right)\right\|_{L^2(\mathbb{R})}$ such that

$$C_3^{-1}\le n_\alpha(t,x)\le C_3 \tag{1.25}$$

holds for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$ and $\alpha=i,e$;

(ii) The wave pattern $\left(N_i^{R_2}(x/t),U_i^{R_2}(x/t),N_e^{R_2}(x/t),U_e^{R_2}(x/t),\Phi_2^{R_2}(x/t)\right)$ given by (1.21) are time-asymptotically nonlinear stable in the sense that

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}\left|\left(n_\alpha(t,x)-N_2^{R_2}(x/t),u_\alpha(t,x)-U_2^{R_2}(x/t)\right)\right|=0,\quad \alpha=i,e \tag{1.26}$$

and

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}\left|\partial_x\left(\phi(t,x)-\frac{T_im_e-T_em_i}{m_i+m_e}\ln N_2^{R_2}(x/t)\right)\right|=0. \tag{1.27}$$

Several remarks concerning our main results Theorem 1.1 and Theorem1.2 are listed below:

(i) Our main results hold for large initial perturbation which extend the previous ones obtained in [2].

(ii) Compared with the Cauchy problem (1.1), (1.2), (1.3) of the two-fluid model, the result obtained in Theorem1.1 for the Cauchy problem (1.4), (1.5), (1.6) of the one-fluid model is better than the one for the two-fluid model. In fact, an extra bound for the self-consistent electric potential $\phi(t,x)$ is derived through the damping effect induced by the Boltzmann relation $n_e = e^{-\phi}$. Moreover, we can obtain the large time behavior of the electric potential $\phi(t,x)$.

(iii) Although in our main results only the rarefaction wave of the second family is shown to be time-asymptotically nonlinear stable for both the Cauchy problem (1.1), (1.2), (1.3) of the two-fluid model and the Cauchy problem (1.4), (1.5), (1.6) of the one-fluid model, similar results also hold for the wave pattern consisting of rarefaction wave of the first family only and the composite wave pattern consisting of rarefaction wave of the first family and rarefaction wave of the second family.

(iv) It is worth to pointing out that the nonlinear stability of ion-acoustic shock profile with small initial perturbation has been established in [4] for the Cauchy problem (1.4), (1.5), (1.6) of the one-fluid model, we are convinced that similar stability result holds also for a class of large initial perturbation with large density oscillation.

(v) For the corresponding non-isentropic NSP system, some results on the nonlinear stability of rarefaction waves, viscous contact discontinuity, and/or their superposition with small initial perturbation have been obtained in [3, 7]. It would be interesting to obtain the corresponding nonlinear stability results with large initial perturbation.

The details of the proofs of Theorem1.1 and Theorem 1.2 will be given in [8].

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