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STUDY ON 2-DIMENSIONAL SUBMANIFOLDS WITH CONSTANT DETERMINANT OF BLASCHKE TENSOR

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Abstract: In this paper, we study the rigidity of 2-dimensional submanifolds in S^{2+p}. Let M^2 be a 2-dimensional submanifold in the $(2+p)$-dimensional unit sphere S^{2+p} without umbilic points. Four basic invariants of M^2 under the Moebius transformation group of S^{2+p} are Moebius metric g, Blaschke tensor A, Moebius form Φ and Moebius second fundamental form B. In this paper, by using inequality estimation, we proved the following rigidity theorem: Let $x : M^2 \to S^{2+p}$ be a $2-$dimensional compact submanifold in the $(2+p)-$dimensional unit sphere S^{2+p} with vanishing Moebius form Φ and $\mathrm{Det}\,A = c(const) > 0$, if $\mathrm{tr}\,A \geq \frac{1}{4}$, then either $x(M^2)$ is Moebius equivalent to a minimal submanifold with constant scalar curvature in S^{2+p}, or $S^1(r) \times S^1(\sqrt{\frac{1}{1+c^2} - r^2})$ in $S^3(\frac{1}{\sqrt{1+c^2}})$, where $r^2 = \frac{2-\sqrt{1-64c}}{4(1+c^2)}$. Our results complement the case 2-dimensional submanifolds in document [3].

Keywords: $2-$dimensional submanifolds; Moebius metric; Moebius form; Moebius second fundamental form; Blaschke tensor

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1 Introduction

Let $x : M^2 \to S^{2+p}$ be a $2-$dimensional submanifold in the $(2+p)-$dimensional unit sphere S^{2+p} without umbilic points. Let $\{e_i\}$ be a local orthonormal basis for the first fundamental form $I = dx \cdot dx$ with dual basis $\{\theta_i\}$. Let $II = \sum\limits_{i,j,\alpha} h^{\alpha}_{ij}\theta_i\theta_j e_\alpha$ be the Moebius second fundamental form and $H = \sum\limits_{\alpha} H^\alpha e_\alpha$ be the mean curvature vector of x, where $\{e_\alpha\}$ is a local orthonormal basis for the normal bundle of x. Define positive function $\rho^2 = 2(\sum\limits_{\alpha,i,j} \left(h^\alpha_{ij}\right)^2 - 2\|H\|^2)$, the g= $\rho^2 I$ is Moebius metric and is a Moebius invariant, the normalized scalar curvature of g will be denoted by R and is called the normalized Moebius scalar curvature. Three basic Moebius invariants of x, Moebius form $\Phi = \sum\limits_{i,\alpha} C^\alpha_i \theta_i e_\alpha$, Blaschke tensor $A = \rho^2 \sum\limits_{i,j} A_{ij}\theta_i\theta_j$ and the Moebius second fundamental form $B = \rho^2 \sum\limits_{i,j} B_{ij}\theta_i\theta_j$, are

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defined by([1])

$$C_i^\alpha = -\rho^{-2}(H_{,i}^\alpha + \sum_j (h_{ij}^\alpha - H^\alpha \delta_{ij}) e_j(\log\rho)), \tag{1.1}$$

$$A_{ij} = -\rho^{-2}(Hess_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho) - \sum_\alpha H^\alpha h_{ij}^\alpha) \tag{1.2}$$

$$-\frac{1}{2}\rho^{-2}\left(\|\nabla\log\rho\|^2 - 1 + \|H\|^2\right)\delta_{ij},$$

$$B_{ij}^\alpha = \rho^{-1}\left(h_{ij}^\alpha - H^\alpha\delta_{ij}\right), \tag{1.3}$$

where $Hess_{ij}$ and ∇ are the Hessian-matrix and the gradient with respect to the induced metric $I = dx \cdot dx$. Let $\nabla^\perp$ be normal connection, and the $H_{,i}^\alpha$ is defined by $\nabla^\perp H = H_{,i}^\alpha \theta_i e_\alpha$. Moreover, we introduce the trace-free Blaschke tensor

$$\widetilde{A} = A - \frac{1}{2}\operatorname{tr} A \cdot \mathrm{g}, \quad \widetilde{A}_{ij} = A_{ij} - \frac{1}{2}\sum_k A_{kk}\delta_{ij}, \quad \|\widetilde{A}\|^2 = \sum_{i,j}\left(\widetilde{A}_{ij}\right)^2, \tag{1.4}$$

$\|\widetilde{A}\| = 0$ if and only if A is isotropic tensor.

Hu and Li studied the dimension of submanifold is $m \geq 3$, the Moebius form $\Phi = 0$, and the constant scalar curvature. In this paper, we proved the dimension of submanifold is $m = 2$, the Moebius form $\Phi = 0$, and $\operatorname{Det} A = c(const) > 0$, we get the following theorem.

Theorem 1.1 Let $x : M^2 \to S^{2+p}$ be a 2−dimensional compact submanifold in the $(2+p)$−dimensional unit sphere S^{2+p} with vanishing Moebius form Φ and $0 < \operatorname{Det} A = c(const) < \frac{1}{64}$, then

$$\int_M \left(\operatorname{tr} A - \frac{1}{4}\right)\|\widetilde{A}\|^2 dM \leq 0. \tag{1.5}$$

In particular, if $\operatorname{tr} A \geq \frac{1}{4}$, then either

$$\widetilde{A} \equiv 0, \tag{1.6}$$

and $x(M^2)$ is Moebius equivalent to a minimal submanifold with constant curvature in S^{2+p}; or

$$\operatorname{tr} A = \frac{1}{4}, \tag{1.7}$$

and $x(M^2)$ is Moebius equivalent to $S^1(r) \times S^1(\sqrt{\frac{1}{1+c^2} - r^2})$ in $S^3(\frac{1}{\sqrt{1+c^2}})$, where $r^2 = \frac{2-\sqrt{1-64c}}{4(1+c^2)}$.

