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ON REFINED YOUNG'S REVERSE INEQUALITIES FOR POSITIVE LINEAR OPERATORS

CAO Hai-song

(*School of Mathematics and Statistics, North China University of Water Resources and Electric Power, Zhengzhou 450045, China*)

Abstract: In this paper, the inverse inequalities of Young for positive linear operators are studied. By using the properties of operators' monotone function and convex function, we obtained some improved scalar versions and corresponding operator versions of Young's inverse inequality with Kantorovich constant, which generalize the conclusions in the literature.

Keywords: Young's reverse inequality; positive linear operators; Kantorovich constant

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1 Introduction

Let $\mathcal{B}(\mathcal{H})$ be the $\mathcal{C}^*-$ algebra of all bounded linear operators on a Hilbert space H equipped with the operator norm, $\mathcal{S}(\mathcal{H})$ the set of all bounded self-adjoint operators, and $\mathbb{P} = \mathbb{P}(\mathrm{H})$ the open convex cone of all positive invertible operators. For $X, Y \in \mathcal{S}(\mathcal{H})$, we write $X \leq Y$ if $Y - X$ is positive, and $X < Y$ if $Y - X$ is positive invertible.

The classical Young inequality says that if $a, b \geq 0$ and $0 \leq v \leq 1$, then

$$a^v b^{1-v} \leq va + (1-v)b \tag{1.1}$$

with equality if and only if $a = b$.

This inequality has been studied, generalized and refined in different directions, see[1-2]. It is worth to mention that in [3], J. L. Wu and J. G. Zhao presented refined and reversed versions of the scalar Young type inequality which can be stated as follows:

$$K(\sqrt{h}, 2)^{r'} a^v b^{1-v} + r(\sqrt{a} - \sqrt{b})^2 \leq va + (1-v)b, \tag{1.2}$$

$$K(\sqrt{h}, 2)^{-r'} a^v b^{1-v} + s(\sqrt{a} - \sqrt{b})^2 \geq va + (1-v)b, \tag{1.3}$$

where $a, b > 0, v \in [0,1] - \{\frac{1}{2}\}, h = \frac{b}{a}, r = \min\{v, 1-v\}, s = \max\{v, 1-v\}, r' = \min\{2r, 1-2r\}$ and $K(\cdot, 2)$ is Kantorovich constant, defined by $K(t, 2) = \frac{(t+1)^2}{4t}$ for $t > 0$.

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Biography: Cao Haisong(1986–), male, born at Dengzhou, Henan, lecturer, major in the numerical linear algebra. E-mail:hscao@ncwu.edu.cn

In [4], a more refined version was presented which can be stated as follows:

(i) If $0 < v \leq \frac{1}{2}$. Then

$$a^{1-v}b^{v} + (1-v)(\sqrt{a}-\sqrt{b})^{2} - r_{0}(\sqrt[4]{ab}-\sqrt{b})^{2} \geq (1-v)a + vb, \tag{1.4}$$

(ii) If $\frac{1}{2} < v < 1$. Then

$$a^{1-v}b^{v} + v(\sqrt{a}-\sqrt{b})^{2} - r_{0}(\sqrt[4]{ab}-\sqrt{a})^{2} \geq (1-v)a + vb, \tag{1.5}$$

where $a, b > 0, v \in (0,1), r = \min\{v, 1-v\}$ and $r_0 = \min\{2r, 1-2r\}$.

Let $A, B \in B(H)$ be two positive operators, $v \in [0,1]$.

$v-$weighted arithmetic mean of A and B, denoted by $A\nabla_{v}B$, is defined as

$$A\nabla_{v}B = (1-v)A + vB.$$

If A is invertible, $v-$ geometric mean of of A and B, denoted by $A\sharp_{v}B$, is defined as

$$A\sharp_{v}B = A^{1/2}(A^{-1/2}BA^{-1/2})^{v}A^{1/2}.$$

For more details, see [5]. When $v = \frac{1}{2}$, we write $A\nabla B$ and $A\sharp B$ for brevity, respectively. It is well known that if A and B are positive invertible operators, then

$$A\nabla_{v}B \geq A\sharp_{v}B, \quad 0 \leq v \leq 1.$$

In [6], S. Furuichi gave a refinement version:

$$A\nabla_{v}B \geq A\sharp_{v}B + 2r(A\nabla B - A\sharp B) \geq A\sharp_{v}B.$$

