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ON SPLIT REGULAR BIHOM-LIE COLOR ALGEBRAS

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Abstract: The aim of this article is to study the structure of split regular BiHom-Lie color algebras. By developing techniques of connections of roots for this kind of algebras, we show that such a split regular BiHom-Lie color algebra L is of the form $L = U + \sum\limits_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$ with U a subspace of the abelian (graded) subalgebra H and any $I_{[\alpha]}$, a well described (graded) ideal of L, satisfying $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$. Under certain conditions, in the case of L being of maximal length, the simplicity of the algebra is characterized and it is shown that L is the direct sum of the family of its simple (graded) ideals.

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1 Introduction

The origin of Hom-structures can be found in the physics literature around 1990, appearing in the study of q-deformations of algebras of vector fields, especially Witt and Virasoro algebras, see for instance [1–3]. So far, many authors have studied Hom-type algebras [4–11]. A BiHom-algebra is an algebra in such a way that the identities defining the structure are twisted by two homomorphisms ϕ, ψ. The notion of BiHom-Lie algebras was introduced in [12], which is intimately related to both Lie algebras and Hom-Lie algebras. The case of $\phi = \psi = \mathrm{Id}$ implies BiHom-Lie algebras are Lie algebras and the other case of $\phi = \psi$ give Hom-Lie algebras. The notion of Lie color algebras was introduced as generalized Lie algebras in 1960 by Ree [13]. In particular, BiHom-Lie color algebras are defined as an extension of BiHom-Lie (super)algebras to Γ-graded algebras, where Γ is any abelian group.

As is well-known, the class of the split algebras is specially related to addition quantum numbers, graded contractions and deformations. Recently, the structure of different classes of

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split algebras have been studied by using techniques of connections of roots (see for instance [14–23]). In the present paper we introduce the class of split BiHom-Lie color algebras of arbitrary dimension as the natural extension of the class of split BiHom-Lie superalgebras studied in [24] and the class of split Lie color algebras studied in [19]. The purpose of this paper is to consider the structure of split regular BiHom-Lie color algebras by the techniques of connections of roots based on some work in [14, 16, 17, 22, 24].

Throughout this paper, split regular BiHom-Lie color algebras L are considered of arbitrary dimension and over an arbitrary base field $\mathbb{K}$. This paper is organized as follows. In section 2, we establish the preliminaries on split regular BiHom-Lie color algebras theory. In section 3, we show that such an arbitrary split regular BiHom-Lie color algebra L with a symmetric root system is of the form $L = U + \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$ with U a subspace of the abelian (graded) subalgebra H and any $I_{[\alpha]}$ a well described (graded) ideal of L, satisfying $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$. In section 4, we show that under certain conditions, in the case of L being of maximal length, the simplicity of the algebra is characterized and it is shown that L is the direct sum of the family of its simple (graded) ideals.

2 Preliminaries

First we recall the definitions of Lie color algebras and Hom-Lie color algebras. The following definition is well-known from the theory of graded algebra.

Definition 2.1 [10] Let Γ be an abelian group. A bi-character on Γ is a map $\varepsilon : \Gamma\times\Gamma \to \mathbb{K}\setminus\{0\}$ satisfying

(1) $\varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha) = 1$,

(2) $\varepsilon(\alpha,\beta+\gamma) = \varepsilon(\alpha,\beta)\varepsilon(\alpha,\gamma)$,

(3) $\varepsilon(\alpha+\beta,\gamma) = \varepsilon(\alpha,\gamma)\varepsilon(\beta,\gamma)$,

for all $\alpha,\beta,\gamma \in \Gamma$.

It is clear that $\varepsilon(\alpha,0) = \varepsilon(0,\alpha) = 1$ for any $\alpha \in \Gamma$, where 0 denotes the identity element of Γ.

Definition 2.2 [16] Let $L = \oplus_{g\in\Gamma} L_g$ be a Γ-graded $\mathbb{K}$-vector space. For a nonzero homogeneous element $v \in L$, denote by $\bar{v}$ the unique group element in Γ such that $v \in L_{\bar{v}}$, which will be called the homogeneous degree of v. We shall say that L is a Lie color algebra if it is endowed with a $\mathbb{K}$-bilinear map $[\cdot,\cdot] : L\times L \to L$ satisfying

(1) $[v,w] = -\varepsilon(\bar{v},\bar{w})[w,v]$, (skew-symmetry)

(2) $[v,[w,t]] = [[v,w],t] + \varepsilon(\bar{v},\bar{w})[w,[v,t]]$, (Jacobi identity)

for all homogeneous elements $v,w,t \in L$.

Lie superalgebras are examples of Lie color algebras with $\Gamma = \mathbb{Z}_2$ and $\varepsilon(i,j) = (-1)^{ij}$, for any $i,j \in \mathbb{Z}_2$. We also note that L_0 is a Lie algebra.

Definition 2.3 [10] A Hom-Lie color algebra is a quadruple $(L,[\cdot,\cdot],\phi,\varepsilon)$ consisting of a Γ-graded $\mathbb{K}$-vector space L, an even bilinear mapping $[\cdot,\cdot] : L\times L \to L$, a homomorphism ϕ and a bi-character ε on Γ satisfying

(1) $[x,y] = -\varepsilon(\bar{x},\bar{y})[y,x]$,

(2) $\varepsilon(\bar{z},\bar{x})[\phi(x),[y,z]]+\varepsilon(\bar{x},\bar{y})[\phi(y),[z,x]]+\varepsilon(\bar{y},\bar{z})[\phi(z),[x,y]]=0$,

for all homogeneous elements $x,y,z\in L$, $\bar{x},\bar{y},\bar{z}$ denote the homogeneous degree of x,y,z, respectively. Furthermore, if ϕ is an algebra automorphism, then it is said that L is a regular Hom-Lie color algebra.

Clearly Hom-Lie algebras and Lie color algebras are examples of Hom-Lie color algebras. Then we recall the definition of BiHom-Lie algebras and give the definition of BiHom-Lie color algebras.

Definition 2.4 [12] A BiHom-Lie algebra over a field $\mathbb{K}$ is a 4-tuple $(L,[\cdot,\cdot],\phi,\psi)$, where L is a $\mathbb{K}$-linear space, $[\cdot,\cdot]:L\times L\to L$ is a bilinear map and $\phi,\psi:L\to L$ are linear mappings satisfying the following conditions:

(1) $\phi\circ\psi=\psi\circ\phi$,

(2) $[\psi(x),\phi(y)]=-[\psi(y),\phi(x)]$, (BiHom-skew-symmetry)

(3) $[\psi^2(x),[\psi(y),\phi(z)]]+[\psi^2(y),[\psi(z),\phi(x)]]+[\psi^2(z),[\psi(x),\phi(y)]]=0$, (BiHom-Jacobi identity)

for any $x,y,z\in L$.