2 Preliminaries

In this section, we give the Moebius invariants and review its structural equations for surfaces in S^{2+p}, for details we refer to [2].

Let R_1^{4+p} be the Lorentzian space with inner product

$$\langle x, y\rangle = -x_0y_0 + x_1y_1 + \cdots + x_{3+p}y_{3+p}, \tag{2.1}$$

where $x = (x_0, x_1, \cdots, x_{3+p})$, and $y = (y_0, y_1, \cdots, y_{3+p})$. Let $x : M^2 \to S^{2+p}$ be an umbilic-free surface immersed in S^{2+p}. We define the Moebius position vector $Y : M^2 \to R_1^{4+p}$ of x by

$$Y = \rho(1, x), \quad \rho^2 = 2(\sum_{\alpha,i,j} \left(h_{ij}^{\alpha}\right)^2 - 2\|H\|^2) > 0. \tag{2.2}$$

Then we have the following.

Theorem 2.1 (see[2]) Two submanifolds x, $\widehat{x} : M \to S^n$ are Moebius equivalent if and only if there exists T in the Lorentz group $O(n+1,1)$ in R_1^{4+p} such that $Y = \widehat{Y}T$.

From Theorem 2.1, we know that the 2-form

$$\mathrm{g} = \langle dY, dY \rangle = \rho^2 dx \cdot dx, \tag{2.3}$$

is a Moebius invariant(see[1]). Let Δ be the Laplace operator with respect to g. Then we have $\langle \Delta Y, \Delta Y \rangle = 1 + 4K$, where K is the sectional curvature of g ([1]). By defining

$$N = -\frac{1}{2}\Delta Y - \frac{1}{8}(1 + 4K)Y, \tag{2.4}$$

then we have([1])

$$\langle \Delta Y, Y \rangle = -2, \quad \langle \Delta Y, dY \rangle = 0, \tag{2.5}$$

$$\langle Y, Y \rangle = 0, \quad \langle N, Y \rangle = 1, \quad \langle N, N \rangle = 0. \tag{2.6}$$

Let $\{E_1, E_2\}$ be a local orthonormal basis for (M^2, g) with dual basis $\{\omega_1, \omega_2\}$, write $Y_i = E_i(Y)$, then

$$\langle Y_i, Y \rangle = \langle Y_i, N \rangle = 0, \quad \langle Y_i, Y_j \rangle = \delta_{ij}, \quad 1 \le i, j \le 2. \tag{2.7}$$

Let V be the orthogonal complement of span $\{Y, N, Y_1, Y_2\}$ in R_1^{4+p}. Then we have the orthogonal decomposition

$$R_1^{4+p} = \mathrm{span}\{Y, N\} \oplus \mathrm{span}\,\{Y_1, Y_2\} \oplus V. \tag{2.8}$$

Let $\{E_\alpha\}$ be an orthonormal basis of V, where

$$E_\alpha = \left(H^\alpha, H^\alpha x + e_\alpha\right), \quad 3 \le \alpha \le 2 + p. \tag{2.9}$$

Then $\{Y, N, Y_1, Y_2, E_3, \cdots, E_{2+p}\}$ forms a moving frame in R_1^{4+p} along M^2. The structure equations are given by

$$dY = Y_1\omega_1 + Y_2\omega_2, \tag{2.10}$$

$$dN = (A_{11}\omega_1 + A_{12}\omega_2)Y_1 + (A_{21}\omega_1 + A_{22}\omega_2)Y_2 + \sum_\alpha (C_1^\alpha \omega_1 + C_2^\alpha \omega_2)E_\alpha, \tag{2.11}$$

$$dY_1 = -(A_{11}\omega_1 + A_{12}\omega_2)Y - \omega_1 N + \omega_{11}Y_1 + \omega_{12}Y_2 + \sum_\alpha (B_{11}^\alpha \omega_1 + B_{12}^\alpha \omega_2)E_\alpha, \tag{2.12}$$

$$dY_2 = -(A_{21}\omega_1 + A_{22}\omega_2)Y - \omega_2 N + \omega_{21}Y_1 + \omega_{22}Y_2 + \sum_\alpha (B_{21}^\alpha \omega_1 + B_{22}^\alpha \omega_2)E_\alpha, \tag{2.13}$$

$$dE_\alpha = -\sum_i C_i^\alpha \omega_i Y - \sum_j (B_{1j}^\alpha \omega_j Y_1 + B_{2j}^\alpha \omega_j Y_2) + \sum_\beta \omega_{\alpha\beta} E_\beta, \tag{2.14}$$

where the coefficients ω_{ij} belong to the connection form of the Moebius metric g, and we have the symmetries $A_{ij} = A_{ji}$, $B_{ij} = B_{ji}$. It is clear that

$$A = A_{11}\omega_1 \otimes \omega_1 + A_{22}\omega_2 \otimes \omega_2 + 2A_{12}\omega_1 \otimes \omega_2, \tag{2.15}$$

$$B = (B_{11}^\alpha \omega_1 \otimes \omega_1 + B_{22}^\alpha \omega_2 \otimes \omega_2 + 2B_{12}^\alpha \omega_1 \otimes \omega_2)E_\alpha, \tag{2.16}$$

$$\Phi = (C_1^\alpha \omega_1 + C_2^\alpha \omega_2)E_\alpha, \tag{2.17}$$

are Moebius invariants, and

$$B_{11}^\alpha + B_{22}^\alpha = 0, \quad \sum_\alpha [(B_{11}^\alpha)^2 + 2(B_{12}^\alpha)^2 + (B_{22}^\alpha)^2] = \frac{1}{2}. \tag{2.18}$$