In [4], J. Zhao and J. Wu presented other improved inequalities:

(i) If $0 < v \leq \frac{1}{2}$. Then

$$A\sharp_{v}B + 2(1-v)(A\nabla B - A\sharp B) - r_{0}(A\sharp B - A\sharp_{\frac{3}{4}}B + B) \geq A\nabla_{v}B, \tag{1.6}$$

(ii) If $\frac{1}{2} < v < 1$. Then

$$A\sharp_{v}B + 2v(A\nabla B - A\sharp B) - r_{0}(A\sharp B - 2A\sharp_{\frac{1}{4}}B + A) \geq A\nabla_{v}B, \tag{1.7}$$

where $A, B \in \mathcal{B}(\mathcal{H})$ are two positive invertible operators, $v \in (0,1), r = \min\{v, 1-v\}$ and $r_0 = \min\{2r, 1-2r\}$.

Since then, many researchers have tried to give new refinements and generalizations of these inequalities and have obtained a series of improvements, one can refer to the references of [7-9].

In this paper, some Young's reverse inequalities with Kantorovich constant for scalars were presented which are improvements of (1.3) $\sim$ (1.5). Then on the base of them, the corresponding variations of recent refinements for positive linear operators were obtained which are refinements of (1.6) and (1.7).

2 Refinements of the Young's Reverse Inequality for Scalars

In this section, some improved Young's reverse inequalities with Kantorovich constant for scalars were presented.

Theorem 2.1 Let a, b be two nonnegative real numbers and $v \in (0,1) - \{\frac{1}{2}\}$.

(i) If $0 < v \leq \frac{1}{4}$, then

$$\begin{aligned}&K(\sqrt[8]{h},2)^{-R'}a^{1-v}b^{v} + (1-v)(\sqrt{a}-\sqrt{b})^2 - 2v(\sqrt[4]{ab}-\sqrt{b})^2 - R(\sqrt[8]{ab^3}-\sqrt{b})^2\\ \geq&(1-v)a + vb,\end{aligned} \tag{2.1}$$

(ii) If $\frac{1}{4} < v < \frac{1}{2}$, then

$$\begin{aligned}&K(\sqrt[8]{h},2)^{-R'}a^{1-v}b^{v} + (1-v)(\sqrt{a}-\sqrt{b})^2 - (1-2v)(\sqrt[4]{ab}-\sqrt{b})^2 - R(\sqrt[8]{ab^3}-\sqrt[4]{ab})^2\\ \geq&(1-v)a + vb,\end{aligned} \tag{2.2}$$

(iii) If $\frac{1}{2} < v \leq \frac{3}{4}$, then

$$\begin{aligned}&K(\sqrt[8]{h},2)^{-R'}a^{1-v}b^{v} + v(\sqrt{a}-\sqrt{b})^2 - (2v-1)(\sqrt[4]{ab}-\sqrt{a})^2 - R(\sqrt[8]{a^3b}-\sqrt[4]{ab})^2\\ \geq&(1-v)a + vb,\end{aligned} \tag{2.3}$$

(iv) If $\frac{3}{4} < v \leq 1$, then

$$\begin{aligned}&K(\sqrt[8]{h},2)^{-R'}a^{1-v}b^{v} + v(\sqrt{a}-\sqrt{b})^2 - (2-2v)(\sqrt[4]{ab}-\sqrt{a})^2 - R(\sqrt[8]{a^3b}-\sqrt{a})^2\\ \geq&(1-v)a + vb,\end{aligned} \tag{2.4}$$

where $h = \frac{b}{a}, r = \min\{v, 1-v\}$, $t = \min\{2r, 1-2r\}$, $R = \min\{2t, 1-2t\}$, $R' = \min\{2R, 1-2R\}$ and $K(\cdot, 2)$ is Kantorovich constant, defined by $K(t,2) = \frac{(t+1)^2}{4t}$ for $t > 0$.