Definition 2.5 A BiHom-Lie color algebra L is a quintuple $(L,[\cdot,\cdot],\phi,\psi,\varepsilon)$ consisting of a Γ-graded space L, an even bilinear mapping $[\cdot,\cdot]:L\times L\to L$, two homomorphisms ϕ,ψ and a bi-character ε on Γ satisfying

(1) $\phi\circ\psi=\psi\circ\phi$,

(2) $[\psi(x),\phi(y)]=-\varepsilon(\bar{x},\bar{y})[\psi(y),\phi(x)]$, (BiHom-skew-symmetry)

(3) $\varepsilon(\bar{z},\bar{x})[\psi^2(x),[\psi(y),\phi(z)]]+\varepsilon(\bar{x},\bar{y})[\psi^2(y),[\psi(z),\phi(x)]]+\varepsilon(\bar{y},\bar{z})[\psi^2(z),[\psi(x),\phi(y)]]=0$, (BiHom-Jacobi identity)

for all homogeneous elements $x,y,z\in L$, $\bar{x},\bar{y},\bar{z}$ denote the homogeneous degree of x,y,z, respectively. Furthermore, if ϕ,ψ are algebra automorphism, then it is said that L is a regular BiHom-Lie color algebra.

Lie color algebra are examples of BiHom-Lie color algebras by taking $\phi=\psi=\mathrm{Id}$. Hom-Lie color algebras are also examples of BiHom-Lie color algebras by considering $\psi=\phi$.

Example 2.6 Let $(L,[\cdot,\cdot])$ be a Lie color algebra, $\phi,\psi:L\to L$ two automorphisms and $\phi\circ\psi=\psi\circ\phi$. If we endow the underlying linear space L with a new product $[\cdot,\cdot]^{'}:L\times L\to L$ defined by $[x,y]^{'}:=[\phi(x),\psi(y)]$ for any $x,y\in L$, then we have that $(L,[\cdot,\cdot]^{'},\phi,\psi)$ becomes a regular BiHom-Lie color algebra.

Throughout this paper we will consider a regular BiHom-Lie color algebra L being of arbitrary dimension and over an arbitrary base field $\mathbb{K}$. $\mathbb{N}$ denotes the set of all non-negative integers and $\mathbb{Z}$ denotes the set of all integers. The usual regularity concepts will be understood in the graded sense. For instance, a subalgebra A of L is a graded subspace $A=\oplus_{g\in\Gamma}A_g$ such that $[A,A]\subset A$ and $\phi(A)=\psi(A)=A$. A graded subspace $I=\oplus_{g\in\Gamma}I_g$ of L is called an ideal if $[I,L]+[L,I]\subset I$ and $\phi(I)=\psi(I)=I$. A BiHom-Lie color algebra L will be called simple if $[L,L]\neq 0$ and its only (graded) ideals are $\{0\}$ and L.

We introduce the concept of split regular BiHom-Lie color algebra in an analogous way. We begin by considering a maximal abelian graded subalgebra $H=\oplus_{g\in\Gamma}H_g$ among the abelian graded subalgebras of L. We observe that H is necessarily a maximal abelian subalgebra of L as the following lemma shows.

Lemma 2.7 Let $H=\oplus_{g\in\Gamma}H_g$ be a maximal abelian graded subalgebra of a BiHom-Lie color algebra L. Then H is a maximal abelian subalgebra of L.

Proof We consider an abelian subalgebra K of L such that $H\subset K$. For any $x\in K$ we have $[x,H_g]=0$ for each $g\in\Gamma$, and so by writing $x=\sum_{i=1}^{n}x_{g_i}$ with $x_{g_i}\in L_{g_i}$ for $i=1,\cdots,n$, being $g_i\in\Gamma$ and $g_i\neq g_j$ if $i\neq j$, by the grading we get $[x_{g_i},H_g]=0$. Hence, for any g_i, $i=1,\cdots n$, we have $(H_{g_i}+\mathbb{K}x_{g_i})\oplus(\oplus_{g\in\Gamma\setminus\{g_i\}}H_g)$ is an abelian graded subalgebra of L containing H and so $x_{g_i}\in H_{g_i}$. From here we get $x\in H$ and then $K=H$.

Let us introduce the class of split algebras in the framework of regular Lie color algebras L. First, we recall that a Lie color algebra $(L,[\cdot,\cdot])$, over a base field $\mathbb{K}$, is called split respect to a maximal abelian subalgebra H of L, if L can be written as the direct sum

$$L=H\oplus(\oplus_{\alpha\in\Delta}L_\alpha)$$

where

$$L_\alpha=\{v_\alpha\in L:[h_0,v_\alpha]=\alpha(h_0)v_\alpha,\ \text{for any } h_0\in H_0\},$$

for a nonzero linear functional α on H_0 such that $L_\alpha\neq 0$.

We introduce the concept of a split regular BiHom-Lie color algebra in an analogous way.

Definition 2.8 We denote by $H=\oplus_{g\in\Gamma}H_g$ a maximal abelian (graded) subalgebra, of a regular BiHom-Lie color algebra L. For a linear functional $\alpha:H_0\to\mathbb{K}$, we define the root space of L (with respect to H) associated to α as the subspace

$$L_\alpha=\{v_\alpha\in L:[h_0,\phi(v_\alpha)]=\alpha(h_0)\phi\psi(v_\alpha),\ \text{for any } h_0\in H_0\}.$$

The elements $\alpha:H_0\to\mathbb{K}$ satisfying $L_\alpha\neq 0$ are called roots of L with respect to H. We denote $\Lambda:=\{\alpha\in(H_0)^*\setminus\{0\}:L_\alpha\neq 0\}$. We say that L is a split regular BiHom-Lie color algebra, with respect to H, if

$$L=H\oplus(\oplus_{\alpha\in\Lambda}L_\alpha).$$

We also say that Λ is the root system of L.

Noting that when $\phi=\psi=\mathrm{Id}$, the split Lie color algebras become examples of split regular BiHom-Lie color algebras and when $\phi=\psi$, the split regular Hom-Lie color algebras become examples of split regular BiHom-Lie color algebras. Hence, the present paper extends the results in [19, 22]. Let us see another example.

Example 2.9 Let $(L=H\oplus(\oplus_{\alpha\in\Delta}L_\alpha),[\cdot,\cdot])$ be a split Lie color algebra, $\phi,\psi:L\to L$ two automorphisms such that $\phi(H)=\psi(H)=H$ and $\phi\circ\psi=\psi\circ\phi$. By the Example 2.6, we know that $(L,[\cdot,\cdot]',\phi,\psi)$, where $[x,y]':=[\phi(x),\psi(y)]$ for any element $x,y\in L$, is a regular BiHom-Lie color algebra. Then it is straightforward to verify that the direct sum

$$L=H\oplus(\oplus_{\alpha\in\Delta}L_{\alpha\psi^{-1}})$$

makes of the regular BiHom-Lie color algebra $(L, [\cdot,\cdot]^{'}, \phi, \psi)$ a split regular BiHom-Lie color algebra, being the root system $\Lambda = \{\alpha\psi^{-1} : \alpha \in \Delta\}$.

From now on $L = H \oplus (\oplus_{\alpha\in\Lambda} L_\alpha)$ denotes a split regular BiHom-Lie color algebras. Also, and for an easier notation, the mappings $\phi|_H, \psi|_H, \phi|_H^{-1}, \psi|_H^{-1} : H \to H$ will be denoted by ϕ, ψ, ϕ^{-1}, ψ^{-1} respectively.