Define the covariant derivatives of A, B and Φ by ([1])

$$A_{ij,1}\omega_1 + A_{ij,2}\omega_2 = dA_{ij} + A_{i1}\omega_{1j} + A_{i2}\omega_{2j} + A_{1j}\omega_{1i} + A_{2j}\omega_{2i}, \tag{2.19}$$

$$B_{ij,1}^\alpha \omega_1 + B_{ij,2}^\alpha \omega_2 = dB_{ij}^\alpha + B_{i1}^\alpha \omega_{1j} + B_{i2}^\alpha \omega_{2j} + B_{1j}^\alpha \omega_{1i} + B_{2j}^\alpha \omega_{2i} + \sum_\beta B_{ij}^\beta \omega_{\beta\alpha}, \tag{2.20}$$

$$C_{i,1}^\alpha \omega_1 + C_{i,2}^\alpha \omega_2 = dC_i^\alpha + C_1^\alpha \omega_{1i} + C_2^\alpha \omega_{2i} + \sum_\beta C_i^\beta \omega_{\beta\alpha}. \tag{2.21}$$

The integrability conditions for the structure equations (2.10) – (2.14) are given by ([1])

$$A_{i1,2} - A_{i2,1} = \sum_\alpha \left(B_{i2}^\alpha C_1^\alpha - B_{i1}^\alpha C_2^\alpha\right), \tag{2.22}$$

$$C_{1,2}^\alpha - C_{2,1}^\alpha = \sum_k \left(B_{1k}^\alpha A_{k2} - B_{k2}^\alpha A_{k1}\right), \tag{2.23}$$

$$B_{i1,2}^\alpha - B_{i2,1}^\alpha = \delta_{i1} C_2^\alpha - \delta_{i2} C_1^\alpha, \tag{2.24}$$

$$R_{1212} = -\frac{1}{4} + \operatorname{tr} A, \tag{2.25}$$

$$R_{\alpha\beta 12} = -2\left(B_{22}^\alpha B_{12}^\beta - B_{12}^\alpha B_{22}^\beta\right), \tag{2.26}$$

$$\sum_i B_{ij,i}^\alpha = -C_j^\alpha, \tag{2.27}$$

where R_{1212} and $R_{\alpha\beta 12}$ denote the sectional curvature of g and the normal curvature of the normal connection. Set $K = R_{1212}$. The second covariant derivative of A_{ij} and B_{ij}^α are defined by ([1])

$$\begin{aligned} A_{ij,k1}\omega_1 + A_{ij,k2}\omega_2 = & dA_{ij,k} + A_{1j,k}\omega_{1i} + A_{2j,k}\omega_{2i} + A_{i1,k}\omega_{1j} + A_{i2,k}\omega_{2j} \\ & + A_{ij,1}\omega_{1k} + A_{ij,2}\omega_{2k}, \end{aligned} \tag{2.28}$$

$$\begin{aligned}B^{\alpha}_{ij,k1}\omega_1 + B^{\alpha}_{ij,k2}\omega_2 =& dB^{\alpha}_{ij,k} + B^{\alpha}_{1j,k}\omega_{1i} + B^{\alpha}_{2j,k}\omega_{2i} + B^{\alpha}_{i1,k}\omega_{1j} + B^{\alpha}_{i2,k}\omega_{2j}\\ &+ B^{\alpha}_{ij,1}\omega_{1k} + B^{\alpha}_{ij,2}\omega_{2k} + \sum_{\beta} B^{\beta}_{ij,k}\omega_{\beta\alpha}.\end{aligned} \tag{2.29}$$

3 Integral Inequality

Let $x : M^2 \to S^{2+p}$ be a submanifold in S^{2+p} without umbilic points, the Moebius metric is $g = \rho^2 dx \cdot dx$, and so the canonical lift of x is given by $Y = \rho(k, x)$([1]). Then along with M^2, we can choose a moving frame $\{Y, N, Y_1, Y_2, E_3, \cdots, E_{2+p}\}$ in R^{4+p}_1, and we replace E_α in (2.9) by $E_\alpha = (H^\alpha k, e_\alpha + H^\alpha x)$. For the Moebius invariants A, B and Φ appearing in the structure equation (2.11) – (2.14), by calculation, we can get the expression (1.1) for Φ, (1.3) for B and (1.2) should be changed to([3])

$$\begin{aligned}A_{ij} =& -\rho^{-2}\left(Hess_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho) - \sum_{\alpha} H^\alpha h^\alpha_{ij}\right)\\ &- \frac{1}{2}\rho^{-2}\left(\|\nabla\log\rho\|^2 - \frac{1}{k^2} + \|H\|^2\right)\delta_{ij}.\end{aligned} \tag{3.1}$$