Proof The proof of inequalities (2.2)∼ (2.4) are similar to that of the inequality (2.1). Thus, we only need to prove the inequality (2.1). For (i), if $0 < v \leq \frac{1}{4}$, then by the inequality (1.2), we have

$$\begin{aligned}&K(\sqrt[8]{h},2)^{-R'}a^{1-v}b^{v} + (1-v)(\sqrt{a}-\sqrt{b})^2 - 2v(\sqrt[4]{ab}-\sqrt{b})^2 - (1-v)a - vb\\ =&K(\sqrt[8]{h},2)^{-R'}a^{1-v}b^{v} + (1-4v)b + 4v\sqrt[4]{ab^3} - 2\sqrt{ab}\\ \geq&K(\sqrt[8]{h},2)^{-R'}a^{1-v}b^{v} + K(\sqrt[8]{h},2)^{R'}b^{1-4v}(\sqrt[4]{ab^3})^{4v} + R(\sqrt[8]{ab^3}-\sqrt{b})^2 - 2\sqrt{ab}\\ =&K(\sqrt[8]{h},2)^{-R'}a^{1-v}b^{v} + K(\sqrt[8]{h},2)^{R'}a^{v}b^{1-v} + R(\sqrt[8]{ab^3}-\sqrt{b})^2 - 2\sqrt{ab}\\ \geq&2\sqrt{ab} + R(\sqrt[8]{ab^3}-\sqrt{b})^2 - 2\sqrt{ab}\\ =&R(\sqrt[8]{ab^3}-\sqrt{b})^2.\end{aligned}$$

This completes the proof.

Remark 2.1 Obviously $R \geq 0$ and $K(\cdot, h) \geq 1$, so the inequalities (2.1) ∼ (2.4) are the improvements of the scalar Young type inequalities (1.3) ∼ (1.4).

3 Operator Inequalities for the Improved Young Inequalities

Based on the improvements of the scalar Young type inequalities (2.1) $\sim$ (2.4), we present corresponding operator inequalities for the improved Young inequalities.

Lemma 3.1 Let $X \in \mathcal{B}(\mathcal{H})$ be self-adjoint and let f and g be continuous real functions such that $f(t) \geq g(t)$ for all $t \in Sp(X)$ (the spectrum of X). Then $f(X) \geq g(X)$.

Theorem 3.1 Let $A, B \in \mathcal{B}(\mathcal{H})$ be two positive invertible operators and $v \in (0,1) - \{\frac{1}{2}\}$.

(i) If $0 < v \leq \frac{1}{4}$, then

$$\begin{aligned}&K(\sqrt[8]{h},2)^{-R'}A\sharp_v B + 2(1-v)(A\nabla B - A\sharp B)\\&- 2v(A\sharp B - A\sharp_{\frac{3}{4}}B + B) - R(A\sharp_{\frac{3}{4}}B - 2A\sharp_{\frac{7}{8}}B + B) \geq A\nabla_v B,\end{aligned} \tag{3.1}$$

(ii) If $\frac{1}{4} < v < \frac{1}{2}$, then

$$\begin{aligned}&K(\sqrt[8]{h},2)^{-R'}A\sharp_v B + 2(1-v)(A\nabla B - A\sharp B)\\&- (1-2v)(A\sharp B - A\sharp_{\frac{3}{4}}B + B) - R(A\sharp_{\frac{3}{4}}B - 2A\sharp_{\frac{5}{8}}B + A\sharp B) \geq A\nabla_v B,\end{aligned} \tag{3.2}$$

(iii) If $\frac{1}{2} < v \leq \frac{3}{4}$, then

$$\begin{aligned}&K(\sqrt[8]{h},2)^{-R'}A\sharp_v B + 2v(A\nabla B - A\sharp B)\\&- (2v-1)(A\sharp B - 2A\sharp_{\frac{1}{4}}B + B) - R(A\sharp_{\frac{1}{4}}B - 2A\sharp_{\frac{3}{8}}B + A\sharp B) \geq A\nabla_v B,\end{aligned} \tag{3.3}$$