It is clear that the root space associated to the zero root L_0 satisfies $H \subset L_0$. Conversely, given any $v_0 \in L_0$ we can write

$$v_0 = h \oplus (\oplus_{i=1}^{n} v_{\alpha_i}),$$

where $h \in H$ and $v_{\alpha_i} \in L_{\alpha_i}$ for $i = 1, \cdots, n$, with $\alpha_i \neq \alpha_j$ if $i \neq j$. Hence

$$0 = [h_0, h \oplus (\oplus_{i=1}^{n} v_{\alpha_i})] = \oplus_{i=1}^{n} \alpha_i(h_0)\psi(v_{\alpha_i}),$$

for any $h_0 \in H_0$. So taking into account the direct character of the sum and that $\alpha_i \neq 0$ gives us $v_{\alpha_i} = 0$ for $i = 1, \cdots, n$. So $v_0 = h \in H$. Consequently,

$$H = L_0. \tag{2.1}$$

Lemma 2.10 Let $L = \oplus_{g\in\Gamma} L_g$ be a split BiHom-Lie color algebra with corresponding root space decomposition $L = H \oplus (\oplus_{\alpha\in\Lambda} L_\alpha)$. If we denote by $L_{\alpha,g} = L_\alpha \cap L_g$, then the following assertions hold.

(1) $L_\alpha = \oplus_{g\in\Gamma} L_{\alpha,g}$ for any $\alpha \in \Lambda \cup \{0\}$.

(2) $H_g = L_{0,g}$. In particular $H_0 = L_{0,0}$.

(3) L_0 is a split BiHom-Lie algebra, respect to H_0, with root space decomposition $L_0 = H_0 \oplus (\oplus_{\alpha\in\Lambda} L_{\alpha,0})$.

Proof (1) By the Γ-grading of L we may express any $v_\alpha \in L_\alpha$, $\alpha \in \Lambda \cup \{0\}$, in the form $v_\alpha = v_{\alpha,g_1} + \cdots + v_{\alpha,g_n}$ with $v_{\alpha,g_i} \in L_{g_i}$ for distinct $g_1, \cdots, g_n \in \Gamma$. If $h_0 \in H_0$ then $[h_0, \phi(v_{\alpha,g_i})] = \alpha(h_0)\phi\psi(v_{\alpha,g_i})$ for $i = 1, \cdots, n$. Hence $L_\alpha = \oplus_{g\in\Gamma}(L_\alpha \cap L_g)$ and we can write $L_\alpha = \oplus_{g\in\Gamma} L_{\alpha,g}$ for any $\alpha \in \Lambda \cup \{0\}$.

(2) Consequence of (2.1) and item 1.

(3) We also have $L_g = H_g \oplus (\oplus_{\alpha\in\Lambda} L_{\alpha,g})$ for any $g \in \Gamma$. By considering $g = 0$ we get $L_0 = H_0 \oplus (\oplus_{\alpha\in\Lambda} L_{\alpha,0})$. Hence, the direct character of the sum and the fact that $\alpha \neq 0$ for any $\alpha \in \Lambda$ gives us that H_0 is a maximal abelian subalgebra of the BiHom-Lie algebra L_0. Hence L_0 is a split BiHom-Lie algebra respect to H_0.

Lemma 2.11 For any $\alpha \in \Lambda \cup \{0\}$, the following assertions hold.

(1) $\phi(L_\alpha) = L_{\alpha\phi^{-1}}$ and $\phi^{-1}(L_\alpha) = L_{\alpha\phi}$.

(2) $\psi(L_\alpha) = L_{\alpha\psi^{-1}}$ and $\psi^{-1}(L_\alpha) = L_{\alpha\psi}$.

Proof (1) For any $h_0 \in H_0$ and $v_\alpha \in L_\alpha$, since

$$[h_0, \phi(v_\alpha)] = \alpha(h_0)\phi\psi(v_\alpha), \tag{2.2}$$

we have that by writing $h_0^{'} = \phi(h_0)$, then

$$\begin{aligned}[h_0^{'}, \phi^2(v_\alpha)] &= \phi([h_0, \phi(v_\alpha)]) = \alpha(h_0)\phi^2\psi(v_\alpha) \\ &= \alpha\phi^{-1}(h_0^{'})\phi^2\psi(v_\alpha) = \alpha\phi^{-1}(h_0^{'})\phi\psi(\phi(v_\alpha)).\end{aligned}$$

Therefore we get $\phi(v_\alpha) \in L_{\alpha\phi^{-1}}$ and so

$$\phi(L_\alpha) \subset L_{\alpha\phi^{-1}}. \tag{2.3}$$

Now, let us show $L_{\alpha\phi^{-1}} \subset \phi(L_\alpha)$. Indeed, for any $h_0 \in H_0$ and $v_\alpha \in L_\alpha$, (2.2) shows $[\phi^{-1}(h_0), v_\alpha] = \alpha(h_0)\psi(v_\alpha)$. From here, we get $[\phi(h_0), v_\alpha] = \alpha\phi^2(h_0)\psi(v_\alpha)$ and

$$\phi^{-1}(L_\alpha) \subset L_{\alpha\phi}. \tag{2.4}$$

Hence, since for any $x \in L_{\alpha\phi^{-1}}$ we can write $x = \phi(\phi^{-1}(x))$ and by (2.4) we have $\phi^{-1}(x) \in L_\alpha$ and $L_{\alpha\phi^{-1}} \subset \phi(L_\alpha)$. This fact together with (2.3) show $\phi(L_\alpha) = L_{\alpha\phi^{-1}}$.

To show $\phi^{-1}(L_\alpha) = L_{\alpha\phi}$, the fact $\phi^{-1}(L_\alpha) \subset L_{\alpha\phi}$ is (2.4), while the fact $L_{\alpha\phi} \subset \phi^{-1}(L_\alpha)$ is consequence of writing any element $x \in L_{\alpha\phi}$ of the form $x = \phi^{-1}(\phi(x))$ and applying (2.3).

(2) To verify

$$\psi(L_\alpha) \subset L_{\alpha\psi^{-1}}, \tag{2.5}$$

we observe that (2.2) gives us $[\psi(h_0), \psi\phi(v_\alpha)] = \alpha(h_0)\psi\phi\psi(v_\alpha)$, and so $[\psi(h_0), \phi\psi(v_\alpha)] = \alpha\psi^{-1}(\psi(h_0))\phi\psi(\psi(v_\alpha))$. Since (2.2) and the identity $\psi^{-1}\phi = \phi\psi^{-1}$ also gives us

$$\psi^{-1}(L_\alpha) \subset L_{\alpha\psi}, \tag{2.6}$$

we conclude as above that $\psi(L_\alpha) = L_{\alpha\psi^{-1}}$. We can argue similarly with (2.5) and (2.6) to get $\psi^{-1}(L_\alpha) = L_{\alpha\psi}$.