Lemma 3.1 For any positive constants $k > a > 0$, the torus $x_{a,k} : M_{a,k} = S^1(a) \times S^1(b) \to S^3(k)$, $a^2 + b^2 = k^2$, choose unit frame field $\{e_1\}$ and $\{e_2\}$ in $S^1(a)$ and $S^1(b)$ respectively, the Moebius invariants components of the torus are as follows:

$$\Phi \equiv 0, \tag{3.2}$$

$$K = 0, \tag{3.3}$$

$$A_{11} = \frac{3}{8} - \frac{a^2}{2k^2},\ A_{22} = -\frac{1}{8} + \frac{a^2}{2k^2},\ A_{12} = 0, \tag{3.4}$$

$$B_{11} = -\frac{1}{2},\ B_{22} = \frac{1}{2},\ B_{12} = 0. \tag{3.5}$$

Proof We write $R^4 = R^2 \times R^2$ and let $x_1 : S^1 \to R^2$ and $x_2 : S^1 \to R^2$ be standard embeddings of the unit sphere, then $x = a(x_1, 0) + b(0, x_2)$. The unit normal vector of $M_{a,k} = S^1(a) \times S^1(b)$ in $S^3(k)$ is given by $e_3 = \frac{b}{k}(x_1, 0) - \frac{a}{k}(0, x_2)$; the second fundamental form of $M_{a,k}$ is given by $II = -dxde_3 = \frac{ab}{k}(-dx_1 \cdot dx_1 + dx_2 \cdot dx_2)$; the Euclidean induced metric of M^2 is given by $I = a^2 dx_1 \cdot dx_1 + b^2 dx_2 \cdot dx_2$. Choose an orthonormal frame $\{e_1, e_2\}$ on TM^2 with dual frame $\{\theta_1, \theta_2\}$ such that $d(ax_1) = \theta_1 e_1$ and $d(bx_2) = \theta_2 e_2$, then we have

$$I = \theta_1^2 + \theta_2^2, \quad II = -\frac{b}{ka}\theta_1^2 + \frac{a}{kb}\theta_2^2, \tag{3.6}$$

$$h_{ij} = \lambda_i \delta_{ij}, \quad \lambda_1 = -\frac{b}{ka}, \quad \lambda_2 = \frac{a}{kb}. \tag{3.7}$$

From (3.7) we see that

$$H := \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{a^2 - b^2}{2abk}, \tag{3.8}$$

$$S:=\lambda_1^2+\lambda_2^2=\frac{a^4+b^4}{a^2b^2k^2}, \tag{3.9}$$

$$\rho^2:=2\left(S-2H^2\right)=\frac{k^2}{a^2b^2}. \tag{3.10}$$

The Moebius metric g is given by

$$\mathrm{g}=\rho^2dx\cdot dx=\frac{k^2}{a^2b^2}\left(\theta_1^2+\theta_2^2\right)=\omega_1^2+\omega_2^2, \tag{3.11}$$

and

$$\omega_i=\frac{k}{ab}\theta_i,\quad E_i=\frac{ab}{k}e_i,\quad Y_i=(0,e_i),\quad E_3=(Hk,e_3+Hx),\quad 1\le i\le 2. \tag{3.12}$$

$$B_{ij}=b_i\delta_{ij},\quad b_1=-\frac{1}{2},\quad b_2=\frac{1}{2}. \tag{3.13}$$

From $(3.1),(3.8),(3.9)$ and (3.10), we have

$$A_{ij}=-\frac{1}{2}\rho^{-2}\left(H^2\delta_{ij}-\frac{1}{k^2}-2Hh_{ij}\right)=a_i\delta_{ij}, \tag{3.14}$$

where

$$a_1=\frac{3}{8}-\frac{a^2}{2k^2},\quad a_2=-\frac{1}{8}+\frac{a^2}{2k^2}. \tag{3.15}$$

Then we have

$$\operatorname{tr}A=a_1+a_2=\frac{1}{4}, \tag{3.16}$$

from $\operatorname{tr}A=\frac{1}{4}(1+4K)$, thus

$$K=0. \tag{3.17}$$

So the conclusion holds.

Now, we suppose that $\Phi=0$, $K=0$ and $\operatorname{tr}A=\frac{1}{4}>0$, then we have

$$A_{i1,2}=A_{i2,1},\quad B^\alpha_{i1,2}=B^\alpha_{i2,1}, \tag{3.18}$$

$$B^\alpha_{i1}A_{1j}+B^\alpha_{i2}A_{2j}=B^\alpha_{1j}A_{1i}+B^\alpha_{2j}A_{2i}. \tag{3.19}$$

From (1.4), we see $\tilde{A}_{ij}=A_{ij}-\frac{1}{2}\operatorname{tr}A\delta_{ij}$, and define

$$\|A\|^2=A_{11}^2+2A_{12}^2+A_{22}^2,\quad \|\widetilde{A}\|^2=\tilde{A}_{11}^2+2\tilde{A}_{12}^2+\tilde{A}_{22}^2, \tag{3.20}$$

then we have

$$\|A\|^2=\|\widetilde{A}\|^2+\frac{1}{2}(\operatorname{tr}A)^2, \tag{3.21}$$

$$\operatorname{tr}A^3=a_1^3+a_2^3. \tag{3.22}$$

Choose a basis $\{E_i\}$ such that (A_{ij}) is diagonalized, i.e.,

$$A_{ij}=a_i\delta_{ij}. \tag{3.23}$$

Then we have

$$\tilde{A}_{ij}=\tilde{a}_i\delta_{ij},\quad \tilde{a}_1=a_1-\frac{1}{2}(a_1+a_2),\quad \tilde{a}_2=a_2-\frac{1}{2}(a_1+a_2), \tag{3.24}$$

$$\|\tilde{A}\|^2=\tilde{a}_1^2+\tilde{a}_2^2,\quad \|A\|^2=a_1^2+a_2^2,\quad \tilde{a}_1+\tilde{a}_2=0, \tag{3.25}$$

$$\|A\|^2=\|\widetilde{A}\|^2+\frac{1}{2}\left(a_1+a_2\right)^2, \tag{3.26}$$

$$a_1^3+a_2^3=\tilde{a}_1^3+\tilde{a}_2^3+\frac{3}{2}\left(a_1+a_2\right)\left(\tilde{a}_1^2+\tilde{a}_2^2\right)+\frac{1}{4}\left(a_1+a_2\right)^3, \tag{3.27}$$

$$\tilde{a}_i=a_i-\frac{1}{8}(1+4K),\quad \operatorname{tr}A=a_1+a_2=\frac{1}{4}(1+4K). \tag{3.28}$$