(iv) If $\frac{3}{4} < v \leq 1$, then

$$\begin{aligned}&K(\sqrt[8]{h},2)^{-R'}A\sharp_v B + 2v(A\nabla B - A\sharp B)\\&- (2-2v)(A\sharp B - 2A\sharp_{\frac{1}{4}}B + A) - R(A\sharp_{\frac{1}{4}}B - 2A\sharp_{\frac{1}{8}}B + A) \geq A\nabla_v B,\end{aligned} \tag{3.4}$$

where $h = \frac{||B||}{||A||}, r = \min\{v, 1-v\}$, $t = \min\{2r, 1-2r\}$, $R = \min\{2t, 1-2t\}$, $R' = \min\{2R, 1-2R\}$ and $K(\cdot, 2)$ is Kantorovich constant, defined by $K(t,2) = \frac{(t+1)^2}{4t}$ for $t > 0$.

Proof The proof of inequalities (3.2)$\sim$ (3.4) are similar to that of inequality (3.1). Thus, we only need to prove the inequality (3.1).

For (i), if $0 < v \leq \frac{1}{4}$, then by the inequality (2.1), we have

$$K(\sqrt[8]{h},2)^{-R'}b^v + (1-v)(1-\sqrt{b})^2 - 2v(\sqrt[4]{b} - \sqrt{b})^2 - R(\sqrt[8]{b^3} - \sqrt{b})^2 \geq (1-v) + vb,$$

for any $b > 0$, which for $X = A^{-1/2}BA^{-1/2}$ and thus for $Sp(X) \subseteq (0, +\infty)$, then

$$\begin{aligned}&K(\sqrt[8]{h},2)^{-R'}X^v + (1-v)(X - 2X^{\frac{1}{2}} + I)\\&- 2v(X^{\frac{1}{2}} - 2X^{\frac{3}{4}} + X) - R(X^{\frac{3}{4}} - 2X^{\frac{7}{8}} + X) \geq (1-v)I + vX.\end{aligned} \tag{3.5}$$

Multiplying both sides of (3.5) by $A^{1/2}$, we get

$$\begin{aligned}&K(\sqrt[8]{h},2)^{-R'}A\sharp_v B + (1-v)(A + B - 2A\sharp B)\\&- 2v(A\sharp B - 2A\sharp_{\frac{3}{4}}B + B) - R(A\sharp_{\frac{3}{4}}B - 2A\sharp_{\frac{7}{8}}B + B) \geq A\nabla_v B.\end{aligned}$$

This is equal to

$$K(\sqrt[8]{h},2)^{-R'}A\sharp_v B+2(1-v)(A\nabla B-A\sharp B)$$
$$-2v(A\sharp B-2A\sharp_{\frac{3}{4}}B+B)-R(A\sharp_{\frac{3}{4}}B-2A\sharp_{\frac{7}{8}}B+B)\geq A\nabla_v B.$$

This completes the proof.

Remark 3.1 Since $f(t)=(\sqrt[8]{t^3}-\sqrt{t})^2$ is a continuous function on $(0,+\infty)$ and $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ is a positive operator, then $Sp(f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}))\subseteq[0,+\infty)$. Then $A^{\frac{1}{2}}f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$ $=A\sharp_{\frac{3}{4}}B-2A\sharp_{\frac{7}{8}}B+B$ is a positive operator. Similarly, we can obtain that the operators $A\sharp_{\frac{3}{4}}B-2A\sharp_{\frac{5}{8}}B+A\sharp B$, $A\sharp_{\frac{1}{4}}B-2A\sharp_{\frac{1}{8}}B+A$ and $A\sharp_{\frac{3}{4}}B-2A\sharp_{\frac{7}{8}}B+B$ are positive operators. Furthermore, $K(\cdot,h)\geq 1$, so the inequalities (3.1)∼ (3.5) are the refinements of (1.6) and (1.7).

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正线性算子的Young 型逆不等式的一个改进

曹海松

(华北水利水电大学数学与统计学院, 河南 郑州 450045)

摘要: 本文研究了正线性算子的Young 型逆不等式的问题. 利用算子单调函数以及凸函数的性质，获得了若干带有Kantorovich 常数的Young 型逆不等式标量形式及相对应的正线性算子的Young 型逆不等式的改进形式, 推广了现有文献中的结论.

关键词: Young 型逆不等式; 正线性算子; Kantorovich常数

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