Lemma 2.12 For any α, $\beta \in \Lambda \cup \{0\}$, we have $[L_\alpha, L_\beta] \subset L_{\alpha\phi^{-1}+\beta\psi^{-1}}$.

Proof For each $h_0 \in H_0$, $v_\alpha \in L_\alpha$ and $v_\beta \in L_\beta$, we can write

$$[h_0, \phi([v_\alpha, v_\beta])] = [\psi^2\psi^{-2}(h_0), \phi([v_\alpha, v_\beta])].$$

So, by denoting $h_0^{'} = \psi^{-2}(h_0)$, we can apply BiHom-Jacobi identity and BiHom-skew-symmetry to get

$$\begin{aligned}
&[\psi^2(h_0^{'}), \phi([v_\alpha, v_\beta])] \\
&= [\psi^2(h_0^{'}), [\psi\psi^{-1}\phi(v_\alpha), \phi(v_\beta)]] \\
&= -\varepsilon(\bar{h_0^{'}}, \bar\alpha + \bar\beta)[\psi\phi(v_\alpha), [\psi(v_\beta), \phi(h_0^{'})]] - \varepsilon(\bar\alpha + \bar{h_0^{'}}, \bar\beta)[\psi^2(v_\beta), [\psi(h_0^{'}), \phi\psi^{-1}\phi(v_\alpha)]] \\
&= -\varepsilon(\bar{h_0^{'}}, \bar\alpha + \bar\beta)(-\varepsilon(\bar\beta, \bar{h_0^{'}}))[\psi\phi(v_\alpha), [\psi(h_0^{'}), \phi(v_\beta)]] - \varepsilon(\bar\alpha + \bar{h_0^{'}}, \bar\beta)[\psi^2(v_\beta), [\psi(h_0^{'}), \phi\psi^{-1}\phi(v_\alpha)]] \\
&= \varepsilon(\bar{h_0^{'}}, \bar\alpha)[\psi\phi(v_\alpha), [\psi(h_0^{'}), \phi(v_\beta)]] - \varepsilon(\bar\alpha + \bar{h_0^{'}}, \bar\beta)[\psi^2(v_\beta), \phi[\phi^{-1}\psi(h_0^{'}), \psi^{-1}\phi(v_\alpha)]] \\
&= \varepsilon(\bar{h_0^{'}}, \bar\alpha)[\psi\phi(v_\alpha), [\psi(h_0^{'}), \phi(v_\beta)]] - \varepsilon(\bar\alpha + \bar{h_0^{'}}, \bar\beta)(-\varepsilon(\bar\beta, \bar\alpha + \bar{h_0^{'}}))[\psi[\phi^{-1}\psi(h_0^{'}), \psi^{-1}\phi(v_\alpha)], \phi\psi(v_\beta)] \\
&= \varepsilon(\bar{h_0^{'}}, \bar\alpha)[\psi\phi(v_\alpha), [\psi(h_0^{'}), \phi(v_\beta)]] + [[\psi^2\phi^{-1}(h_0^{'}), \phi(v_\alpha)], \phi\psi(v_\beta)] \\
&= \varepsilon(\bar{h_0^{'}}, \bar\alpha)\beta\psi(h_0^{'})[\psi\phi(v_\alpha), \phi\psi(v_\beta)] + \alpha(\psi^2\phi^{-1}(h_0^{'}))[\phi\psi(v_\alpha), \phi\psi(v_\beta)] \\
&= \varepsilon(\bar{h_0^{'}}, \bar\alpha)(\beta\psi + \alpha\psi^2\phi^{-1})(h_0^{'})[\psi\phi(v_\alpha), \phi\psi(v_\beta)] \\
&= \varepsilon(\bar{h_0^{'}}, \bar\alpha)(\beta\psi + \alpha\psi^2\phi^{-1})(h_0^{'})[\phi\psi(v_\alpha), \phi\psi(v_\beta)] \\
&= \varepsilon(\bar{h_0^{'}}, \bar\alpha)(\beta\psi + \alpha\psi^2\phi^{-1})(h_0^{'})\phi\psi([v_\alpha, v_\beta]) \\
&= (\beta\psi + \alpha\psi^2\phi^{-1})(h_0^{'})\phi\psi([v_\alpha, v_\beta]).
\end{aligned}$$

Taking into account $h_0^{'} = \psi^{-2}(h_0)$ we have shown that

$$[h_0, \phi([v_\alpha, v_\beta])] = (\beta\psi^{-1} + \alpha\phi^{-1})(h_0)\phi\psi([v_\alpha, v_\beta]).$$

From here, $[L_\alpha, L_\beta] \subset L_{\alpha\phi^{-1}+\beta\psi^{-1}}$.

From Lemma 2.12 we can assert that

$$[L_{\alpha,g_1}, L_{\beta,g_2}] \subset L_{\alpha\phi^{-1}+\beta\psi^{-1},g_1+g_2}$$

for any $g_1, g_2 \in \Gamma$.

Lemma 2.13 If $\alpha \in \Lambda$, then $\alpha\phi^{-z_1}\psi^{-z_2} \in \Lambda$ for any $z_1, z_2 \in \mathbb{Z}$.

Proof This is a consequence of Lemma 2.11 (1) and (2).

Definition 2.14 A root system Λ of a split BiHom-Lie color algebra is called symmetric if it satisfies that $\alpha \in \Lambda$ implies $-\alpha \in \Lambda$.

3 Decompositions

In the following, let L be a split regular BiHom-Lie color algebra with a symmetric root system Λ and $L = H \oplus (\oplus_{\alpha\in\Lambda} L_\alpha)$ the corresponding root decomposition. We begin by developing the techniques of connections of roots in this section.

Definition 3.1 Let α and β be two nonzero roots. We shall say that α is connected to β if there exists $\alpha_1, \cdots, \alpha_k \in \Lambda$ such that

If $k = 1$, then $\alpha_1 \in \{\alpha\phi^{-n}\psi^{-r} : n, r \in \mathbb{N}\} \cap \{\pm\beta\phi^{-m}\psi^{-s} : m, s \in \mathbb{N}\}$.

If $k \geq 2$, then

(1) $\alpha_1 \in \{\alpha\phi^{-n}\psi^{-r} : n, r \in \mathbb{N}\}$.

(2) $\alpha_1\phi^{-1} + \alpha_2\psi^{-1} \in \Lambda$,

$$\begin{aligned}&\alpha_1\phi^{-2} + \alpha_2\phi^{-1}\psi^{-1} + \alpha_3\psi^{-1} \in \Lambda,\\ &\quad\vdots\\ &\alpha_1\phi^{-i} + \alpha_2\phi^{-i+1}\psi^{-1} + \alpha_3\phi^{-i+2}\psi^{-1} + \cdots + \alpha_i\phi^{-1}\psi^{-1} + \alpha_{i+1}\psi^{-1} \in \Lambda,\\ &\quad\vdots\\ &\alpha_1\phi^{-k+2} + \alpha_2\phi^{-k+3}\psi^{-1} + \alpha_3\phi^{-k+4}\psi^{-1} + \cdots + \alpha_{k-2}\phi^{-1}\psi^{-1} + \alpha_{k-1}\psi^{-1} \in \Lambda.\end{aligned}$$

(3) $\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+2}\psi^{-1} + \alpha_3\phi^{-k+3}\psi^{-1} + \cdots + \alpha_i\phi^{-k+i}\psi^{-1} + \cdots + \alpha_{k-1}\phi^{-1}\psi^{-1} + \alpha_k\psi^{-1} \in \{\pm\beta\phi^{-m}\psi^{-s} : m, s \in \mathbb{N}\}$.