Proposition 3.2 Let $x: M^2\to S^{2+p}$ be a compact submanifold in the unit sphere S^{2+p} with vanishing Moebius form Φ, $\|\nabla A\|^2=\sum\limits_{i,j,k}A_{ij,k}^2$, we have the following equation

$$\frac{1}{2}\Delta\|A\|^2=\|\nabla A\|^2-\sum_\alpha\left[\operatorname{tr}(AB^\alpha)\right]^2+\operatorname{tr}A(\|A\|^2-2\operatorname{Det}A)+\sum_{i,j}A_{ij}(\operatorname{tr}A)_{ij}. \tag{3.29}$$

Proof

$$\begin{aligned}A_{ij,11}+A_{ij,22}=&A_{ij,11}-A_{i1,j1}+A_{i1,j1}-A_{i1,1j}+A_{1i,1j}-A_{11,ij}+A_{ij,22}\\&-A_{i2,j2}+A_{i2,j2}-A_{i2,2j}+A_{2i,2j}-A_{22,ij}+(\operatorname{tr}A)_{ij},\end{aligned} \tag{3.30}$$

$$\begin{aligned}\sum_{i,j}(A_{ij}A_{ij,11}+A_{ij}A_{ij,22})=&-2\sum_{i,j}A_{ij}C_i^\alpha C_j^\alpha-2\sum_\alpha\|AB^\alpha-B^\alpha A\|^2-\sum_\alpha\left[\operatorname{tr}(AB^\alpha)\right]^2\\&-\sum_{i,j,}A_{ij}B_{ij}^\alpha(C_{1,1}^\alpha+C_{2,2}^\alpha)+2\operatorname{tr}A^3-\operatorname{tr}A\|A\|^2\\&-2\sum_{i,j}A_{ij}(C_1^\alpha B_{ij,1}^\alpha+C_2^\alpha B_{ij,2}^\alpha)+\sum_\alpha(\operatorname{tr}A)[(C_1^\alpha)^2+(C_2^\alpha)^2]\\&+\sum_{i,j}A_{ij}(\operatorname{tr}A)_{ij},\end{aligned} \tag{3.31}$$

then

$$\begin{aligned}\frac{1}{2}\Delta\|A\|^2=&\|\nabla A\|^2+\sum_{i,j}A_{ij}\Delta A_{ij}\\=&\|\nabla A\|^2-2\sum_\alpha\sum_{i,j}\left(A_{ij}-\frac{1}{2}\operatorname{tr}A\delta_{ij}C_i^\alpha C_j^\alpha\right)-2\sum_\alpha\|AB^\alpha-B^\alpha A\|^2\\&-\sum_\alpha\left[\operatorname{tr}(AB^\alpha)\right]^2-\sum_\alpha\sum_{i,j,k}A_{ij}B_{ij}^\alpha(C_{1,1}^\alpha+C_{2,2}^\alpha)+2\operatorname{tr}A^3-\operatorname{tr}A\|A\|^2\\&-2\sum_\alpha\sum_{i,j,k}A_{ij}(C_1^\alpha B_{ij,1}^\alpha+C_2^\alpha B_{ij,2}^\alpha)+\sum_{i,j}A_{ij}(\operatorname{tr}A)_{ij}.\end{aligned} \tag{3.32}$$

From $\Phi=0$, we obtain

$$\frac{1}{2}\Delta\|A\|^2=\|\nabla A\|^2-\sum_{\alpha}\left[\operatorname{tr}(AB^{\alpha})\right]^2+\operatorname{tr}A(\|A\|^2-2\operatorname{Det}A)+\sum_{i,j}A_{ij}(\operatorname{tr}A)_{ij}. \tag{3.33}$$

Lemma 3.3 Let $x: M^2\to S^{2+p}$ be a compact submanifold in the unit sphere S^{2+p} with vanishing Moebius form Φ, we have

$$0\geq\int_M\|\nabla A\|^2-\|\nabla\operatorname{tr}A\|^2+2(\operatorname{tr}A-\frac{1}{4})\|\widetilde{A}\|^2, \tag{3.34}$$

where equality holds if and only if $B^{\alpha}=\lambda_{\alpha}\widetilde{A}(\widetilde{A}\neq 0)$.

Proof Let $L: C^{\infty}(M)\to C^{\infty}(M)$ be L operator([5]), and $L(f)$ is defined by $L(f):=(A_{ij}-\operatorname{tr}A\delta_{ij})f_{,ij}$, from(3.18) we have

$$\sum_j A_{ij,j}-(\operatorname{tr}A)_j=\sum_j A_{ij,j}-\sum_i A_{ii,j}=0, \tag{3.35}$$

thus, L^2 self-adjointness, i.e., $(g,L(f))=(f,L(g))$. In particular, $(L(f),1)=0$, i.e., $\int_M L(f)=0$. Then

$$\begin{aligned}L(\operatorname{tr}A)&=(A_{ij}-\operatorname{tr}A\delta_{ij})(\operatorname{tr}A)_{ij}=A_{ij}(\operatorname{tr}A)_{ij}-\operatorname{tr}A\Delta\operatorname{tr}A\\&=A_{ij}(\operatorname{tr}A)_{ij}-\frac{1}{2}\Delta(\operatorname{tr}A)^2+\|\nabla\operatorname{tr}A\|^2,\end{aligned} \tag{3.36}$$

then we have

$$\frac{1}{2}\Delta\left[\|A\|^2-(\operatorname{tr}A)^2\right]=\|\nabla A\|^2-\|\nabla\operatorname{tr}A\|^2-\sum_{\alpha}(\operatorname{tr}(AB^{\alpha}))^2+2\operatorname{tr}A^3-\operatorname{tr}A\|A\|^2+L(\operatorname{tr}A). \tag{3.37}$$

Integrating both sides of the above equation, according to the properties of Δ and L, we get

$$0=\int_M[\|\nabla A\|^2-\|\nabla\operatorname{tr}A\|^2-\sum_{\alpha}(\operatorname{tr}(AB^{\alpha}))^2+\operatorname{tr}A(\|A\|^2-2\operatorname{Det}A)]dM. \tag{3.38}$$

From Cauchy-Schwarz Inequality, we see

$$-\left(\operatorname{tr}(AB^{\alpha})\right)^2=-\left(\operatorname{tr}\left(\widetilde{A}B^{\alpha}\right)\right)^2\geq-\frac{1}{2}\|\widetilde{A}\|^2, \tag{3.39}$$

where the equality holds in (3.39) if and only if $B^{\alpha}=\lambda_{\alpha}\widetilde{A}(\widetilde{A}\neq 0)$. It is clear that $\|A\|^2-2\operatorname{Det}A=2\|\widetilde{A}\|^2$. Thus, we have

$$0\geq\int_M\|\nabla A\|^2-\|\nabla\operatorname{tr}A\|^2+2(\operatorname{tr}A-\frac{1}{4})\|\widetilde{A}\|^2.$$

Lemma 3.4 Let $x: M^2\to S^{2+p}$ be a compact submanifold in the unit sphere S^{2+p} with vanishing Moebius form Φ, $\operatorname{Det}A=c>0$, we have

$$\|\nabla A\|^2-\|\nabla\operatorname{tr}A\|^2\geq 0, \tag{3.40}$$

where equality holds if and only if $\nabla A = 0$.

Proof

$$\|A\|^2 - (\operatorname{tr} A)^2 = -2\operatorname{Det} A = -2c, \tag{3.41}$$

i.e.,

$$A_{11}^2 + 2A_{12}^2 + A_{22}^2 = (A_{11} + A_{22})^2 - 2c. \tag{3.42}$$

Take the derivative of the left hand side of the equation, we have

$$\sum_{i,j} A_{ij,k}^2 = 2\sum_{i,j} A_{ij,k}A_{ij}. \tag{3.43}$$

Take the derivative of the right hand side of the equation, we have

$$(\sum_i A_{ii})_k^2 = 2(\operatorname{tr} A)(\operatorname{tr} A)_k, \tag{3.44}$$

thus

$$\sum_{i,j,k} A_{ij,k}A_{ij} = \operatorname{tr} A(\operatorname{tr} A)_k. \tag{3.45}$$

Square both ends of the above equation, we have

$$(\sum_{i,j,k} A_{ij,k}A_{ij})^2 = \|\nabla \operatorname{tr} A\|^2(\operatorname{tr} A)^2. \tag{3.46}$$

On the left hand side, we use the Cauchy-Schwarz inequality to get

$$(\sum_{i,j,k} A_{ij,k}A_{ij})^2 \le \|\nabla A\|^2\|A\|^2, \tag{3.47}$$

taking (3.47) in (3.46), we have

$$\|\nabla A\|^2\|A\|^2 \ge \|\nabla \operatorname{tr} A\|^2(\operatorname{tr} A)^2, \tag{3.48}$$

$$\|\nabla A\|^2\|A\|^2 \ge \|\nabla \operatorname{tr} A\|^2(\|A\|^2 + 2c), \tag{3.49}$$

$$(\|\nabla A\|^2 - \|\nabla \operatorname{tr} A\|^2)\|A\|^2 \ge 2c\|\nabla \operatorname{tr} A\|^2 \ge 0. \tag{3.50}$$

Then we have

$$\|\nabla A\|^2 - \|\nabla \operatorname{tr} A\|^2 \ge 0, \tag{3.51}$$

where the equality holds if and only if $\nabla A = 0$.

From the above lemma, we can get the following theorem:

Lemma 3.5 Let $x : M^2 \to S^{2+p}$ be a compact submanifold in the unit sphere S^{2+p} with vanishing Moebius form Φ, $\operatorname{Det} A = c(const) > 0$, we have the following inequality

$$\int_M \left(\operatorname{tr} A - \frac{1}{4}\right)\|\widetilde{A}\|^2 dM \le 0.$$

In particular, when $\operatorname{tr} A \geq \frac{1}{4}$, then either $\widetilde{A} = 0$ or $\operatorname{tr} A = \frac{1}{4}$, and we have $B^\alpha = \lambda_\alpha \widetilde{A},\ \nabla A = 0$.

4 Integral Inequality

Theorem 4.1 Let $x: M^2 \to S^{2+p}$ be a $2-$dimensional compact submanifold in the $(2+p)-$dimensional unit sphere S^{2+p} with vanishing Moebius form Φ, and $0 < \operatorname{Det} A = c(const) < \frac{1}{64}$, then

$$\int_M \left(\operatorname{tr} A - \frac{1}{4}\right) \|\widetilde{A}\|^2 dM \leq 0. \tag{4.1}$$

In particular, if $\operatorname{tr} A \geq \frac{1}{4}$, then either

$$\widetilde{A} \equiv 0, \tag{4.2}$$

and $x(M^2)$ is Moebius equivalent to a minimal submanifold with constant curvature in S^{2+p}; or

$$\operatorname{tr} A = \frac{1}{4}, \tag{4.3}$$

and $x(M^2)$ is Moebius equivalent to $S^1(r) \times S^1(\sqrt{\frac{1}{1+c^2} - r^2})$ in $S^3(\frac{1}{\sqrt{1+c^2}})$, where $r^2 = \frac{4+\sqrt{1-64c}}{8}$.