We shall also say that $\{\alpha_1, \cdots, \alpha_k\}$ is a connection from α to β.

Our next goal is to show that the connection is an equivalence relation on Λ.

Proposition 3.2 The relation $\sim$ in Λ, defined by $\alpha \sim \beta$ if and only if α is connected to β, is an equivalence relation.

Proof This can be proved completely analogously to [14, Corollary 2.1].

For any $\alpha \in \Lambda$, we denote by

$$\Lambda_\alpha := \{\beta \in \Lambda : \beta \sim \alpha\}.$$

Clearly if $\beta \in \Lambda_\alpha$ then $-\beta \in \Lambda_\alpha$ and, by Proposition , if $\gamma \notin \Lambda_\alpha$ then $\Lambda_\alpha \cap \Lambda_\gamma = \emptyset$.

Our next goal is to associate an adequate ideal L_{Λ_α} of L to any Λ_α. For Λ_α, $\alpha \in \Lambda$, we define $H_{\Lambda_\alpha} := \mathrm{span}_{\mathbb{K}}\{[L_{\beta\psi^{-1}}, L_{-\beta\phi^{-1}}] : \beta \in \Lambda_\alpha\}$, and $V_{\Lambda_\alpha} := \oplus_{\beta\in\Lambda_\alpha} L_\beta$. We denote by L_{Λ_α} the following graded subspace of L, $L_{\Lambda_\alpha} := H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}$.

Proposition 3.3 For any $\alpha \in \Lambda$, the linear subspace L_{Λ_α} is a subalgebra of L.

Proof First we have to check that L_{Λ_α} satisfies $[L_{\Lambda_\alpha}, L_{\Lambda_\alpha}] \subset L_{\Lambda_\alpha}$. Taking into account $H = L_0$, then $[H_{\Lambda_\alpha}, H_{\Lambda_\alpha}] = 0$ and

$$[L_{\Lambda_\alpha}, L_{\Lambda_\alpha}] = [H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}, H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}] \subset [H_{\Lambda_\alpha}, V_{\Lambda_\alpha}] + [V_{\Lambda_\alpha}, H_{\Lambda_\alpha}] + \Sigma_{\beta,\gamma\in\Lambda_\alpha}[L_\beta, L_\gamma]. \quad (3.1)$$

Let us consider the first summand in (3.1). Given $\beta \in \Lambda_\alpha$, we have $[H_{\Lambda_\alpha}, L_\beta] \subset [L_0, L_\beta] \subset L_{\beta\psi^{-1}}$, being $\beta\psi^{-1} \in \Lambda_\alpha$ by Lemma 2.13. Hence,

$$[H_{\Lambda_\alpha}, V_{\Lambda_\alpha}] \subset V_{\Lambda_\alpha}. \quad (3.2)$$

Similarly, we can also get

$$[V_{\Lambda_\alpha}, H_{\Lambda_\alpha}] \subset V_{\Lambda_\alpha}. \quad (3.3)$$

We consider now the third summand $\Sigma_{\beta,\gamma\in\Lambda_\alpha}[L_\beta, L_\gamma]$. Given $\beta, \gamma \in \Lambda_\alpha$ such that $[L_\beta, L_\gamma] \neq 0$, if $\beta\phi^{-1} + \gamma\psi^{-1} = 0$, then clearly $[L_\beta, L_\gamma] \subset H_{\Lambda_\alpha}$. Supposing that $\beta\phi^{-1} + \gamma\psi^{-1} \neq 0$, since $[L_\beta, L_\gamma] \neq 0$ together with Lemma 2.12 ensures that $\beta\phi^{-1}+\gamma\psi^{-1} \in \Lambda$, we have that $\{\beta, \gamma\}$ is a connection from β to $\beta\phi^{-1}+\gamma\psi^{-1}$. The transitivity of $\sim$ gives now that $\beta\phi^{-1}+\gamma\psi^{-1} \in \Lambda_\alpha$ and so

$$[L_\beta, L_\gamma] \subset L_{\beta\phi^{-1}+\gamma\psi^{-1}} \subset V_{\Lambda_\alpha}. \quad (3.4)$$

From (3.1)-(3.4), we conclude that $[L_{\Lambda_\alpha}, L_{\Lambda_\alpha}] \subset L_{\Lambda_\alpha}$.

Secondly, we have to verify that $\phi(L_{\Lambda_\alpha}) = L_{\Lambda_\alpha}$ and $\psi(L_{\Lambda_\alpha}) = L_{\Lambda_\alpha}$. But this is a direct consequence of Lemma 2.11.

Proposition 3.4 If $\gamma \notin \Lambda_\alpha$, then $[L_{\Lambda_\alpha}, L_{\Lambda_\gamma}] = 0$.

Proof We have

$$[L_{\Lambda_\alpha}, L_{\Lambda_\gamma}] = [H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}, H_{\Lambda_\gamma} \oplus V_{\Lambda_\gamma}] \subset [H_{\Lambda_\alpha}, V_{\Lambda_\gamma}] + [V_{\Lambda_\alpha}, H_{\Lambda_\gamma}] + [V_{\Lambda_\alpha}, V_{\Lambda_\gamma}]. \quad (3.5)$$

We consider the above third summand $[V_{\Lambda_\alpha}, V_{\Lambda_\gamma}]$ and suppose that there exist $\beta \in \Lambda_\alpha$ and $\eta \in \Lambda_\gamma$ such that $[L_\beta, L_\eta] \neq 0$. As necessarily $\beta\phi^{-1} \neq -\eta\psi^{-1}$, then $\beta\phi^{-1} + \eta\psi^{-1} \in \Lambda$. So $\{\beta, \eta, -\beta\phi^{-2}\psi\}$ is a connection between β and η. By the transitivity of the connection relation we have $\gamma \in \Lambda_\alpha$, a contradiction. Hence $[L_\beta, L_\eta] = 0$ and so

$$[V_{\Lambda_\alpha}, V_{\Lambda_\gamma}] = 0. \quad (3.6)$$

We consider now the first summand $[H_{\Lambda_\alpha}, V_{\Lambda_\gamma}]$ in (3.5) and suppose there exist $\beta \in \Lambda_\alpha$ and $\eta \in \Lambda_\gamma$ such that

$$[[L_{\beta\psi^{-1}}, L_{-\beta\phi^{-1}}], \phi^2(L_\eta)] \neq 0.$$

By BiHom-skew-symmetry, $[\psi^2(L_\eta), [L_{-\beta\psi^{-1}}, L_{\beta\phi^{-1}}]] \neq 0$. Hence, there exist $i, j, k \in \Gamma$ such that $[\psi^2(L_{\eta,i}), [\psi(L_{-\beta,j}), \phi(L_{\beta,k})]] \neq 0$. By BiHom-Jacobi identity, we get either

$[\psi(L_{\beta,k}),\phi(L_{\eta,i})]\neq 0$ or $[\psi(L_{\eta,i}),\phi(L_{-\beta,j})]\neq 0$. From here $[V_{\Lambda_\alpha},V_{\Lambda_\gamma}]\neq 0$ in any case, which contradicts (3.6). Hence $[H_{\Lambda_\alpha},V_{\Lambda_\gamma}]=0$. Finally, we note that the same above argument shows, $[V_{\Lambda_\alpha},H_{\Lambda_\gamma}]=0$. By (3.5), we conclude $[L_{\Lambda_\alpha},L_{\Lambda_\gamma}]=0$.