Lemma 4.2 Let $x: M^2 \to S^{2+p}$ be a $2-$dimensional compact submanifold in the $(2+p)-$dimensional unit sphere S^{2+p} with vanishing Moebius form Φ ,if $\|\widetilde{A}\| \neq 0, \operatorname{tr} A = \frac{1}{4}$ and $0 < \operatorname{Det} A = c(const) < \frac{1}{64}$, then the $x(M^2)$ is Moebius equivalent to $S^1(r) \times S^1(\sqrt{\frac{1}{1+c^2} - r^2})$ in $S^3(\frac{1}{\sqrt{1+c^2}})$, where $r^2 = \frac{2-\sqrt{1-64c}}{4(1+c^2)}$.

Proof From $\operatorname{tr} A = \frac{1}{4},\ \operatorname{Det} A = c$, we have

$$a_1 + a_2 = \frac{1}{4}, \quad a_1 a_2 = c. \tag{4.4}$$

Which implies

$$a_1 = \frac{1+\sqrt{1-64c}}{8}, \quad a_2 = \frac{1-\sqrt{1-64c}}{8}, \tag{4.5}$$

then we get

$$\widetilde{a}_1 = \frac{\sqrt{1-64c}}{8}, \quad \widetilde{a}_2 = -\frac{\sqrt{1-64c}}{8}. \tag{4.6}$$

From Lemma 3.5, we have

$$B^\alpha = \lambda_\alpha \widetilde{A}, \tag{4.7}$$

i.e,

$$(B_{11}^\alpha)^2 + 2(B_{12}^\alpha)^2 + (B_{22}^\alpha)^2 = \sum_\alpha \lambda_\alpha^2 [(\widetilde{A}_{11})^2 + (\widetilde{A}_{22})^2] = \frac{1-64c}{32} \sum_\alpha \lambda_\alpha^2, \tag{4.8}$$

by use of (2.18), we see

$$(\frac{1}{32} - 2c) \sum_\alpha \lambda_\alpha^2 = \frac{1}{2}. \tag{4.9}$$

We claim that we can choose the normal frame field $\{E_\alpha\}$, such that

$$B^3_{ij} = \lambda \widetilde{A}_{ij}, \quad B^\alpha_{ij} = 0, \quad \lambda = \frac{4}{\sqrt{1-64c}}, \quad \forall i, j; \alpha \geq 4. \tag{4.10}$$

In fact, we can choose a new orthonormal frame $\{\bar{e}_\alpha\}$ in the normal bundle $N(M^2)$ such that $\bar{e}_3 = \frac{\sum\limits_\alpha \lambda_\alpha e_\alpha}{\sqrt{\sum\limits_\alpha \lambda^2_\alpha}}$, and then define a new orthonormal frame $\{\bar{E}_\alpha\}$ in the Moebius normal bundle by $\bar{E}_\alpha = \left(\bar{H}_\alpha, \bar{H}^\alpha x + \bar{e}_\alpha\right)$, where $\sum\limits_\alpha H^\alpha e_\alpha = \sum\limits_\alpha \bar{H}^\alpha \bar{e}_\alpha = H$ is the mean curvature vector of M^2, then $\bar{E}_3 = \frac{\sum\limits_\alpha \lambda_\alpha E_\alpha}{\sqrt{\sum\limits_\beta \lambda^2_\beta}}$ and with respect to $\bar{E}_\alpha$. If $\{e_\alpha\}$, $\{\bar{e}_\alpha\}$ are two orthonormal frames in the normal bundle with $e_\alpha = \sum \sigma_{\alpha\beta}\bar{e}_\beta$, where $(\sigma_{\alpha\beta})$ is an orthogonal matrix, then we have $E_\alpha = \sum\limits_\beta \sigma_{\alpha\beta}\bar{E}_\beta$. From $B^\alpha_{ij} = \rho^{-1}\left(h^\alpha_{ij} - H^\alpha \delta_{ij}\right)$, when $\alpha \geq 4$, we have

$$h^\alpha_{ij} = H^\alpha \delta_{ij}, \tag{4.11}$$

that means that span$\{e_4, e_5, \cdots, e_{2+p}\}$ is totally umbilical in the normal bundle $N(M^2)$. From (4.10) we have

$$\omega_{3\alpha} = \theta_{3\alpha} \equiv 0, \quad \forall \alpha \tag{4.12}$$

where $\theta_{\alpha\beta}$ is the Euclidean normal connection of $N(M^2)$([1]).

From (2.18), we have $B^3_{11} + B^3_{22} = 0, \frac{1}{2} = (B^3_{11})^2 + (B^3_{22})^2$, which implies

$$B^3_{11} = -\frac{1}{2}, \quad B^3_{22} = \frac{1}{2}. \tag{4.13}$$

From (2.19) and $\nabla A = 0$, we have

$$A_{11}\omega_{12} + A_{22}\omega_{21} = 0, \tag{4.14}$$

$$(a_1 - a_2)\omega_{12} = 0, \tag{4.15}$$

thus, we have

$$\omega_{12} = 0, \tag{4.16}$$

$$d\omega_1 = 0, \quad d\omega_2 = 0. \tag{4.17}$$

Hence, by [1,Theorem 1] and (4.11), (4.12), we can conclude that $x(M^2)$ is located in some sphere $S^3(\frac{1}{\sqrt{1+c^2}})$ which are totally umbilical submanifold of S^n.

Since a Riemannian universal coverage is locally equidistant and not general, we can assume that M^2 is simply connected. From above, TM^2 has the following decomposition $TM^2 = V_1 \oplus V_2$, where V_1 and V_2 are the 1-dimensional eigenspace of the Blaschke tensor A with eigenvalues a_1 and a_2.

Form (4.17), we can get that the eigenspaces V_1 and V_2 are both integrable. We can write $M = M_1 \times M_2$, where M_1 and M_2 are both 1−dimensional submanifold. We can define $\mathrm{g}_1 = \omega^2_1$ and $\mathrm{g}_2 = \omega^2_2$, then we have

$$(M^2, \mathrm{g}) = (M_1, \mathrm{g}_1) \times (M_2, \mathrm{g}_2). \tag{4.18}$$

From (2.12) and (4.16), we have

$$dY_1 = (-a_1 Y - N + b_1 E_3)\omega_1, \tag{4.19}$$

$$dY_2 = (-a_2 Y - N + b_2 E_3)\omega_2, \tag{4.20}$$

$$\langle dY_1, dY_1\rangle = (2a_1 + b_1^2)(\omega_1)^2 = \frac{2+\sqrt{1-64c}}{4}(\omega_1)^2, \tag{4.21}$$

$$\langle dY_2, dY_2\rangle = (2a_2 + b_2^2)(\omega_2)^2 = \frac{2-\sqrt{1-64c}}{4}(\omega_2)^2. \tag{4.22}$$

Since M^2 is compact submanifold, M_1 and M_2 are also compact submanifold.

Then we consider $\bar{x} : S^1(r_1)\times S^1(r_2) \to S^3(\frac{1}{\sqrt{1+c^2}})$ being a torus, where $r_1^2 = \frac{2-\sqrt{1-64c}}{4(1+c^2)}$, $r_2^2 = \frac{2+\sqrt{1-64c}}{4(1+c^2)}$. Let $\widetilde{x} : M^2 \to S^3(\frac{1}{\sqrt{1+c^2}})$, from (3.13), (3.15), (4.5), (4.13), $\widetilde{x}$ and $\bar{x}$ have the same Moebius shape. From the proof of Lemma 3.1 we see that the Moebius metric $\bar{\mathrm{g}}$ of $\bar{x}$ is given by $\bar{\mathrm{g}} = \bar{\rho}^2 d\bar{x}\cdot d\bar{x} = \bar{\mathrm{g}}_1 + \bar{\mathrm{g}}_2$, where for $\bar{x} = r_1\overline{(x_1,0)} + r_2\overline{(0,x_2)}$,

$$\bar{\mathrm{g}}_1 = \frac{4}{2+\sqrt{1-64c}}d\bar{x}_1\cdot d\bar{x}_1, \quad \bar{\mathrm{g}}_2 = \frac{4}{2-\sqrt{1-64c}}d\bar{x}_2\cdot d\bar{x}_2. \tag{4.23}$$

From (4.21) − (4.23), we have a 1−dimensional manifold with the same curvature as $(S^1(r_1), \bar{\mathrm{g}}_1)$ and (M_1, g_1), $(s^1(r_2), \bar{\mathrm{g}}_2)$ and (M_2, g_2) are also 1−dimensional manifolds with the same curvature. Thus there exist isometries $\psi_1 : (M_1, \mathrm{g}_1) \to (S^1(r_1), \bar{\mathrm{g}}_1)$ and $\psi_2 : (M_2, \mathrm{g}_2) \to (S^1(r_2), \bar{\mathrm{g}}_2)$, let $\psi = (\psi_1, \psi_2)$, then $\psi : M^2 \to S^1(r_1)\times S^1(r_2)$ holds the Moebius metric and the Moebius shape operator. (M^2, g) and $(S^1(r_1)\times S^1(r_2), \bar{\mathrm{g}})$ have the same Moebius metric, Moebius second fundamental form, Moebius shape operator, Blaschke tensor, so the $x(M^2)$ is Moebius equivalent to $S^1(r)\times S^1(\sqrt{\frac{1}{1+c^2} - r^2})$ in $S^3(\frac{1}{\sqrt{1+c^2}})$, where $r^2 = \frac{2-\sqrt{1-64c}}{4(1+c^2)}$.

This completes the proof of the main theorem.

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Blaschke 张量的行列式为常数的2维子流形的研究

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摘要: 本文研究了 S^{2+p} 中 2 维子流形的莫比乌斯刚性问题. 设 M^2 是 $2+p$ 维单位球 S^{2+p} 中的无脐子流形, M^2 在 S^{2+p} 的莫比乌斯变换群下的四个莫比乌斯基本量为莫比乌斯度量 g, Blaschke 张量 A, 莫比乌斯形式 Φ 以及莫比乌斯第二基本形式 B, 利用不等式估计, 证明了下列刚性定理: 设 $x: M^2 \to S^{2+p}$ 是 $2+p$ 维单位球 S^{2+p} 中莫比乌斯形式消失的 2 维紧致子流形, Blaschke 张量 A 的行列式 $\operatorname{Det} A = c(const) > 0$, 若 $\operatorname{tr} A \geq \frac{1}{4}$, 那么$x(M^2)$ 莫比乌斯等价于 S^{2+p} 中常曲率极小子流形或者 $S^3(\frac{1}{\sqrt{1+c^2}})$ 中环面 $S^1(r) \times S^1(\sqrt{\frac{1}{1+c^2} - r^2})$, 其中 $r^2 = \frac{2-\sqrt{1-64c}}{4(1+c^2)}$. 本文的证明补充了文献 [3] 中 2 维子流形情形.

关键词: 2维子流形; 莫比乌斯度量; 莫比乌斯形式; 莫比乌斯第二基本形式; Blaschke张量

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