Theorem 3.5 The following assertions hold.

(1) For any $\alpha\in\Lambda$, the subalgebra $L_{\Lambda_\alpha}=H_{\Lambda_\alpha}\oplus V_{\Lambda_\alpha}$ of L associated to Λ_α is an ideal of L.

(2) If L is simple, then there exists a connection from α to β for any $\alpha,\beta\in\Lambda$ and $H=\sum_{\alpha\in\Lambda}[L_{\alpha\psi^{-1}},L_{-\alpha\phi^{-1}}]$.

Proof (1) Since $[L_{\Lambda_\alpha},H]=[L_{\Lambda_\alpha},L_0]\subset V_{\Lambda_\alpha}$, taking into account Propositions 3.3 and 3.4, we have

$$[L_{\Lambda_\alpha},L]=[L_{\Lambda_\alpha},H\oplus(\oplus_{\beta\in\Lambda_\alpha}L_\beta)\oplus(\oplus_{\gamma\notin\Lambda_\alpha}L_\gamma)]\subset L_{\Lambda_\alpha}.$$

In a similar way we get $[L,L_{\Lambda_\alpha}]\subset L_{\Lambda_\alpha}$. Finally, by Lemma 2.11, we also have $\phi(L_{\Lambda_\alpha})=L_{\Lambda_\alpha}$ and $\psi(L_{\Lambda_\alpha})=L_{\Lambda_\alpha}$. So we conclude that L_{Λ_α} is an ideal of L.

(2) The simplicity of L implies $L_{\Lambda_\alpha}=L$. From here, it is clear that $\Lambda_\alpha=\Lambda$ and $H=\sum_{\alpha\in\Lambda}[L_{\alpha\psi^{-1}},L_{-\alpha\phi^{-1}}]$.

Theorem 3.6 For a vector space complement U of $\mathrm{span}_{\mathbb{K}}\{[L_{\alpha\psi^{-1}},L_{-\alpha\phi^{-1}}]:\alpha\in\Lambda\}$ in H, we have

$$L=U+\sum_{[\alpha]\in\Lambda/\sim}I_{[\alpha]},$$

where any $I_{[\alpha]}$ is one of the ideals L_{Λ_α} of L described in Theorem 3.5(1), satisfying $[I_{[\alpha]},I_{[\beta]}]=0$, whenever $[\alpha]\neq[\beta]$.

Proof By Proposition 3.2, we can consider the quotient set $\Lambda/\sim:=\{[\alpha]:\alpha\in\Lambda\}$. Let us denote by $I_{[\alpha]}:=L_{\Lambda_\alpha}$. We obtain that $I_{[\alpha]}$ is well defined and by Theorem 3.5(1), an ideal of L. Therefore

$$L=U+\sum_{[\alpha]\in\Lambda/\sim}I_{[\alpha]}.$$

By applying Proposition 3.4 we also obtain $[I_{[\alpha]},I_{[\beta]}]=0$ if $[\alpha]\neq[\beta]$.

Let us denote by $\mathrm{Z}(L):=\{x\in L:[x,L]+[L,x]=0\}$ the center of L.

Corollary 3.7 If $\mathrm{Z}(L)=0$ and $H=\sum_{\alpha\in\Lambda}[L_{\alpha\psi^{-1}},L_{-\alpha\phi^{-1}}]$, then L is the direct sum of the ideals given in Theorem 3.5,

$$L=\oplus_{[\alpha]\in\Lambda/\sim}I_{[\alpha]}.$$

Furthermore $[I_{[\alpha]},I_{[\beta]}]=0$, whenever $[\alpha]\neq[\beta]$.

Proof Since $H=\sum_{\alpha\in\Lambda}[L_{\alpha\psi^{-1}},L_{-\alpha\phi^{-1}}]$, we get $L=\oplus_{[\alpha]\in\Lambda/\sim}I_{[\alpha]}$. We show the direct character of the sum. Given $x\in I_{[\alpha]}\cap\sum_{\substack{[\beta]\in\Lambda/\sim\\ [\beta]\neq[\alpha]}}I_{[\beta]}$, by using again the equation $[I_{[\alpha]},I_{[\beta]}]=0$, for $[\alpha]\neq[\beta]$, we obtain

$$[x,I_{[\alpha]}]+[x,\sum_{\substack{[\beta]\in\Lambda/\sim\\ [\beta]\neq[\alpha]}}I_{[\beta]}]=0,$$

$$[I_{[\alpha]}, x] + [\sum_{\substack{[\beta]\in\Lambda/\sim \\ [\beta]\neq[\alpha]}} I_{[\beta]}, x] = 0.$$

It implies $[x, L] + [L, x] = 0$, that is, $x \in \mathrm{Z}(L) = 0$. Thus $x = 0$, as desired.

4 The Simple Components

In this section, we study the sufficient conditions for the decomposition of L into direct sums of simple ideals. Under certain conditions we give an affirmative answer.

Lemma 4.1 Let $L = H \oplus (\oplus_{\alpha\in\Lambda} L_\alpha)$ be a split regular BiHom-Lie color algebra. If I is an ideal of L, then $I = (I \cap H) \oplus (\oplus_{\alpha\in\Lambda}(I \cap L_\alpha))$.

Proof We can see $L = H \oplus (\oplus_{\alpha\in\Lambda} L_\alpha)$ as a weight module with respect to the split BiHom-Lie algebra L_0, with maximal abelian subalgebra H_0, in the natural way. The character of ideal of I gives us that I is a submodule of L. It is well-known that a submodule of a weight module is again a weight module. From here, I is a weight module with respect to L_0 (and H_0) and so $I = (I \cap H) \oplus (\oplus_{\alpha\in\Lambda}(I \cap L_\alpha))$.

Taking into account the above lemma, we observe that the grading of I and Lemma 2.10(1) let us write

$$I = \oplus_{g\in\Gamma} I_g = \oplus_{g\in\Gamma}\big((I_g \cap H_g) \oplus (\oplus_{\alpha\in\Lambda}(I_g \cap L_{\alpha,g})\big). \tag{4.1}$$

Lemma 4.2 Let L be a split regular BiHom-Lie color algebra with $\mathrm{Z}(L) = 0$ and I an ideal of L. If $I \subset H$, then $I = \{0\}$.

Proof We suppose that there exists a nonzero ideal I of L such that $I \subset H$. We get $[I, H] \subset [H, H] = 0$ and $[I, \oplus_{\alpha\in\Lambda} L_\alpha] \subset I \subset H$. Then taking into account $H = L_0$, we have $[I, \oplus_{\alpha\in\Lambda} L_\alpha] \subset H \cap (\oplus_{\alpha\in\Lambda} L_\alpha) = 0$ and $[\oplus_{\alpha\in\Lambda} L_\alpha, I] \subset (\oplus_{\alpha\in\Lambda} L_\alpha) \cap H = 0$. From here $I \subset \mathrm{Z}(L) = 0$, which is a contradiction.

Let us introduce the concepts of root-multiplicativity and maximal length in the framework of split Hom-Lie color algebras. For each $g \in \Gamma$, we denote $\Lambda_g := \{\alpha \in \Lambda : L_{\alpha,g} \neq 0\}$.

Definition 4.3 A split regular BiHom-Lie color algebra L is root-multiplicative if given $\alpha \in \Lambda_{g_i}$ and $\beta \in \Lambda_{g_j}$, with $g_i, g_j \in \Gamma$, such that $\alpha + \beta \in \Lambda$, then $[L_{\alpha,g_i}, L_{\beta,g_j}] \neq 0$.

Definition 4.4 A split regular BiHom-Lie color algebra L is of maximal length if for any $\alpha \in \Lambda_g, g \in \Gamma$, we have $\dim L_{\kappa\alpha,\kappa g} = 1$ for $\kappa \in \{\pm 1\}$.

If L is of maximal length, according to (4.1) we assert that given any nonzero ideal I of L then

$$I = \oplus_{g\in\Gamma}\big((I_g \cap H_g) \oplus (\oplus_{\alpha\in\Lambda_g^I} L_{\alpha,g})\big). \tag{4.2}$$

where $\Lambda_g^I := \{\alpha \in \Lambda : I_g \cap L_{\alpha,g} \neq 0\}$ for each $g \in \Gamma$.

Theorem 4.5 Let L be a split regular BiHom-Lie color algebra of maximal length, root multiplicative and $\mathrm{Z}(L) = 0$. Then L is simple if and only if it has all of its nonzero roots connected and $H = \sum_{\alpha\in\Lambda}[L_{\alpha\psi^{-1}}, L_{-\alpha\phi^{-1}}]$.

Proof The first implication is Theorem 3.5(2). To prove the converse, we consider I a nonzero ideal of L. By Lemma 4.2 and (4.2) we can write $I = \oplus_{g\in\Gamma}\big((I_g \cap H_g) \oplus (\oplus_{\alpha\in\Lambda_g^I} L_{\alpha,g})\big)$

with $\Lambda_g^I \subset \Lambda_g$ for any $g \in \Gamma$ and some $\Lambda_g^I \neq \emptyset$. Hence, we may choose $\alpha_0 \in \Lambda_g^I$ such that

$$0 \neq L_{\alpha_0,g} \subset I. \tag{4.3}$$

Since $\phi(I) = I$, $\psi(I) = I$, and by Lemma 2.11, we can assert that if $\alpha \in \Lambda_I$, then $\{\alpha\phi^{z_1}\psi^{z_2} : z_1, z_2 \in \mathbb{Z}\} \subset \Lambda_I$. In particular,

$$\{L_{\alpha_0\phi^{z_1}\psi^{z_2},g} : z_1, z_2 \in \mathbb{Z}\} \subset I.$$

Now, let us take any $\beta \in \Lambda$ satisfying $\beta \notin \{\pm\alpha_0\phi^{z_1}\psi^{z_2} : z_1, z_2 \in \mathbb{Z}\}$. Since α_0 and β are connected, we have a connection $\{\alpha_1, \cdots, \alpha_k\}$, $k \geq 2$, from α_0 to β satisfying:

$\alpha_1 = a_0\phi^{-n}\psi^{-r}$, for some $n, r \in \mathbb{N}$,
$\alpha_1\phi^{-1} + \alpha_2\psi^{-1} \in \Lambda$,
$\alpha_1\phi^{-2} + \alpha_2\phi^{-1}\psi^{-1} + \alpha_3\psi^{-1} \in \Lambda$,
$\vdots$
$\alpha_1\phi^{-i} + \alpha_2\phi^{-i+1}\psi^{-1} + \alpha_3\phi^{-i+2}\psi^{-1} + \cdots + \alpha_i\phi^{-1}\psi^{-1} + \alpha_{i+1}\psi^{-1} \in \Lambda$,
$\vdots$
$\alpha_1\phi^{-k+2} + \alpha_2\phi^{-k+3}\psi^{-1} + \alpha_3\phi^{-k+4}\psi^{-1} + \cdots + \alpha_{k-2}\phi^{-1}\psi^{-1} + \alpha_{k-1}\psi^{-1} \in \Lambda$,
$\alpha_1\phi^{-k+1}+\alpha_2\phi^{-k+2}\psi^{-1}+\alpha_3\phi^{-k+3}\psi^{-1}+\cdots+\alpha_i\phi^{-k+i}\psi^{-1}+\cdots+\alpha_{k-1}\phi^{-1}\psi^{-1}+\alpha_k\psi^{-1} = \epsilon\beta\phi^{-m}\psi^{-s}$ for some $m, s \in \mathbb{N}$ and $\epsilon \in \{\pm 1\}$.

Since $\alpha_2 \in \Lambda$, there exists $g_1 \in \Gamma$ such that $L_{\alpha_2,g_1} \neq 0$ and so $\alpha_2 \in \Lambda_{g_1}$. From here, we have $\alpha_1 \in \Lambda_g$ and $\alpha_2 \in \Lambda_{g_1}$, such that $\alpha_1\phi^{-1} + \alpha_2\psi^{-1} \in \Lambda_{g+g_1}$. The root-multiplicativity and maximal length of L show $0 \neq [L_{\alpha_1,g}, L_{\alpha_2,g_1}] = L_{\alpha_1\phi^{-1}+\alpha_2\psi^{-1},g+g_1}$. Since $0 \neq L_{\alpha_1,g} \subset I$ as the consequence of (4.3) we get

$$0 \neq L_{\alpha_1\phi^{-1}+\alpha_2\psi^{-1},g+g_1} \subset I.$$

We can argue in a similar way from $\alpha_1\phi^{-1} + \alpha_2\psi^{-1}$, α_3 and $\alpha_1\phi^{-2} + \alpha_2\phi^{-1}\psi^{-1} + \alpha_3\psi^{-1}$ to get

$$0 \neq L_{\alpha_1\phi^{-2}+\alpha_2\phi^{-1}\psi^{-1}+\alpha_3\psi^{-1},g_2} \subset I$$

for some $g_2 \in \Gamma$. Following this process with the connection $\{\alpha_1, \cdots, \alpha_k\}$, we obtain that

$$0 \neq L_{\alpha_1\phi^{-k+1}+\alpha_2\phi^{-k+2}\psi^{-1}+\cdots+\alpha_k\psi^{-1},g_3} \subset I$$

and so either $0 \neq L_{\beta\phi^{-m}\psi^{-s},g_3} \subset I$ or $0 \neq L_{-\beta\phi^{-m}\psi^{-s},g_3} \subset I$ for some $g_3 \in \Gamma$. That is,

$$0 \neq L_{\epsilon\beta\phi^{-m}\psi^{-s},g_3} \subset I \quad \text{for} \quad \text{some} \quad \epsilon \in \{\pm 1\}, \quad \text{some} \quad g_3 \in \Gamma$$

and for any $\beta \in \Lambda$. By Lemma 2.11, we can get

$$0 \neq L_{\epsilon\beta,g_3} \subset I \quad \text{for} \quad \text{some} \quad \epsilon \in \{\pm 1\}, \quad \text{some} \quad g_3 \in \Gamma. \tag{4.4}$$

Taking into account $H = \sum_{\beta\in\Lambda}[L_{\beta\psi^{-1}}, L_{-\beta\phi^{-1}}]$, the grading of L gives

$$H_0 = \sum_{\gamma\in\Lambda, g\in\Gamma} [\psi(L_{\gamma,g}), \phi(L_{-\gamma,-g})].$$

From here, there exists $\gamma \in \Lambda$ and $g_4 \in \Gamma$ such that

$$[[\psi(L_{\gamma,g_4}),\phi(L_{-\gamma,-g_4})],\psi^2\psi^{-2}\phi(L_{\epsilon\beta,g_3})] \neq 0. \tag{4.5}$$

By the BiHom-Jacobi identity either

$$[\psi(L_{-\gamma,-g_4}),\phi(\psi^{-2}\phi(L_{\epsilon\beta,g_3}))] \neq 0 \quad \text{or} \quad [\psi(\psi^{-2}\phi(L_{\epsilon\beta,g_3})),\phi(L_{\gamma,g_4})] \neq 0$$

and so

$$L_{-\gamma\psi^{-1}\phi^{-1}+\epsilon\beta\psi\phi^{-2},-g_4+g_3} \neq 0 \quad \text{or} \quad L_{\gamma\psi^{-1}\phi^{-1}+\epsilon\beta\psi\phi^{-2},g_4+g_3} \neq 0.$$

That is

$$0 \neq L_{\kappa\gamma\psi^{-1}\phi^{-1}+\epsilon\beta\psi\phi^{-2},\kappa g_4+g_3} \subset I \tag{4.6}$$

for some $\kappa \in \{\pm 1\}$. Since $\epsilon\beta \in \Lambda_{g_3}$, by the maximal length of L we have $-\epsilon\beta \in \Lambda_{-g_3}$. By (4.6) and the root-multiplicativity and maximal length of L we obtain

$$0 \neq [L_{\kappa\gamma\psi^{-1}\phi^{-1}+\epsilon\beta\psi\phi^{-2},\kappa g_4+g_3}, L_{-\epsilon\beta\phi^{-3}\psi^{2},-g_3}] = L_{\kappa\gamma\psi^{-1}\phi^{-2},\kappa g_4} \subset I. \tag{4.7}$$

By Lemma 2.11(1), we can get

$$L_{\kappa\gamma,\kappa g_4} \subset I. \tag{4.8}$$

Taking into account (4.7) and (4.5) we get $\beta\phi^{-1}([\psi(L_{\gamma,g_4}),\phi(L_{-\gamma,-g_4})]) \neq 0$. For any $g_5 \in \Gamma$ such that $L_{\epsilon\beta,g_5} \neq 0$, we have

$$0 \neq [[L_{\gamma\psi^{-1},g_4}, L_{-\gamma\phi^{-1},-g_4}],\phi(L_{\epsilon\beta,g_5})] = L_{\epsilon\beta\phi^{-1}\psi^{-1},g_5} \subset I$$

and so $L_{\epsilon\beta} \subset I$. That is, we can assert that

$$L_{\epsilon\beta} \subset I \tag{4.9}$$

for any $\beta \in \Lambda$ and some $\epsilon \in \{\pm 1\}$. Since $H = \sum_{\beta\in\Lambda}[L_{\beta\psi^{-1}}, L_{-\beta\phi^{-1}}]$, we get

$$H \subset I. \tag{4.10}$$

Now, given any $-\epsilon\beta \in \Lambda$, by the facts $-\epsilon\beta \neq 0$, $H \subset I$ and the maximal length of L we have

$$[H, L_{-\epsilon\beta\psi}] = L_{-\epsilon\beta} \subset I. \tag{4.11}$$

From (4.9)-(4.11) we conclude that $I = L$. Consequently L is simple.

Theorem 4.6 Let L be a split regular BiHom-Lie color algebra of maximal length, root multiplicative and satisfying $\mathrm{Z}(L) = 0$, $H = \sum_{\alpha\in\Lambda}[L_{\alpha\psi^{-1}}, L_{-\alpha\phi^{-1}}]$. Then $L = \oplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$, where any $I_{[\alpha]}$ is a simple (split) ideal having its roots system $\Lambda_{I_{[\alpha]}}$, and all of its elements are connected.

Proof By corollary 3.7, $L = \oplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$ is the direct sum of the ideals $I_{[\alpha]} = H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}$=$(\sum_{\beta\in[\alpha]}[L_{\beta\psi^{-1}}, L_{-\beta\phi^{-1}}]) \oplus (\oplus_{\beta\in[\alpha]} L_\beta)$ having any $I_{[\alpha]}$ its root system, $\Lambda_{I_{[\alpha]}} := [\alpha]$. It is easy to check that $\Lambda_{I_{[\alpha]}}$ has all of its roots $\Lambda_{I_{[\alpha]}}$-connected, (connected through roots in

$\Lambda_{I_{[\alpha]}}$). We also have that any of the $I_{[\alpha]}$ is root-multiplicative as consequence of the root-multiplicativity of L. Clearly $I_{[\alpha]}$ is of maximal length, and finally $\mathrm{Z}_{I_{[\alpha]}}(I_{[\alpha]})=0$, (where $\mathrm{Z}_{I_{[\alpha]}}(I_{[\alpha]})$ denotes the center of $I_{[\alpha]}$ in $I_{[\alpha]}$), as consequence of $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$, (Theorem 3.6), and $\mathrm{Z}(L) = 0$. We can apply Theorem 4.5 to any $I_{[\alpha]}$ so as to conclude that $I_{[\alpha]}$ is simple. It is clear that the decomposition $L = \oplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$ satisfies the assertions of the theorem.

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分裂的正则双Hom-李Color代数

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摘要: 本文研究了任意分裂的正则双Hom-李color 代数的结构. 利用此种代数的根连通, 得到了带有对称根系的分裂的正则双Hom-李color 代数. L可以表示成$L = U + \sum\limits_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$, 其中$U$是交换(阶化)子代数$H$ 的子空间, 任意$I_{[\alpha]}$为L的理想, 并且满足当$[\alpha] \neq [\beta]$时, $[I_{[\alpha]}, I_{[\beta]}] = 0$. 在一定条件下，定义$L$的最大长度和根可积, 证明$L$可分解为单(阶化)理想族的直和.

关键词: 双Hom-李color代数; 分裂; 根空间; 根系

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